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The Acoustic Wave Propagation Equation: Discontinuous Galerkin Time Domain Solution Approach

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Abstract

This paper discusses a finite element method, discontinuous Galerkin time domain approach that solves the 2-D acoustic wave equation in cylindrical coordinates. The method is based on discretization of the wave field into a grid of r and θ where r is the distance from the centre of the domain and θ is the radial angle. The Galerkin formulation is used to approximate the solution of the acoustic wave equation for the r and θ derivatives. The boundary conditions applied at the boundaries of the numerical grid are the free surface boundary condition at $r = 1$ and the absorbing boundary condition applied at the edges of the grid at $r = 2$. The solution is based on considering wave motion in the direction normal to the boundary, which in this case is the radial direction over radial angle $\theta \in [0^o, 30^o]$. The exact solution is described in terms of Bessel function of the first kind, which forms the basis of the boundary conditions for the values of pressure and eventually sufficient accuracy of the numerical solution. The algorithm generated in Matlab is tested against the known analytical solution, which demonstrates that, pressure of the wave increases as the radius increases within the same radial angle. The domain was discretized using linear triangular elements. The main advantage of this method is the ability to accurately represent the

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wave propagation in the free surface boundary with absorbing boundary condition at the edges of the grid, hence the method can handle wave propagation on the surface of a cylindrical domain. The resulting numerical algorithm enables the evaluation of the effects of cavities on seismograms recorded in boreholes or in cylindrical shaped tunnels.

Keywords: acoustic wave equation, wave propagation, cylindrical domain, triangular elements, Galerkin method, numerical solution, exact solution, discontinuous Galerkin method

1 Introduction

Acoustic or Sound wave propagation is motion of sound waves in heterogeneous media (fluids and solids). There are numerous numerical methods for solving different types of partial differential equations(PDEs) that describes the physical dynamics of the world, for instance PDEs are used to understand fluid flow in aerodynamics, wave dynamics for seismic exploration, orbital mechanics among others. [1], presents a mathematical model of linear acoustic wave propagation in fluids. The approach is based on an analytical solution to the homogenous wave equation for fluid medium. The propagation of acoustic pressure wave by the normal mode analysis in a medium with 2-D spatially-variable acoustic properties has been explained. The normal mode method analysis gives exact solutions without assumed restrictions on pressure and velocity. In [2] there is operator upscaling for acoustic wave equation, upscaling is the process of redefining the physical system's parameters up to a coarser grid, forming effective or equivalent parameters. Modelling of wave propagation in a heterogeneous medium requires input data that varies on many different spatial and temporal scales. Operator based upscaling captures the effect of the fine scales on a coarser domain without solving the full fine-scale problem. The method applied to the constant density, variable sound velocity acoustic wave equation, consists of solving small independent problems for approximate fine-scale information internal to each coarse block and using these sub-grid solutions to define an upscaled operator on the coarse grid. Galerkin method was invented by a Russian Mathematician, Boris Grigoryerich Galerkin. Galerkin methods are a class of methods for converting continuous operator problem (such as a differential equation) to a discrete problem. In principle, it is equivalent to applying the method of variation of parameters to a function space with a finite set of basis functions. Galerkin methods developed in engineering have now been used in many diverse applications including meteorology, oceanology and many other scientific disciplines that require tracking various wave phenomena. Considerable research has been undertaken in recent times to solve hyperbolic problems to develop optimal methods with respect to local polynomial degree p(REFERENCE PLEASE)[14]. The resulting methods have hence been termed hp-finite element methods. The comparison between continuous piecewise polynomials and their discontinuous versions can be found in [3], where a least squares stabilization method is proposed for discontinuous Galerkin methods. The discontinuous Galerkin (DG) method was originally introduced by Reed and Hill [4]. Erickson and Johnson [5] published a series of papers analyzing the DG method applied to parabolic problems where they focused on the heat equation by adopting the DG method in time and the standard Galerkin method space. Cockburn and Shu [6] extended the DG method to solve first-order hyperbolic partial differential equations of conservation laws. The authors developed later the local discontinuous Galerkin method for convention-diffusion problems. Grote [7] presented the symmetric interior penalty discontinuous Galerkin (SIPDG) method for the numerical discretization of second order scalar wave equation. They used the SIPDG finite element method in space while leaving the time dependence continuous. Celiker and Cockburn [8] studied the discontinuous Galerkin, Petro-Galerkin and hybridized mixed methods for convection-diffusion problems in one space. Wang [9] performed a study on Finite Difference and Discontinuous Galerkin methods for wave equations. Wave propagation problems can be modelled by partial differential equations. Wave propagation in fluids and in solids is modelled by the acoustic wave equation and the elastic wave equation respectively. In real world applications, wave often propagates in heterogeneous media with complex geometries, which makes it impossible to derive exact solutions to the governing equations. An efficient numerical method produces accurate approximation at low computational cost. The finite difference method is conceptually simple and easy, but has difficulties in handling complex geometries of the computational domain. However, Discontinuous Galerkin method is flexible with complex geometries, making it suitable for multi-physics problems. An energy based Discontinuous Galerkin method is used to solve a coupled acoustic-elastic problem. The idea of the DG method is to decompose the original problem into a set of sub problems that are connected using an appropriate transmission condition (known as the numerical flux). For geometric partitioning of the computational domain, the DG method uses standard disjoint finite element meshes. In the DG method, each element of the computational mesh determines a single sub problem. By setting the material properties for each sub problem to be constant, the solution is calculated separately for each element of the computational mesh. The solution for the whole computational domain is achieved by summing over all the elements of the mesh. In this paper we endeavour to do this by solving a 2-D acoustic wave equation over the cylindrical domain.

