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On Optimal Reinsurance with Stochastic Premium

Antonella Campana

Department of Economics
University of Molise
Campobasso, Italy

Paola Ferretti

Department of Economics
Ca' Foscari University of Venice
Venice, Italy

Abstract

In this paper we study optimal reinsurance models from the perspective of an insurer by minimizing the total risk exposure under a distortion risk measure in the hypothesis of a stochastic reinsurance premium. This assumption is consistent with reinsurance practice in which reinsurance premiums frequently depend on the recorded claims rate, therefore a random component results. For example, it is consistent with reinstatement clauses which are widely used in the industry.

Mathematics Subject Classification: 62P05

Keywords: non-proportional reinsurance, risk measures, VaR, $CVaR$, risk premium.

1 Introduction

The problem of optimal reinsurance model has been extensively studied from various perspectives, since the fundamental works of Borch [6] and Arrow [1].

Borch [6] showed that stop-loss reinsurance is optimal because it minimizes the variance of the insurer's retained loss under the assumption of a reinsurance premium following the Expected Principle. Under the same hypothesis,

Arrow [1] proved that the stop-loss reinsurance maximizes the expected utility of the terminal wealth of a risk-averse insurer.
Subsequently, Literature investigated the optimal reinsurance problem by using different premium principles or different risk measures. Just to name a few, Kaluszka [13, 14] generalized Borch's result by using the mean-variance premium principle and convex premium principles. Young [18] generalized Arrow's result by considering Wang's premium principle. Then, two classes of optimal reinsurance models have been introduced ([7], [8], [16]) in which the Value at Risk (VaR) and the Conditional Value at Risk ($CVaR$) of the insurer's total risk exposure have been minimized. More recently, Chi and Weng [11] studied optimal reinsurance design problem by minimizing the risk-adjusted value of an insurer's liability; Asimit et al. [3] studied the optimal non-life reinsurance problem by minimizing the risk exposure under Solvency II regime.

Optimal reinsurance model is still an interesting topic for both researchers and practitioners, in fact, some studies are devoted to investigate the optimal reinsurance model by extending both the premium principles and the risk measures.

In this paper we start from the work of Chi and Tan [10] in which two specific risk-measure based optimal reinsurance models are considered and the robustness of the optimal reinsurance over a prescribed class of premium principles is analysed. We tackle the issue of extending their results by assuming that the reinsurance premium is stochastic: this may shed light on some particular insurance situations. In fact, often in reinsurance treaties the reinsurance premium is a function of loss amounts, consequently it is a random variable; again, clauses as sliding scale premium, profit commission and paid reinstatements make the reinsurance premium random as is described in [17].

Although clauses making the reinsurance premium random are quite common in practice, literature dealing with a rigorous and quantitative approach to the subject is quite limited. Moreover, the current literature mainly focuses on the calculation of reinstatement premiums under different premium principles (see e.g. Sundt [15] and Walhin et al. [17]).

In Section 2, we present some notations and preparatory results, in order to illustrate the model of reinsurance introduced in [10] and here generalised. In Section 3, following the idea of a stochastic reinsurance premium, we prove optimality of layer reinsurance in VaR and $CVaR$ frameworks. In Section 4 some concluding remarks are presented.

2 Preparatory settings and results

Some notations, abbreviations and conventions used throughout the paper are the following. F_X denotes the one-dimensional cumulative distribution function (cdf) of the real-valued random variable (r.v.) X, with $F_X(x) = Pr\{X \le x\}$. We assume $E[X] < \infty$.

We will refer to the usual definition of the inverse of a distribution function, that is for any real $k \in [0,1]$:

$$F_X^{-1}(k) = \inf\{x \in \Re : F_X(x) \ge k\}, \tag{1}$$

where conventionally it is $\inf \varnothing = +\infty$.
Inverse distribution function formulation is particularly interesting with reference to some very well-known risk measures as *Value-at-Risk* and *Conditional Value at Risk*. Formally, the *Value-at-Risk*, VaR_α, of a r.v. X at a confidence level $1-\alpha$ $(0 < \alpha < 1)$ corresponds to the $1-\alpha$ quantile of X and is defined as

$$VaR_\alpha(X) = F_X^{-1}(1-\alpha). \tag{2}$$

We assume that $0 < \alpha < 1 - F_X(0)$: if $\alpha \ge 1 - F_X(0)$, then $VaR_\alpha(X) = 0$.

