Error Estimates and Rate of Convergence for Some Theorems Concerning Fixed Points in Generalized Metric Spaces

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Abstract

Through this paper we make extensions of some theorems concerning fixed points in generalized metric spaces. We obtained some results for priori and posteriori error estimates and rate of convergence.

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1. Introduction

In 2000, Branciari, A. [4] introduced the concept of a generalized metric space (gms) where the triangle inequality of a metric space has been replaced by an inequality
involving three terms (tetrahedral inequality) instead of two terms. In 2007, P. Das, and L. Kanta Dey obtained a theorem about the uniqueness of fixed points in gms. In 2009 and 2010, A. Al-Bsoul, A. Fora, and A. Bellour obtained two theorems concerning the same subject. In this paper we shall extend some theorems raised in [1], [2], and [3] to find error estimates and rate of convergence.

Throughout this paper, \( \mathbb{R}^+ \) denotes the set of all nonnegative real numbers, \( x^* \) denotes a fixed point, and \( X \) denotes a nonempty set.

Let us start with some main definitions and theorems related to our paper.

**Definition 1.1** [4]. Let \( X \) be a nonempty set and let \( d: X \times X \to \mathbb{R}^+ \) be a mapping such that for all \( x, y, z, w \in X \) with \( z \neq x, z \neq w, w \neq y \), we have the following properties:
1) \( d(x, y) = 0 \) if \( x = y \),
2) \( d(x, y) = d(y, x) \) (symmetry),
3) \( d(x, y) \leq d(x, z) + d(z, w) + d(w, y) \) (tetrahedral inequality),
then we say that \( d \) is a generalized metric and \((X, d)\) is called a generalized metric space or gms.

**Definition 1.2** [1]. Let \( T: X \to X \) be a mapping where \((X, d)\) is a gms. For each \( x \in X \), the set \( O(x) = \{ x, Tx, T^2 x, \ldots \} \) will be called the orbit at \( x \).

**Definition 1.3** [1]. An orbit \( O(x) \) is called \( T \)-orbitally complete if every Cauchy sequence in \( O(x) \) converges to a point in \( X \).

**Definition 1.4** [2]. A mapping \( T: X \to X \) on a generalized metric space \((X, d)\) is called locally contractive if for every \( x \in X \) there exists \( \epsilon_x > 0 \) and \( \lambda_x \in [0, 1) \) such that for all \( p, q \in \{ y: d(x, y) \leq \epsilon_x \} \) the relation \( d(T_p, T_q) \leq \lambda_x \) hold.

**Definition 1.5** [2]. A mapping \( T: X \to X \) on a generalized metric space \((X, d)\) is called \((\epsilon, \lambda)\) uniformly locally contractive if it is a locally contractive at all points \( x \in X \) and \( \epsilon, \lambda \) do not depend on \( x \), i.e,
\[
d(x, y) < \epsilon \Rightarrow d(Tx, Ty) < \lambda d(x, y),
\]
for all \( x, y \in X \).

Let \( \Phi \) denotes the class of all functions \( \phi: \mathbb{R}^+_+ \to \mathbb{R}^+ \) which are continuous, nondecreasing, \( \phi(t, t, t) < t \) and \( \sum_{k=1}^{\infty} \phi^k(t, t, t) < \infty \) for all \( t > 0 \), where \( \phi^k \) is the function \( \phi^n: \mathbb{R}^+_+ \to \mathbb{R}^+ \) for \( n \geq 2 \) defined by
\[
\phi^n(t_1, t_2, t_3, t_4) = \phi(\phi^{n-1}(t_1, t_1, t_1), \phi^{n-1}(t_2, t_2, t_2), \phi^{n-1}(t_3, t_3, t_3), \phi^{n-1}(t_4, t_4, t_4)).
\]
Theorem 1.6 [1]. Let \((X, d)\) be a gms, and let \(T: X \rightarrow X\) be a mapping such that

\[ d(Tx, Ty) \leq \phi(d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)), \]

where \(\phi \in \Phi\). If there exists \(x \in X\) such that \(O(x)\) is orbitally complete, then \(T\) has a unique fixed point in \(X\).