2 Mathematical Formulation

2.1 A 2-D acoustic wave propagation equation

In one dimension, the wave equation that describes the behaviour of sound is

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{1}$$

where p the acoustic pressure and c is the speed of propagation. A two-dimensional acoustic wave equation can be found using Euler's equation and the equation of continuity as given by Ahmad[10].

$$\frac{\partial p}{\partial t} + \rho c^2 \nabla . u = 0 \qquad \text{Continuity} \tag{2}$$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\nabla p = 0 \qquad \text{Euler} \tag{3}$$

where u is the particle velocity, p is the acoustic pressure, $\rho = \rho(x, z)$ is the density and $c = c(x, z)$ is the velocity of the acoustic wave in the acoustic media. Manipulation of (2) and (3) yields

$$\frac{\partial^2 p}{\partial t^2} - \rho c^2 \left[\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial p}{\partial z}\right)\right] \tag{4}$$

and assuming constant density, (4) is simplified to

$$\frac{\partial^2 p}{\partial t^2} - c^2\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2}\right] \tag{5}$$

The discontinuous Galerkin time domain method is applied directly to acoustic wave problems for partial differential equations of the form

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \qquad \text{in} \quad \Omega \tag{6}$$

where Ω is the domain. Here $p(x, z, t)$ denotes the wave disturbance at horizontal (lateral) coordinate x, vertical (depth) coordinate z (where z-axis points downward) and time t respectively. c is the medium velocity. The two dimensional domain Ω is bounded by the boundary $\partial\Omega$.

2.2 The Acoustic Wave Propagation in Cylindrical coordinates

The acoustic wave equation models sound propagation in the sea in the presence of cylindrical symmetry as domain. The numerical algorithm is based on the solution of the acoustic wave equation in 2-D cylindrical coordinates (r, θ) which leads to

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{7}$$

where r is the radius, θ is the radial angle, p is the pressure field and t denotes time[11]. For the numerical algorithm, we recast equation (7) as a system of three coupled first order equations given by

$$\frac{\partial}{\partial t}\begin{pmatrix} p \\ r\frac{\partial p}{\partial r} \\ \frac{\partial p}{\partial \theta} \end{pmatrix} = A\begin{pmatrix} p \\ r\frac{\partial p}{\partial r} \\ \frac{\partial p}{\partial \theta} \end{pmatrix} + B\begin{pmatrix} p \\ r\frac{\partial p}{\partial r} \\ \frac{\partial p}{\partial \theta} \end{pmatrix} \tag{8}$$

where

$$A = \begin{pmatrix} 0 & \frac{c^2}{r} & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & \frac{c^2}{r^2} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The numerical algorithm solves equation (8) with the free surface boundary condition at r=a (see Figure 1) and with the absorbing boundary condition at the edges of the grid at r=b. The variables are discretized on a spatial grid which is non-uniform in the r direction and uniform in the θ variable.