The risk measure *Conditional Value at Risk*, $CVaR$, also called Tail Value at Risk, Expected Shortfall or Average Value at Risk, is well known among practitioners and academicians. The $CVaR_\alpha(X)$ of a r.v. X at a confidence level $1-\alpha$, where $0 < \alpha < 1$, is defined as

$$CVaR_\alpha(X) = E[X \mid X \ge VaR_\alpha(X)]. \tag{3}$$

Equivalently, $CVaR_\alpha(X)$ can be defined in terms of $VaR_\alpha(X)$

$$CVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_s(X)ds. \tag{4}$$

Risk measures such as $VaR_\alpha(X)$ and $CVaR_\alpha(X)$ are used extensively within banking and insurance sectors for quantifying market risks, in portfolio optimization and for setting regulatory capital.

A key advantage of $CVaR_\alpha(X)$ over $VaR_\alpha(X)$ is that $CVaR_\alpha(X)$ is a coherent risk measure while $VaR_\alpha(X)$ fails to satisfy subadditivity property (see [2] for a detailed discussion of these properties). Moreover, VaR and $CVaR$ are distortion risk measures, that is they obey the properties of translation invariance, additivity for comonotonic risks and positive homogeneity (see [12]).

In the insurance framework, the loss initially assumed by an insurer in the absence of reinsurance is represented by a non-negative r.v. X.

Let f be a non-negative function defined for all possible outcomes of X, representing the reinsured amount. The function is known as *ceded loss function* and satisfies the condition $0 \le f(x) \le x$. In this way the insurance company exposure to loss is reduced by passing part of the risk of loss to a reinsurer (or a group of reinsurers). Let $R_f(x)$ be the *retained loss function*, that is $R_f(x) = x - f(x)$.

In this way, a simple reinsurance contract can be represented by the risk sharing scheme $(f(X), R_f(X))$, where $f(X)$ denotes the amount ensured by the reinsurer and $X - f(X)$ is the residual loss covered by the ceding company.
In order to exclude the moral hazard, it is generally assumed that $f(x)$ and $R_f(x)$ are non-decreasing functions on the set of all the possible outcomes x of X: both the partners of the risk sharing scheme $(f(X), R_f(X))$ have to bear more if the claim amount increases. For this reason, the random vector $(f(X), R_f(X))$ is comonotonic. Note that the non-decreasing condition on both ceded and retained loss functions ensures that $f(x)$ and $R_f(x)$ are Lipschitz continuous (see [9]).
The set of admissible ceded loss functions is then defined as

$$\mathcal{C} = \{f : 0 \le f(x) \le x \wedge \text{ both } R_f(x) \text{ and } f(x) \text{ are non-decreasing functions}\}. \tag{5}$$

Under the reinsurance arrangement, the risk exposure of the ceding company is no longer captured by X but equals the sum of the retained loss and the incurred reinsurance premium. Let π denote the (deterministic) reinsurance premium principle, that is $\pi : \Psi \to \mathfrak{R}_+$, where Ψ is the set of all non-negative random variables with finite expectation. The reinsurance premium is a function of the loss ceded to the reinsurer, namely, it is given by $\pi(f(X))$.

The total risk exposure $T_f(X)$ of the insurer is consequently given by

$$T_f(X) = R_f(X) + \pi(f(X)). \tag{6}$$

Recently, the problem of optimal reinsurance has been studied with reference to different risk measures related to insurer risk exposure and under different premium principles: in this framework, a ceded loss function is called to be *optimal* if it minimizes the (appropriately chosen) risk measure of $T_f(X)$ under a given premium principle for the reinsurance premium $\pi(f(X))$.

In the model originally proposed in [7, 8] and subsequently analysed in [9, 10], the risk measures VaR and $CVaR$ of the insurer risk exposure are minimised, under the hypothesis of a deterministic reinsurance premium computed throught a principle satisfying three basic properties: distribution invariance, risk loading and stop-loss ordering preserving.

Given a value α, the related $Var_\alpha(X)$ and a ceded loss function $f \in \mathcal{C}$, a layer reinsurance contract is so defined:

$$h_f(x) = \min\{(x-a)_+, b\} \tag{7}$$

where $(x-a)_+ = \max\{x-a, 0\}$, the deductible $a \ge 0$ and the upper limit $b \ge 0$ are respectively defined by

$$a = VaR_\alpha(X) - f(VaR_\alpha(X)) \tag{8}$$

$$b = f(VaR_\alpha(X)) = VaR_\alpha(X) - a. \tag{9}$$

The set $\mathcal{C}_\nu$ of all the layer reinsurance functions $h_f(x)$ is the subset of $\mathcal{C}$ so defined

$$\mathcal{C}_\nu = \{\min\{(x - VaR_\alpha(X) - f(VaR_\alpha(X)))_+, VaR_\alpha(X) - a\} : 0 \leq a \leq VaR_\alpha(X)\}. \tag{10}$$

In [10] it is proved that the layer reinsurance contract (7) is VaR-optimal, namely it is

$$VaR_\alpha(T_{h_f}(X)) \leq VaR_\alpha(T_f(X)), \ \ \forall f \in \mathcal{C}. \tag{11}$$

