Numerical-Analytical Solution of Nonlinear Fractional-Order Lorenz’s System

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1 Introduction

Fractional calculus is three centuries old. The beauty of this part of the science is that fractional derivatives (integrals) are not a local (point) property (quantity). The idea of fractional calculus has been known since the regular calculus, with the first reference probably being associated with Liebniz and L’Hospital in 1695 where half-order derivative was mentioned.

At present, the number of applications of fractional calculus rapidly grows, these mathematical phenomena allow us to describe and model a real object more accurately than the classical "integer" methods. [3] The fractional-order calculus plays an important role in physics, thermodynamics, electrical circuits theory, mechatronic systems, signal processing, chemical mixing, chaos theory etc. Abel was the first who wrote a fractional equation for solving the tautochrone problem. In this paper we focus on the nonlinear fractional systems of the form:

\[ D_{\alpha}^{\nu} x_i(t) = f_i(x_1(t), x_2(t), ..., x_n(t), t) \]
\[ x_i(0) = c_i, i = 1, 2, ..., n \]  \hspace{1cm} (1)

where the \( c_i \) are the initial conditions, or in its vector representation:

\[ D^\nu \mathbf{x} = \mathbf{f(x)} \] \hspace{1cm} (2)

where \( \nu = [\nu_1, \nu_2, ..., \nu_n]^T \) for \( 0 < \nu_i < 2, (i = 1, 2, ..., n) \) and \( \mathbf{x} \in \mathbb{R}^n \). The equilibrium points of the system (2) are calculated via solving following equation: \( \mathbf{f(x)} = \mathbf{0} \).

Those kind of systems are very interesting to engineers, physicists and mathematicians, because most real physical systems are inherently nonlinear in nature, especially we will discus about an well-known nonlinear system which exhibit chaos, the Lorenz system and its application. The exact solution of those kind of systems in not possible to be found, so we solve them numerically by using different kind of methods. We use (FDTM), an analytic and numerical method, currently used, as a technique for analytic calculating the power series of the solution.

This paper is organized as follows. Section 2 is a brief introduction of fractional calculus. Section 3 is on the fractional-order system of Lorenz containing the conditions of its stability. Section 4 is on the numerical method (FDTM), the part of theory and numerical results and the Section 5 is on some applications [1, 3].
2 Fractional Calculus

Here, we should mention the basics of the fractional calculus, fractional integral and derivative.

Fractional integral of order $v$ for function $f(t)$ can be expressed as follows:

$$ I_v^t f(t) = D^{-v}_t = \frac{1}{\Gamma(v)} \int_0^t (t-\tau)^{v-1} f(\tau) d\tau $$

(3)

Riemann-Liouville definition of fractional derivative of order $v$ is:

$$ D_v^t f(t) = \frac{1}{\Gamma(n-v)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{v-1} f(\tau) d\tau $$

(4)

Caputo definition of fractional derivative of order $q$ is:

$$ D_v^t f(t) = \frac{1}{\Gamma(n-v)} \int_0^t (t-\tau)^{v-1} f^{(n)}(\tau) d\tau $$

(5)

Is chosen to use Caputo fractional derivative, because it allows initial and boundary conditions to be included in the formulation of the problem, even that for homogeneous initial conditions, these two operators coincide.

Lorenz system of fractional-order [2, 7]

$$ D_v^{\alpha_1} x(t) = a(y(t) - x(t)) $$
$$ D_v^{\alpha_2} y(t) = cx(t) - x(t)z(t) - y(t) $$
$$ D_v^{\alpha_3} z(t) = x(t)y(t) - bz(t) $$

(6)

$x(0) = c_1, y(0) = c_2, z(0) = c_3$, where $D_v^{\alpha_i}$ are Caputo fractional derivative for $i = 1, 2, 3$, $a, b, c$ are real parameters and $\alpha_i \in (0, 1]$ the fractional order. In the continuous work we will discuss about the system where $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$.