3 Method of Solution

3.1 Galerkin methods

Galerkin Methods belong to a class of solution methods for PDE's where the solution residue is minimized giving rise to well-known weak formulation of problems. In this approach, according to Prem[12] a basis function of the form

$$u(x, t) = \phi_0(x) + \sum_{j=1}^{N} c_j(t)\phi_j(x) \tag{9}$$

is chosen where, $\phi_j(x)$ is the finite number of basis functions and $c(t)$ are the unknown coefficients. For a given differential equation of the form $L[u] = f$,

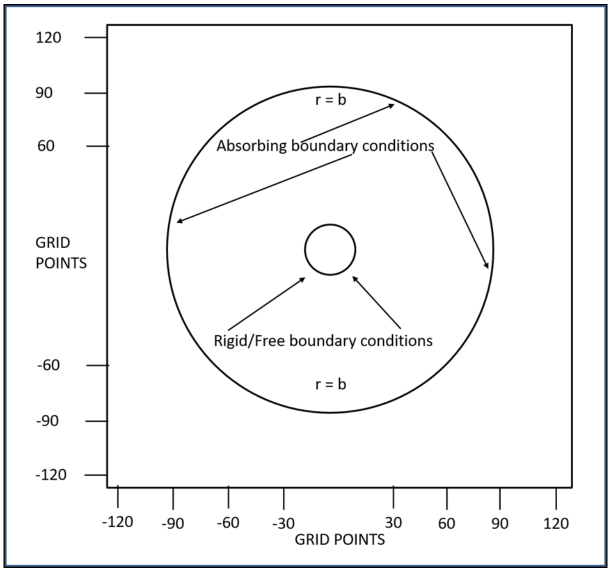

Figure 1: Configuration of boundary conditions

defined on a region Ω where L is a linear spatial differential operator and f is a given function, subject to boundary conditions $u = g(s)$ on Γ_1 and

$$\frac{\partial u}{\partial t} + k(s)u = h(s) \tag{10}$$

on Γ_2 where $\Gamma = \Gamma_1 + \Gamma_2$ is the boundary of the region Ω. A space of functions V is chosen in which elements u and v will reside. The function u is written as a linear combination of the basis functions of the space,

$$u = \sum_i a_i \phi_i \tag{11}$$

and v is chosen from amongst the basis functions. The measure of the residual (error) associated with an approximate solution is defined as $R[u] = L[u] - f$ should then theoretically be zero. That is,

$$\int_\Omega R[u]v dx = 0, \quad \forall v \in C_0^1(\Omega)$$

or

$$\sum_i a_i \int_\Omega L[\phi_i]\phi_j dx = \int_\Omega f\phi_j dx \quad \forall j \tag{12}$$

The Galerkin Method requires that the residual be orthogonal with respect to the basis functions ϕ_i i.e.

$$< R[u], \phi_i >= 0 \tag{13}$$

The infinite sums according to Matt[13], must be truncated at some large N, the integrals evaluated and re-written as a large N-dimensional system of equations to be solved for the unknowns $a_i's$,

$$Ka = f \tag{14}$$

According to Prem[12], the formulation can be generalized to a 2-D case which becomes;

$$\int\int_{\Omega} \{L[u] - f\}\phi_i dxdy = 0, \quad i = 1, \ldots, N \tag{15}$$

or

$$\sum_{j=1}^{N} a_i \int\int \phi_i L\phi_j dxdy = \int\int_{\Omega} f dxdy \tag{16}$$