Moreover, it is

$$\min_{f \in \mathcal{C}} VaR_\alpha(T_f(X)) = \min_{f \in \mathcal{C}_\nu} VaR_\alpha(T_f(X)) = \min_{0 \leq a \leq VaR_\alpha(X)} \{a + \pi(h_f(X))\}. \tag{12}$$

In the case of the risk measure $CVaR$, by considering the layer function k_f so defined

$$k_f(x) = \min\{(x - VaR_\alpha(X) + f(VaR_\alpha(X))_+, b\} \tag{13}$$

where $b \geq f(VaR_\alpha(X))$ is determined by the condition

$$CVaR_\alpha(f(X)) = CVaR_\alpha(k_f(X)) \tag{14}$$

the layer reinsurance treaty results to be $CVaR$-optimal (see [10]), that is

$$CVaR_\alpha(T_{k_f}(X)) \leq CVaR_\alpha(T_f(X)), \ \ \forall f \in \mathcal{C}. \tag{15}$$

3 VaR and $CVaR$ minimization model with a stocastic reinsurance premium

Let us assume that the reinsurance premium is stochastic and let us denote it by $\widetilde{\pi}(f(X))$ or any ceded loss function $f \in \mathcal{C}$. It is assumed that $\widetilde{\pi}$ is a non-decreasing and l.c. function. Then the random vectors $(R_f(X), \widetilde{\pi}(f(X)))$ and $(R_{h_f}(X), \widetilde{\pi}(h_f(X)))$ are comonotonic.
Let $\widetilde{T}_f(X)$ denote the total risk exposure of the insurer with corresponding stochastic reinsurance premium $\widetilde{\pi}(f(X))$:

$$\widetilde{T}_f(X) = R_f(X) + \widetilde{\pi}(f(X)). \tag{16}$$

The next theorem follows.

Theorem 3.1. *The layer reinsurance (7) is VaR-optimal, namely it is*

$$VaR_\alpha(\widetilde{T}_{h_f}(X)) \le VaR_\alpha(\widetilde{T}_f(X)), \ \ \forall f \in \mathcal{C}. \tag{17}$$

Moreover, it is

$$\min_{f\in\mathcal{C}} VaR_\alpha(\widetilde{T}_f(X)) = \min_{f\in\mathcal{C}_\nu} VaR_\alpha(\widetilde{T}_f(X)) = \min_{0\le a\le VaR_\alpha(X)} \{a + \widetilde{\pi}(h_f(X))\} \tag{18}$$

Proof. It is $h_f(VaR_\alpha(X)) = f(VaR_\alpha(X))$ and $h_f \in \mathcal{C}$.
Furthermore, since the ceded loss function $f(x)$ is non-negative, non-decreasing and Lipschitz-continuous, it is

$$h_f(x) \le f(x), \ \ \forall x \ge 0. \tag{19}$$

Then $h_f(X)$ is smaller than $f(X)$ in the stochastic order

$$h_f(X) \le_{st} f(X) \tag{20}$$

and $\widetilde{\pi}(h_f(X)) \le_{st} \widetilde{\pi}(f(X))$.
Hence it is

$$VaR_\alpha(\widetilde{\pi}(h_f(X))) \le VaR_\alpha(\widetilde{\pi}(f(X))) \quad \text{for all } \alpha \in (0,1).$$

On additivity of VaR on comonotonic random variables and on Theorem 2 and Lemma 1 in [12], the sequence of relations follows

$$\begin{aligned}
VaR_\alpha(\widetilde{T}_f(X)) &= VaR_\alpha(R_f(X)) + VaR_\alpha(\widetilde{\pi}(f(X))) \text{ by comonotonic additivity}\\
&= R_f(VaR_\alpha(X)) + VaR_\alpha(\widetilde{\pi}(f(X))) \text{ by Lemma 1 in [12]}\\
&= VaR_\alpha(X) - f(VaR_\alpha(X)) + VaR_\alpha(\widetilde{\pi}(f(X))) \text{ by definition of } R_f\\
&= VaR_\alpha(X) - h_f(VaR_\alpha(X)) + VaR_\alpha(\widetilde{\pi}(f(X))) \text{ by definition of } h_f\\
&\ge VaR_\alpha(X) - h_f(VaR_\alpha(X)) + VaR_\alpha(\widetilde{\pi}(h_f(X))) \text{ by Theorem 2 in [12]}\\
&= R_{h_f}(VaR_\alpha(X)) + VaR_\alpha(\widetilde{\pi}(h_f(X))) \text{ by definition of } R_f\\
&= VaR_\alpha(R_{h_f}(X)) + VaR_\alpha(\widetilde{\pi}(h_f(X))) \text{ by Lemma 1 in [12]}\\
&= VaR_\alpha(\widetilde{T}_{h_f}(X)) \text{ by comonotonic additivity.}
\end{aligned}$$