Let \(\Psi\) denote the class of all nondecreasing upper semi-continuous functions \(\psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+\) such that \(\sum_{n=1}^{\infty} \psi^n(t) < \infty\) for all \(t > 0\).

Theorem 1.7 [3]. Let \((X, d)\) be a gms, and let \(T: X \rightarrow X\) be a mapping such that

\[ d(Tx, Ty) \leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}), \]

where \(\psi \in \Psi\). If there exists \(x \in X\) such that \(O(x)\) is orbitally complete, then \(T\) has a unique fixed point in \(X\).

Lemma 1.8 [3]. Let \(f: \mathbb{R}_+ \rightarrow \mathbb{R}_+\) be a nondecreasing function such that the sequence \((f^n(t))\) converges to 0 for all \(t > 0\). Then

\( i \) \( f(t) < t \) for all \( t > 0 \);

\( ii \) \( f(0) = 0 \).

Definition 1.9 [5]. Let \(T: X \rightarrow X\) be a mapping on a metric space \((X, d)\). For each \(x \in X\), we define \(T^n(x)\) recursively by

\[ T^0(x) = x \text{ and } T^n(x) = T(T^{n-1}(x)). \]

We call \(T^n(x)\) the \(n^{th}\) iteration of \(x\). In order to simplify the notations, we will often use \(Tx\) instead of \(T(x)\). The sequence \((x_n)\) defined by

\[ x_n = Tx_{n-1} = T^n x_0, n = 1, 2, \ldots \]

is called the Picard iteration associated to \(T\).

2. Error estimates, and rate of convergence of some theorems concerning fixed points in generalized metric space (gms).

In this section, we shall extend theorems 1.6 and 1.7 to find priori and posteriori error estimates and rate of convergence. To do this,
\( \phi \left( d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n) \right) \) will be denoted by \( \phi_{n-1,n} \).

\( \psi \left( d(x_m, x_k) \right) \) will be denoted by \( \psi_{m,k} \).

Next theorem is an extension to theorem 1.6

**Theorem 2.1.** Let \((X, d)\) be a gms, and let \(T: X \to X\) be a mapping such that

\[
    d(Tx, Ty) \leq \phi \left( d(x, y), d(x, Tx), d(y, Ty), d(y, Tx) \right),
\]

where \( \phi \in \Phi \). If there exists \( x \in X \) such that \( O(x) \) is orbitally complete, then

(i) The priori and posteriori error estimates

\[
    d(x_n, x^*) < M \frac{d(x_{n-1}, x_n)}{1 - d(x_{n-1}, x_n)}, \quad n = 1, 2, \ldots
\]

and

\[
    d(x_n, x^*) < M \frac{d(x_0, x_1)}{1 - d(x_0, x_1)}, \quad n = 0, 1, 2, \ldots
\]

hold, where \( M = 1 - \lim_{p \to \infty} \phi_{n-1,n}^p \); \( p \) is odd, \( d(x_{n-1}, x_n) \neq 1, n = 1, 2, \ldots \).

(ii) The priori and posteriori error estimates

\[
    d(x_{n+1}, x^*) < K \frac{d(x_{n-1}, x_n)}{1 - d(x_{n-1}, x_n)}, \quad n = 1, 2, \ldots
\]

and

\[
    d(x_{n+1}, x^*) < K \frac{d(x_0, x_1)}{1 - d(x_0, x_1)}, \quad n = 0, 1, 2, \ldots
\]

hold, where \( K = 1 - \lim_{p \to \infty} \phi_{n-1,n}^{p-1} \); \( p \) is even, \( d(x_{n-1}, x_n) \neq 1, n = 1, 2, \ldots \).