Stability as an extremely important property of dynamical systems can be investigated in various domains, one of them is the frequency domain, but if we deal with incommensurate fractional-order systems, it is important to bear in mind that $P(s^\alpha), \alpha \in R$ is a multivalued function of $s^\alpha, \alpha = u/v$, the domain of which can be viewed as a Riemann surface with finite number of Riemann sheets $v$, where the origin is a branch point and the branch cut is assumed at $R^+$. It is fact that in multivalued functions only the first Riemann sheet has its physical significance [7].

Generally, for the multivalued function defined as $w = s^{1/v}$, where $v \in N(v = 1, 2, 3, ...) \ we get the v sheet in the Riemann surface. The relationship between the $w$-plane and the $v$ sheets of the Riemann surface where the sector $-\frac{\pi}{v} < \arg(w) \leq \frac{\pi}{v}$ corresponds to $\Omega$.

Mapping the poles from the $s^\alpha$-plane into the $w$-plane, where $\alpha \in Q$ is such
that \( q = \frac{k}{m} \) for \( k, m \in \mathbb{N} \) and \( |\arg(w)| = |\phi| \), can be done by the following rule:

If we assume \( k = 1 \), then the mapping from \( s \)-plane to \( w \)-plane is independent of \( k \). Unstable region from the \( s \)-plane transforms to sector \( \frac{\pi}{2m} < |\phi| < \frac{\pi}{m} \) and stable region transforms to sector \( \frac{\pi}{m} < |\phi| < \frac{\pi}{2m} \). The region where \( |\phi| > \frac{\pi}{m} \) is not physical. Therefore, the system will be stable if all roots in the \( w \)-plane lie in the region \( |\phi| > \frac{\pi}{2m} \) [4, 7].

**Definition 2.1.** The trajectory \( x(t) = 0 \) of the system (1) is \( t^{-q} \) asymptotically stable if there is a positive real \( q \) such that: \( \forall \|x(t)\| \) with \( t \leq t_0, \exists N(x(t)), \) such that \( \forall t \geq t_0, \|x(t)\| \leq Nt^{-q} \) [7].

The fact that the components of \( x(t) \) slowly decay towards 0 following \( t^{-q} \) leads to fractional systems sometimes being called long memory systems. The power law stability \( t^{-q} \) is a special case of Mittag-Leffler stability.

According to the stability theorem, the equilibrium points asymptotically stable for \( q_1 = q_2 = ... = q_n = q \) if all the eigenvalues \( \lambda_i, (i = 1, 2, ..., n) \) of the Jacobian matrix \( J = \frac{df}{dx} \), where \( f = [f_1, f_2, ..., f_n]^T \), evaluated at the equilibrium, satisfy the condition:

\[
\arg(eig(J)) = |\arg(\lambda_i)| > q \frac{\pi}{2}, \quad i = 1, 2, ..., n
\]

Lorenz's system has three equilibria, where one is obviously in origin \( E_1(0; 0; 0) \) and the other two are: \( E_2(\sqrt{bc-c}; \sqrt{bc-c}; c-1) \), \( E_3(-\sqrt{bc-c}; -\sqrt{bc-c}; c-1) \). The Jacobian matrix of Lorenz’s system (6) at the equilibrium point \( E^*(x^*, y^*, z^*) \) is given by:

\[
J = \begin{bmatrix}
-a & a & 0 \\
c - z^* & -1 & -x^* \\
y^* & x^* & -b
\end{bmatrix}
\]

We will investigate the fractional-order Lorenz’s system by changing the initial conditions and showing their numerical results by using two numerical methods [4].