which in the matrix form is written as $[A]\{c\} = \{b\}$, where,

$$A_{ij} = \int\int_{\Omega} \phi_i L\phi_j dxdy$$

,

$$b_i = \int\int_{\Omega} f\phi_i dxdy$$

also

$$[c] = \left\{ \begin{array}{c} c_i \\ c_j \end{array} \right\}$$

4 Solution of the acoustic wave propagation

The formulation developed above will be solved using dimensionless cylindrical coordinates, where the $x = r\cos\theta$, $z = r\sin\theta$ within the domain $r \in [1, 2], \theta \in [0, \frac{\pi}{6}]$. The acoustic wave equation in cylindrical coordinates (r, θ, t) is as given by (8). When solved using the appropriate initial and boundary conditions given and on discretizing, the equation reduces to; $[M]\{(p\} + c^2 + [K]\{p\} = 0$ where K and M are stiffness and mass matrices respectively, $K_e = \int_e (\nabla N)^T \nabla N d\Omega_e$ and $M_e = \int_e N^T d\Omega_e$ where Ω_e is the domain of the element and N is the basis function matrix, the exact solution for this problem, using Bessel function of the first kind for cylindrical coordinates is;

$$p(r, \theta, t) = 100 J_0(r) \sin(\frac{\pi t}{4}) \sin 3\theta \tag{17}$$

Hence at t=0. the initial condition for pressure becomes $p(r, \theta, 0) = 0$.

5 The appropriate boundary conditions

There are two types of boundary conditions exclusively used in the seismic simulation, the free surface boundary condition and the absorbing boundary condition. The numerical simulation is carried out on a bounded domain whose boundaries are either the physical sea surface, landform or the fields far away from the domain of interest. The absorbing boundary condition does not mimic any physical scenarios, but is used to truncate the open domain problem into a finite one so that the numerical method can handle. To solve equation (8) the free surface boundary condition at $r = a$ (see figure 1) and the absorbing boundary condition at the edges of the grid at $r = b$ are used. The variables are discretized on a spatial grid which is non-uniform in the r direction and uniform in the θ variable. With changing radius at $r = 0$ the values of pressure at the right hand boundary of the domain at t=1 are formulated. With changing radius at $\theta = \frac{\pi}{6}$ the values of pressure at the left hand boundary of the domain at $t = 1$ are also formulated.

6 The appropriate grid

The figure 2 below shows finite mesh showing discretization of the unit cell for the cylindrical bar using triangular elements. The unit bar above is $r = 2$ and $\theta = \frac{\pi}{6}$. The free surface boundary is at $r = 1$ and the absorbing boundary is at the edges of the grid at $r = 2$. The domain is discretised into 8 triangular elements.

7 Results

The solution is based on considering wave motion in the direction normal to the boundary, which in this case is the radial direction over radial angle $\theta \in [0^0, 30^0]$. The solution of the acoustic wave equation was obtained by solving the pressure variable over the domain using the exact solution which was obtained from the Bessel function of the first kind. The boundary conditions for pressure were obtained from the exact solution. For the numerical solution, the domain was discretised using linear triangular elements as in figure 3 above. Time integration was done using finite difference. Other ordinary differential equation solvers can also be used for instance ode45. The Table above illustrates the numerical solution, the exact solution over the nodes 1-9 and the absolute error. A comparison between numerical solution and analytical solution for pressure against $r \in [1, 2]$ is represented graphically in the figure below. A surface response for pressure over radial angle in radians and the radius was also generated as shown in the diagram below.