(iii) The rate of convergence of the Picard iteration is given by

\[
    d(x_n, x^*) \leq \phi \left( d(x_{n-1}, x^*), d(x_{n-1}, x^*), d(x_{n-1}, x^*) + d(x_n, x^*), d(x_{n-1}, x^*) \right).
\]

**Proof.**

(i) By tetrahedral inequality we have

\[
    d(x_{n+p}, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}). \quad (1)
\]

For the right side of inequality (1), it is evident to see that
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\[ d(x_n, x_{n+1}) \leq \phi_{n-1,n}, \text{ and } d(x_{n+1}, x_{n+2}) \leq \phi_{n-1,n}^2. \]

By induction on \( n \), we conclude that

\[ d(x_{n+p-1}, x_{n+p}) \leq \phi_{n-1,n}^p, \]

thus,

\[ d(x_{n+p}, x_n) \leq \phi_{n-1,n} \frac{1 - \phi_{n-1,n}^p}{1 - \phi_{n-1,n}}, \]

since \( \lim_{p \to \infty} \phi(t, t, t) < \lim_{p \to \infty} \sum_{k=1}^{p} \phi^k(t, t, t) < \infty \), \( \phi(t, t, t) < t \) for all \( t > 0 \), and \( d(x_{n-1}, x_n) > 0 \), then

\[ \left( 1 - \lim_{p \to \infty} \phi_{n-1,n}^p \right) \]

is a finite real number, say \( M \), and \( \phi_{n-1,n} < d(x_{n-1}, x_n) \), thus as \( p \to \infty \) we get

\[ d(x_n, x^*) < M \frac{d(x_{n-1}, x_n)}{1 - d(x_{n-1}, x_n)}, n = 1, 2, \ldots, \]

with \( d(x_{n-1}, x_n) \neq 1 \). But \( d(x_{n-1}, x_n) = d(T^{n-1}x_0, T^n x_0) \) by induction on \( n \). Thus

\[ d(T^{n-1}x_0, T^n x_0) \leq \phi_{n-1} \left( d(x_0, x_1), d(x_0, x_1), d(x_0, x_1), d(x_0, x_1) \right). \]

Now, denote \( \phi^k(t, t, t) \) by \( \phi^k(t) \) for \( k \geq 1 \), so \( \phi^k(t) < t \), hence,

\[ d(x_{n-1}, x_n) \leq \phi_{n-1} \left( d(x_0, x_1), d(x_0, x_1), d(x_0, x_1), d(x_0, x_1) \right). \]

So,

\[ d(x_n, x^*) < M \frac{d(x_{n-1}, x_1)}{1 - d(x_{n-1}, x_1)}, n = 0, 1, 2, \ldots, \]

(ii) If \( p \) is even, then by tetrahedral inequality we get

\[ d(x_{n+p}, x_{n+1}) < \phi_{n-1,n} \frac{1 - \phi_{n-1,n}^{p-1}}{1 - \phi_{n-1,n}}. \]

Since \( \lim_{p \to \infty} \phi^{p-1}(t, t, t) < \lim_{p \to \infty} \sum_{k=1}^{p-1} \phi^k(t, t, t) < \infty \), \( \phi(t, t, t) < t \) for all \( t > 0 \), and \( d(x_{n-1}, x_n) > 0 \), then

\[ \left( 1 - \lim_{p \to \infty} \phi_{n-1,n}^p \right) \]

is a finite real number, say \( M \), and \( \phi_{n-1,n} < d(x_{n-1}, x_n) \), thus as \( p \to \infty \) we get

\[ d(x_{n+1}, x^*) < K \frac{d(x_{n-1}, x_n)}{1 - d(x_{n-1}, x_n)}, n = 1, 2, \ldots, \]

with \( d(x_{n-1}, x_n) \neq 1 \). Similarly, we get

\[ d(x_{n+1}, x^*) < K \frac{d(x_{n-1}, x_1)}{1 - d(x_{n-1}, x_1)}, n = 0, 1, 2, \ldots \]
(iii) Since \(d(Tx, Ty) \leq \phi(d(x, y), d(x, Tx), d(y, Ty), d(y, Tx))\), \(d(y, Ty) \leq d(y, x) + d(x, Tx) + d(Tx, Ty)\) and \(\phi\) is nondecreasing, so \(d(Tx, Ty) \leq \phi(d(x, y), d(x, Tx), d(y, x) + d(x, T x) + d(T x, T y), d(y, Tx))\). Take \(x = x^*, y = x_{n-1}\), by considering \(0 \leq d(x_{n-1}, x^*)\), and \(\phi\) is nondecreasing, we get
\[
d(x_n, x^*) \leq \phi(d(x_{n-1}, x^*), d(x_{n-1}, x^*), d(x_{n-1}, x^*), d(x_{n-1}, x^*)).
\]

