A Construction of Special Orthogonal Matrices

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Abstract

We give a construction of orthogonal matrices such that the sum of elements in each column equals 1. These matrices can be applied to statistical analysis.

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1 Introduction

Many parametric statistical procedures (Student’s t-test, the Pearson correlation coefficient, regression analysis, etc.) rely on computing the inner product of vectors and the average of numbers. The results of these procedures are invariant under the data transformation which preserve an inner product and an average (see, e.g., [1, 2]). Hence, substituting the transformed data, we can obtain appropriate statistical results without access to the original data.

In this paper, by modifying the Gram-Schmidt orthogonalization, we give a construction of orthogonal matrices which preserve an inner product and an average as a linear transformation.

2 Main Result

Denote by $v^t$ the transpose of a vector $v$ and by $v \cdot w$ the inner product of vectors $v$ and $w$. We define the vectors $v_1, \ldots, v_n$ recursively as follows:
• \( v_1 = (a_{1,1}, \ldots, a_{n-1,1}, b_{n,1})^t \), where \( a_{1,1}, \ldots, a_{n-1,1} \) are any positive numbers and \( b_{n,1} \) is the solution of

\[
(b_{n,1} + \sum_{i=1}^{n-1} a_{i,1})^2 = b_{n,1}^2 + \sum_{i=1}^{n-1} a_{i,1}^2.
\]  

(1)

• Let \( j \leq n - 1 \). Suppose that \( \{v_k = (v_{1,k}, \ldots, v_{n,k})^t \mid 1 \leq k \leq j \} \) are orthogonal to each other and \( (\sum_{i=1}^j v_{i,k})^2 = v_k \cdot v_k \) for \( 1 \leq k \leq j \). Put

\[
v_{j+1} = (a_{1,j+1}, \ldots, a_{n-1,j+1}, b_{n,j+1})^t
\]

and

\[
v_{j+1} = v_{j+1} - t_j v_j - \cdots - t_1 v_1,
\]

where \( t_k = v_{j+1} \cdot v_k / v_k \cdot v_k \) for \( 1 \leq k \leq j \), \( a_{i,j+1} \) is any number such that

\[
v_{i,j+1} = a_{i,j+1} - \sum_{k=1}^j t_k v_{i,k} > 0
\]

for \( 1 \leq i \leq n - 1 \), and \( b_{n,j+1} \) is the solution of

\[
\left( \sum_{i=1}^n v_{i,j+1} \right)^2 = \left( b_{n,j+1} - \sum_{k=1}^j t_k v_{n,k} \right)^2 + \sum_{i=1}^{n-1} \left( a_{i,j+1} - \sum_{k=1}^j t_k v_{i,k} \right)^2
\]

\[
= v_{j+1} \cdot v_{j+1}.
\]  

(2)

It is easily verified that (1) and (2) have a solution. Now we define

\[
w_j = e_j v_j / \sqrt{v_j \cdot v_j} \text{ for } 1 \leq j \leq n,
\]

where \( e_j = 1 \) if \( \sum_{i=1}^n v_{i,j} > 0 \) and \( e_j = -1 \) if \( \sum_{i=1}^n v_{i,j} < 0 \). From the construction of \( w_1, \ldots, w_n \), the following is immediately derived.

**Theorem 1** Let \( W_n = (w_{i,j}) = (w_1 \cdots w_n) \). Then we have:

(i) \( W^t W = I_n \), where \( I_n \) is the \( n \times n \) identity matrix.

(ii) \( \sum_{i=1}^n w_{i,j} = 1 \) for \( 1 \leq j \leq n \).

By Theorem 1, for vectors \( x = (x_1, \ldots, x_n)^t, y = (y_1, \ldots, y_n)^t \), we have the desired properties

\[
W_n x \cdot W_n y = x \cdot y, \quad \overline{W_n x} = \frac{\sum_{j=1}^n \sum_{i=1}^n w_{i,j} x_j}{n} = \frac{\sum_{j=1}^n x_j}{n} = \overline{x}.
\]

Moreover, it is impossible to recover \( x \) from \( W_n x \) without the information about \( W_n \).
References


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