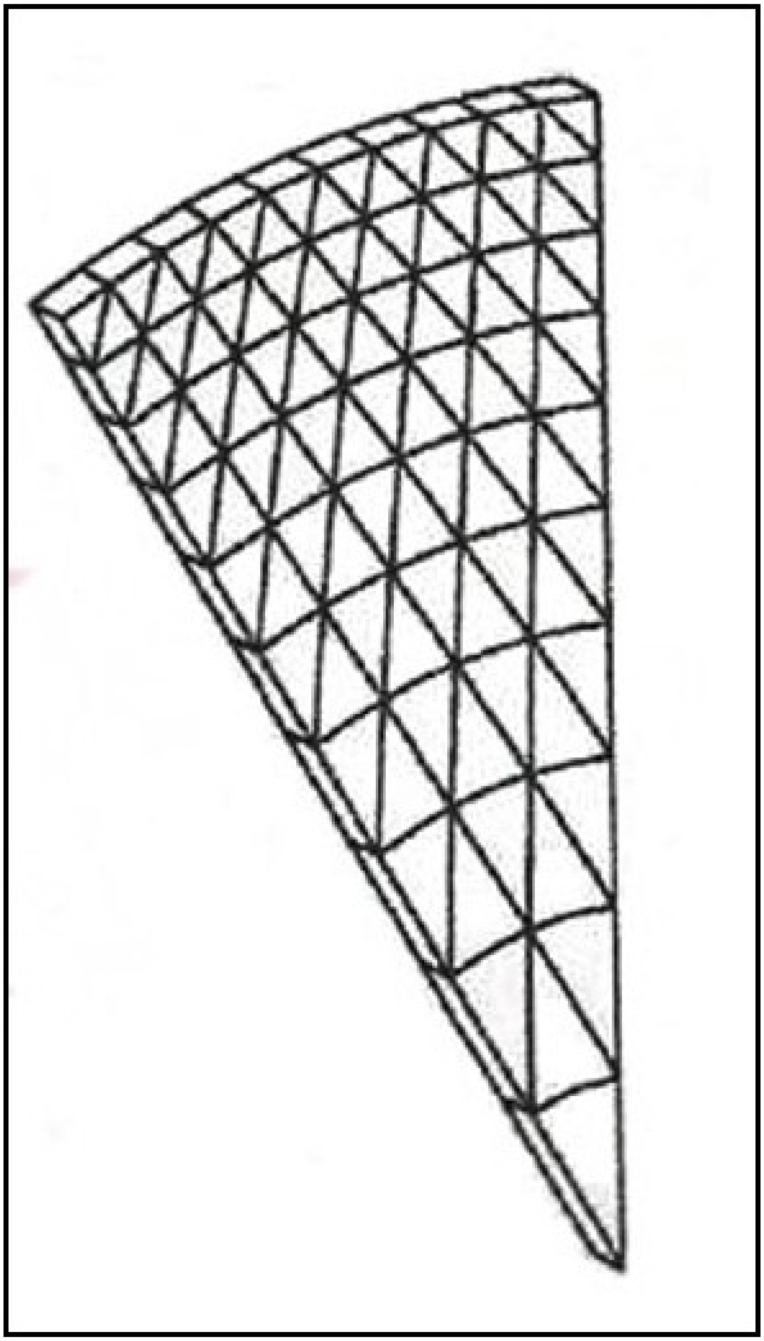

Figure 2: Finite element mesh showing discretization of the unit cell

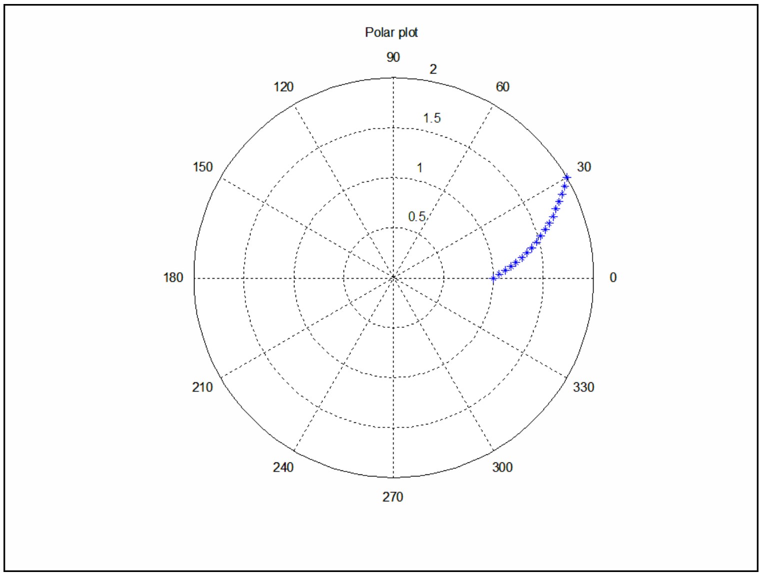

Figure 3: The polar plot of the numerical grid

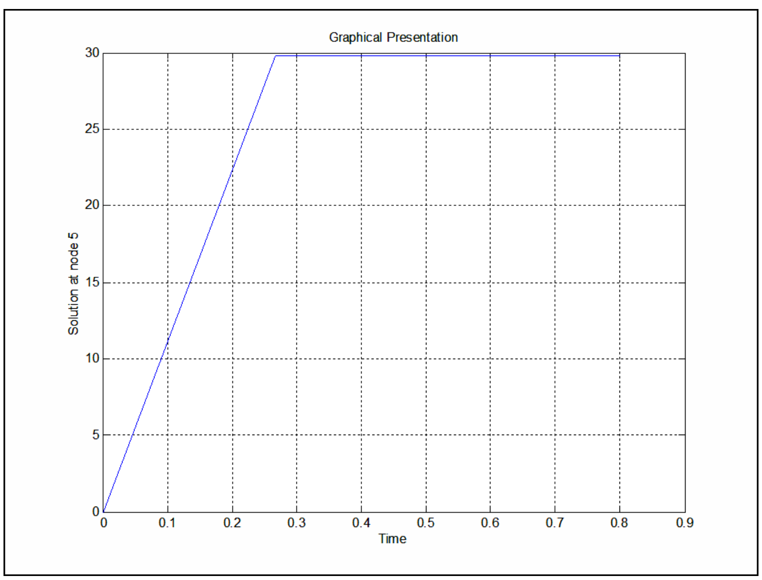

Figure 4: Graphical representation of the numerical solution at node 5 with time

Table 1: The Numerical and the exact solutions over nodes 1-9 and the absolute error

Node	Numerical	Exact	Absolute error
1	57.9164	57.9164	0.0000
2	32.6593	38.6481	5.9888
3	0.0000	0.0000	0.0000
4	34.1447	34.1447	0.0000
5	29.7743	30.1721	0.3978
6	0.0000	0.0000	0.0000
7	3.7800	3.7860	0.0060
8	20.3786	18.6147	1.7639
9	0.0000	0.0000	0.0000