**Example 2.1.** Let \( a = 10, b = \frac{8}{3} \); \( c \) is varied but the system exhibits chaotic behavior for \( c = 28 \). The initial conditions \( x(0) = 0.1, y(0) = 0.1 \) and \( z(0) = 0.1 \), then the equilibrium points of the system with the above parameters are: \( E_1(0, 0, 0) \), \( E_2(8.4853, 8.4853, 27) \) and \( E_3(-8.4853, -8.4853, 27) \). In this case the minimal commensurate order is \( \nu > 0.9941 \) and stability condition \( |\arg(\lambda)| > \gamma \frac{\pi}{2} \) for \( \gamma = \frac{1}{m} \).
3 Numerical Methods

When fractional differential equations of the system are concerned, it turns out that there is a close connection between the type of the initial condition and the type of the fractional derivative. This is actually also the reason for us to choose the Caputo derivative and not the Riemann-Liouville derivative that is more commonly used in pure mathematics: For the Riemann-Liouville case, one would have to specify the values of certain fractional derivatives (and integrals) of the unknown solution at the initial point [5, 10]. However, when we are dealing with a concrete physical application then the unknown quantity will typically have a certain physical meaning (e.g. a dislocation), but it is not clear what the physical meaning of a fractional derivative of the equation is, and hence it is also not clear how such a quantity can be measured. In other words, the required data simply will not be available in practice. When we deal with Caputo derivatives however, the situation is different [10]. We may specify the initial values, the function value itself and integer-order derivatives. These data typically have a well understood physical meaning and can be measured.

3.1 Fractional Multistep Differential Transformation Method

An important part of the paper is to present approximate analytical solutions for Lorenz system with fractional-order (6). The fractional multistep differential transform method (FDTM) is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial [1, 9]. The traditional high order Taylor series method requires symbolic computation. However, the differential transform method obtains a polynomial series solution by means of an iterative procedure [1, 2]. Firstly, expand the analytic function \( f(t) \) in terms of fractional power series as follows:

\[
f(t) = \sum_{k=0}^{\infty} F(k)(t - t_0)^kv
\]

where \( 0 < v \leq 1 \) is the order of fractional derivative and \( F(k) \) is the fractional differential transform of \( f(t) \) (9), given as

\[
F(k) = \frac{1}{\Gamma(vk + 1)}[(D^v_{t_0})^k(f(t_0))]
\]

where \( (D^v_{t_0})^k = D^v_{t_0} \cdot D^v_{t_0} ... D^v_{t_0} \) the \( k \)-times-differential Caputo fractional derivative (5).

In our application, we will approximate the function \( f(t) \) by the finite series, so the finite form of (9):

\[
f(t) = \sum_{k=0}^{N} F(k)(t - t_0)^kv
\]
The following are the basic properties of the Caputo fractional derivative and the differential transformation [1]:

(1) Let \( f \in C^{n-1} \) then \( D^v f \) for \( 0 \leq v \leq n \) is well defined and \( D^{v+1} f \in C_{-1} \)

(2) If \( f(t) = g(t) \pm h(t) \), then \( F(k) = G(k) \pm H(k) \)

(3) If \( f(t) = g(h) \), then \( F(z) = \sum_{l=0}^{k} G(l) H(k-l) \)

(4) If \( (t-t_0)^p \), then \( F(z) = \delta(k-vp) \), where \( \delta(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \)

(5) If \( f(t) = D^{v+1} g(t) \), then \( F(k) = \frac{\Gamma(v(k+1)+1)}{\Gamma(vk+1)} G(v+1) \)

We apply the (FDTM) for the fractional-order system 6 and the differential transformation for this system is:

\[
\frac{\Gamma(v(k+1)+1)}{\Gamma(vk+1)} X(k + 1) = a(Y[k] - X[k])
\]

\[
\frac{\Gamma(v(k+1)+1)}{\Gamma(vk+1)} Y(k + 1) = cX(k) - \sum_{l=0}^{k} X(l) Z(k - l) - Y(k)
\]

\[
\frac{\Gamma(v(k+1)+1)}{\Gamma(vk+1)} Z(k + 1) = \sum_{l=0}^{k} X(l) Y(k - l) - bZ(k)
\]

where \( X(0) = c_1, Y(0) = c_2, Z(0) = c_3 \).