Next theorem is an extension to theorem 1.7

**Theorem 2.2.** Let \((X, d)\) be a gms, and let \(T: X \to X\) be a mapping such that
\[
d(Tx, Ty) \leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}),
\]
where \(\psi \in \Psi\). If there exists \(x \in X\) such that \(O(x)\) is orbitally complete, then

(i) The priori and posteriori error estimates
\[
d(x_n, x^*) < M^* \frac{d(x_{n-1}, x_n)}{1 - d(x_{n-1}, x_n)}, \quad n = 1, 2, \ldots
\]
and
\[
d(x_n, x^*) < M^* \frac{d(x_0, x_1)}{1 - d(x_0, x_1)}, \quad n = 0, 1, 2, \ldots
\]
hold, where \(M^* = 1 - \lim_{p \to \infty} \psi_{n-1,n}^p; p\) is odd, \(d(x_{n-1}, x_n) \neq 1, n = 1, 2, \ldots\).

(ii) The priori and posteriori error estimates
\[
d(x_{n+1}, x^*) < k^* \frac{d(x_{n-1}, x_n)}{1 - d(x_{n-1}, x_n)}, \quad n = 1, 2, \ldots
\]
and
\[
d(x_{n+1}, x^*) < k^* \frac{d(x_0, x_1)}{1 - d(x_0, x_1)}, \quad n = 0, 1, 2, \ldots
\]
hold, where \(k^* = 1 - \lim_{p \to \infty} \psi_{n-1,n}^{p-1}; p\) is even, \(d(x_{n-1}, x_n) \neq 1, n = 1, 2, \ldots\).

(iii) The rate of convergence of the Picard iteration is given by
\[
d(x_n, x^*) \leq \psi(d(x_{n-1}, x^*) + d(x_n, x^*)).
\]

**Proof.**

(i) By tetrahedral inequality we have
\[
d(x_{n+p}, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}).
\]
For the right side of the inequality (1), It is clear that

\[ d(x_n, x_{n+1}) \leq \psi_{n-1,n}, \text{ and } d(x_{n+1}, x_{n+2}) \leq \psi_{n-1,n}^2. \]

By induction on \( n \), we conclude that

\[ d(x_{n+p-1}, x_{n+p}) \leq \psi_{n-1,n}^p, \]

thus,

\[ d(x_{n+p}, x_n) \leq \psi_{n-1,n} \frac{1-\psi_{n-1,n}^p}{1-\psi_{n-1,n}}. \]

Since \( \lim_{p \to \infty} \psi_{n-1,n}^p(t) < \lim_{p \to \infty} \sum_{k=1}^{p-1} \psi^k(t) < \infty \), \( \psi(t) < t \) for all \( t > 0 \), and \( d(x_{n-1}, x_n) > 0 \), then

\[ (1 - \lim_{p \to \infty} \psi_{n-1,n}^p) \] is a finite real number, say \( M^* \), and \( \psi_{n-1,n} < d(x_{n-1}, x_n) \), thus as \( p \to \infty \) we get

\[ d(x_n, x^*) < M^* \frac{d(x_{n-1}, x_n)}{1-d(x_{n-1}, x_n)}, n = 1, 2, \ldots \]

but \( d(x_{n-1}, x_n) = d(T^{n-1}x_0, T^nx_0) \) by induction on \( n \). Thus,

\[ d(T^{n-1}x_0, T^nx_0) \leq \psi_{n-1}(d(x_0, x_1)). \]

Now, denote \( \psi^k(t) \) by \( \psi^k_t \) for \( k \geq 1 \), so, \( \psi^k_t < t \), hence,

\[ d(x_{n-1}, x_n) < d(x_0, x_1), \]

so,

\[ d(x_n, x^*) < M^* \frac{d(x_0, x_1)}{1-d(x_0, x_1)}, n = 0, 1, 2, \ldots \]

(ii) If \( p \) is even, then by tetrahedral inequality we get

\[ d(x_{n+p}, x_{n+1}) \leq d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{n+p-1}, x_{n+p}), \]

so,

\[ d(x_{n+p}, x_{n+1}) \leq \psi_{n-1,n} \frac{1-\psi_{n-1,n}^p}{1-\psi_{n-1,n}^p}. \]