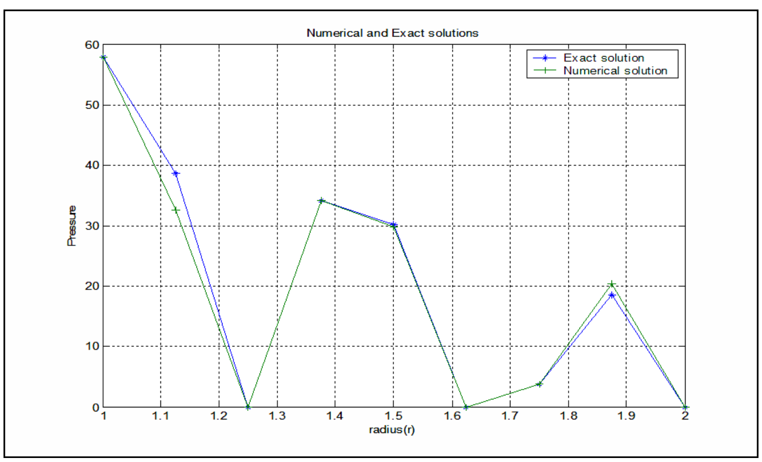

Figure 5: Graphical comparison between the numerical and the exact solutions

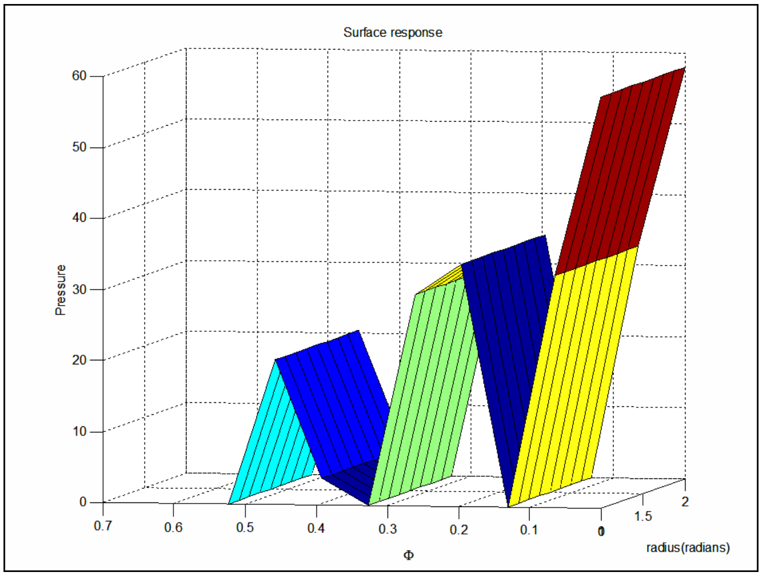

Figure 6: Surface response showing pressure over radial angle ϕ in radians

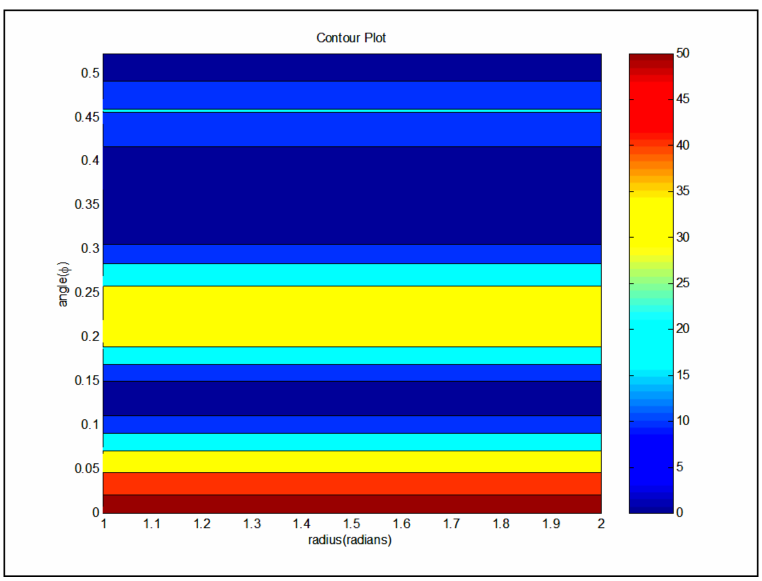

Figure 7: Contour plot

8 Discussion and Conclusion

8.1 Discussion of results

The analytical solution described in terms of Bessel function of the first kind forms the basis of sufficient accuracy of the numerical solution. Table 1 shows the numerical and exact values of pressure. The comparison between the numerical solution and the exact solution shows that the absolute error is zero (0) in nodes 1,3,4,6 and 9. In node 2, the absolute error is higher than in node 5,7 and 8 but very minimal in node 7. The variations are due to the curved boundaries which basically decreases as the radius increases. Fig 4, shows the graphical representation of the numerical solution of pressure at node 5 over the function time, which means at $t \in [0.2, 0.3]$ the numerical solution is obtained for generally all the nodes. Fig 5, shows a comparison between numerical solution and exact solution of pressure values against $r \in [1, 2]$. This means there is generally a perfect match between the numerical and exact results. A surface response for pressure over radial angle in radians and the radius was also generated as shown in fig 6, which shows that there is an increase in pressure as the radius increases. Fig 7, shows the contour plots of the numerical domain which further shows that the pressure is greatest at the greatest value of radius.

8.2 Conclusion

The Garlerkin numerical method with the use of MATLAB code has been used to solve the acoustic wave equation in 2-D cylindrical coordinates. The method is very useful in study of the pressure effects in wave propagation in fluids. The solution scheme is based on describing the exact solution in terms of Bessel function of the first kind and then developing the numerical solution. From the findings, a comparison between numerical solution and analytical solution for pressure against r?[1,2], shows that there is an almost perfect match between the numerical solution and analytical results. This demonstrates that the method can very accurately handle wave propagation in homogenous medium, including propagation on the surface of a cylindrical object. From the findings we can also conclude that, pressure of the wave increases as the radius increases within the same radial angle. The domain was discretized using linear triangular elements, however the solutions at the curved boundaries are slightly less accurate from the values of absolute errors obtained. The errors can be improved by refining the triangular mesh by using more triangular elements over the numerical grid or domain, carefully choosing the time variable and by use of higher order elements i.e quadratic elements. However, the only problem with small elements is that they can lead to small time step which can increase the overall computational cost in terms of time.

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