The \( N \)-th order solutions with the inverse transformation are:

\[
x(t) = \sum_{m=0}^{N} X(m)(t-t_0)^{mv}
\]

\[
y(t) = \sum_{m=0}^{N} Y(m)(t-t_0)^{mv}
\]

\[
z(t) = \sum_{m=0}^{N} Z(m)(t-t_0)^{mv}
\]

If we consider the fractional-order series \( f(t) = \sum_{k=0}^{N} F(k)(t-t_0)^{\alpha k} \), then the transformed method has the following form:

\[
\frac{\Gamma(v(k+1)+1)}{\Gamma(v(k\alpha+1)+1)} X(k + 1) = a(Y[k] - X[k])
\]

\[
\frac{\Gamma(v(k+1)+1)}{\Gamma(v(k\alpha+1)+1)} Y(k + 1) = cX(k) - \sum_{l=0}^{k} X(l) Z(k - l) - Y(k)
\]

\[
\frac{\Gamma(v(k+1)+1)}{\Gamma(v(k\alpha+1)+1)} Z(k + 1) = \sum_{l=0}^{k} X(l) Y(k - l) - bZ(k)
\]

Numerical results for system (6) are taken using the 10-order solutions with the inverse transformation. We will name them like (FDTM) and (FDTM1).
4 Numerical results

As we said in Example 2.1., we will consider the values \(a = 10, b = \frac{8}{3}, c = 28\), an usual case, but investigated in different ways. We took the risk to show how the (FDTM) and (FDTM1) methods will act, on the initial conditions \(x(0) = 0.1, y(0) = 0.1\) and \(z(0) = 0.1\) for a fractional-order from interval \((0, 1]\), which is \(q = 0.8\), even that the conditions to have chaos is \(q > 0.994\). We observe the approximated solutions of the system (6), with above mentioned parameters, using two numerical methods (FDTM) and (FDTM1). Purpose, is to compare the results for each fractional-order and showing that in which conditions they are in good agreement with each other by plotting them, using Mathematica 11.0 Package.

We will share the results on time interval \([0, 10]\), with step size \(h = 0.005\), for \(x(t), y(t)\) and \(z(t)\). The second part is to show the same result with the same conditions for the time interval \([0, 20]\). [2, 5, 8, 10]

5 Historical Implementation and Conclusion

The Lorenz attractor was named after Edward N. Lorenz, who derived it from the simplified equations of convection rolls arising in the equations of the atmosphere in 1963. He for the first time used the term “butterfly effect”, which in chaos theory means sensitive dependence on initial conditions. Lorenz wrote a paper in 1979 entitled Predictability: Does the Flap of a Butterfly’s Wings in Brazil Set Off a Tornado in Texas? Small variations of the initial condition of a dynamical system may produce large variations in the long-term behavior of the system. The phrase refers to the idea that a butterfly’s wings might create tiny changes in the atmosphere that may ultimately alter the path of a tornado or delay, accelerate or even prevent the occurrence of a tornado in a certain location. The apping wing represents a small change in the initial condition of the system, which causes a chain of events leading to large-scale alterations of events [5, 7].
Figure 1: Plots of system (6), when $q = 0.8$, step size $h = 0.005$ and $t \in [0, 10]$
Figure 2: Plots of $\text{Abs}[(\text{FDTM})-(\text{FDTM1})]$ for the system 6, when $q = 0.8$, step size $h = 0.005$ and $t \in [0, 10]$
Figure 3: Plots of system (6), when $q = 0.8$, step size $h = 0.005$ and $t \in [0, 20]$
Figure 4: Plots of $\text{Abs}[(\text{FDTM})-(\text{FDTM1})]$ for the system 6, when $q = 0.8$, step size $h = 0.005$ and $t \in [0, 20]$
References


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