Therefore, it is

$$\begin{aligned}
\min_{f\in\mathcal{C}} VaR_\alpha(\widetilde{T}_f(X)) &= \min_{f\in\mathcal{C}_\nu} VaR_\alpha(\widetilde{T}_f(X))\\
&= \min_{f\in\mathcal{C}_\nu} \{VaR_\alpha(X) - f(VaR_\alpha(X)) + VaR_\alpha(\widetilde{\pi}(h_f(X)))\}\\
&= \min_{0\le a\le VaR_\alpha(X)} \{a + VaR_\alpha\left(\widetilde{\pi}(\min\{(x-a)_+, VaR_\alpha(X) - a\})\right)\}.
\end{aligned}$$

where $\mathcal{C}_\nu$ has been defined in (10).

$\square$

By assuming a random reinsurance premium $\widetilde{\pi}(f(X))$ in the $CVaR$-model, it is possible to state the

Theorem 3.2. *The layer reinsurance (13) is $CVaR$-optimal in the case of a random reinsurance premium $\widetilde{\pi}(f(X))$, that is*

$$CVaR_\alpha(\widetilde{T}_{k_f}(X)) \leq CVaR_\alpha(\widetilde{T}_f(X)), \quad \forall f \in \mathcal{C}. \tag{21}$$

Proof. Translation invariance and additivity for comonotonic risks are properties satisfied by $CVaR$. Moreover, $CVaR$ is a concave distortion risk measure, then it preserves stop-loss order. Given that (see [10])

$$k_f(X) \leq_{sl} f(X)$$

and $\widetilde{\pi}$ is a non-decreasing and l.c. function, it follows

$$\widetilde{\pi}(k_f(X)) \leq_{sl} \widetilde{\pi}(f(X)).$$

Moreover,

$$CVaR_\alpha(\widetilde{\pi}(k_f(X))) \leq_{sl} CVaR_\alpha(\widetilde{\pi}(f(X)))$$

and the following sequence of relations can be deduced

$$\begin{aligned} CVaR_\alpha(\widetilde{T}_{k_f}(X)) &= CVaR_\alpha(R_{k_f}(X)) + CVaR_\alpha(\widetilde{\pi}(k_f(X))) \\ &\quad \text{by comonotonic additivity} \\ &= CVaR_\alpha(X) - CVaR_\alpha(k_f(X)) + CVaR_\alpha(\widetilde{\pi}(k_f(X))) \\ &\quad \text{by definition of } R_{k_f} \text{ and comonotonic additivity} \\ &= CVaR_\alpha(X) - CVaR_\alpha(f(X)) + CVaR_\alpha(\widetilde{\pi}(k_f(X))) \\ &\quad \text{by definition of } k_f \\ &\leq CVaR_\alpha(X) - CVaR_\alpha(f(X)) + CVaR_\alpha(\widetilde{\pi}(f(X))) \\ &\quad \text{by Theorem 2 in [12]} \\ &= CVaR_\alpha(\widetilde{T}_f(X)) \\ &\quad \text{by comonotonic additivity.} \end{aligned}$$

Lastly, it is

$$\begin{aligned} &\min_{f \in \mathcal{C}} CVaR_\alpha(\widetilde{T}_f(X)) = \min_{f \in \mathcal{C}_\nu} CVaR_\alpha(\widetilde{T}_f(X)) \\ &= \min_{\substack{0 \leq a \leq VaR_\alpha(X) \\ a+b \geq VaR_\alpha(X)}} \left\{ a + \frac{1}{\alpha} E[(X-(a+b))_+] + CVaR_\alpha(\widetilde{\pi}(\min\{(X-a)_+, b\})) \right\} \end{aligned}$$

where $\mathcal{C}_\nu := \{\min\{\ (X-a)_+, b\} : 0 \leq a \leq VaR_\alpha(X) \leq a+b\}$.

$\square$

4 Concluding remarks

In reinsurance practice, reinsurance premiums are often composed of a random component, in different ways: accordingly, insurer and reinsurer share the results of the loss ratio in the reinsurance relationship. In the case of a deterministic reinsurance premium, the reinsurance contract may be a priori more expensive for the insurer and the agreement between the parties could vanish. Starting from the work of Chi and Tan in 2013, we focus on the limited stop-loss reinsurance contract and we prove its optimality for the transferring company according to the Var and CVaR criteria even in the presence of a stochastic reinsurance premium. Following this proposal, other models in the reinsurance literature can be subsequently analysed.

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