Since \( \lim_{p \to \infty} \psi_{n-1,n}^p(t) < \lim_{p \to \infty} \sum_{k=1}^{p-1} \psi^k(t) < \infty \), \( \psi(t) < t \) for all \( t > 0 \), and \( d(x_{n-1}, x_n) > 0 \), then

\[ (1 - \lim_{p \to \infty} \psi_{n-1,n}^p) \] is a finite real number, say \( M^* \), and \( \psi_{n-1,n} < d(x_{n-1}, x_n) \), thus as \( p \to \infty \) we get

\[ d(x_{n+1}, x^*) < K^* \frac{d(x_{n-1}, x_n)}{1-d(x_{n-1}, x_n)}, n = 1, 2, \ldots \]

with \( d(x_{n-1}, x_n) \neq 1 \).
Similarly, we get
\[ d(x, x^*) < K^* \frac{d(x_0, x_1)}{1 - d(x_0, x_1)}. \]

(iii) Since
\[ d(Tx, Ty) \leq \psi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\}), \]
\[ d(y, Ty) \leq d(y, x) + d(x, Tx) + d(Tx, Ty) \] and \(\psi\) is nondecreasing, so
\[ d(Tx, Ty) \leq \psi(\max\{d(x, y), d(x, Tx), d(y, x) + d(x, Tx) + d(Tx, Ty), d(y, Tx)\}). \]

Take \(x = x^*, y = x_{n-1}\), then we get
\[ d(x, x^*) \leq \psi\left( d(x_{n-1}, x^*) + d(x, x^*) \right). \]

\[ \Box \]

3. A result concerning fixed points in \((\epsilon, \lambda)\) uniformly locally contractive mappings

In this section, we shall extend a theorem about fixed points of locally contractive mappings which raised by Das.

Next, we give Das’s theorem.

**Theorem 3.1[2].** If \(T\) is an \((\epsilon, \lambda)\) uniformly locally contractive mapping defined on a \(T\)-orbitally complete, \(\frac{\epsilon}{2}\) - chainable gms \((X, d)\) satisfying the following condition:
\[ d(x, y) < \frac{\epsilon}{2}, d(y, z) < \frac{\epsilon}{2} \] implies \(d(x, z) < \epsilon\) for all \(x, y, z \in X\), then \(T\) has a unique fixed point in \(X\).

The following theorem is an extension of Theorem 3.1

**Theorem 3.2.** If \(T\) is an \((\epsilon, \lambda)\) uniformly locally contractive mapping defined on a \(T\)-orbitally complete, \(\frac{\epsilon}{2}\) - chainable gms \((X, d)\) satisfying the following condition:
\[ d(x, y) < \frac{\epsilon}{2}, d(y, z) < \frac{\epsilon}{2} \] implies \(d(x, z) < \epsilon\), For all \(x, y, z \in X\) then
\[ d(x, x^*) < m \frac{\lambda}{2(1-\lambda)}, \]

where \(x_{n-1} = y_0, y_1, \ldots, y_m = Tx_{n-1} = x_n\)

**Proof.** For \(p \geq 3\), if \(p\) is odd, by the tetrahedral inequality, we get
\[ d(x_{n+p}, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \]

For the right side of inequality (1), it is obvious that
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\[ d(x_n, x_{n+1}) < \frac{\lambda m \varepsilon}{2}, \text{ and } d(x_{n+1}, x_{n+2}) < \frac{\lambda^2 m \varepsilon}{2}. \]

By induction on \( n \), we conclude that

\[ d(x_{n+p-1}, x_{n+p}) < \frac{\lambda^p m \varepsilon}{2}, \]

thus, as \( p \to \infty \), we get

\[ d(x_n, x^*) < m \varepsilon \frac{\lambda}{2(1-\lambda)}. \]

Now, for \( p \geq 4 \), if \( p \) is even. Similarly, as \( p \to \infty \) we get

\[ d(x_n, x^*) < m \varepsilon \frac{\lambda}{2(1-\lambda)}. \]

References


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