Partitions into Perfect Powers

Elementary Methods

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Abstract

In this article we prove the following theorem by use of very elementary methods. Let us consider a strictly increasing sequence \( A_i \) \( (i \geq 1) \) of positive integers, such that \( A_1 = 1 \). Let \( \varphi(x) \sim c_1 \sqrt[3]{x} \) be the number of these positive integers not exceeding \( x \), where \( c_1 \) is a positive constant and \( k \geq 1 \) is a positive integer. Let \( p(n) \) be the number of partitions of \( n \) into the numbers of the sequence \( A_i \). Then we have

\[ e^{c_2 k + \sqrt{n}} \leq p(n) \leq e^{c_3 k + \sqrt{n}} \]

where \( c_2 \) and \( c_3 \) are positive constants.

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1 Introduction and Main Results

Let us consider a strictly increasing sequence \( A_n \) of positive integers such that the number of \( A_n \) not exceeding \( x \) is \( \varphi(x) \). That is, \( \varphi(x) \) is the distribution function of the sequence. The number of partitions of the positive integer \( n \) in distinct parts (where the parts pertain to the sequence \( A_n \)) will be denoted \( p'(n) \) and the total number of partitions will be denoted \( p(n) \). It is easy to see that \( p'(n) \) and \( p(n) \) is very related with the distribution function \( \varphi(x) \) of the sequence \( A_n \), since we have the trivial upper bounds

\[ p'(N) \leq \binom{\varphi(N)}{0} + \binom{\varphi(N)}{1} + \cdots + \binom{\varphi(N)}{\varphi(N)} = 2^{\varphi(N)} = e^{(\log 2)\varphi(N)} \]
and
\[ p(N) \leq \left( \frac{\varphi(N)}{0} \right) N^0 + \left( \frac{\varphi(N)}{1} \right) N^1 + \cdots + \left( \frac{\varphi(N)}{N^{\varphi(N)}} \right) N^{\varphi(N)} \leq e^{c_7 \log N \varphi(N)} \]

In this article we are interested in \( p(n) \) when \( \varphi(x) \sim c_1 \sqrt{x} \), where \( s \) is a positive integer and \( c_1 \) a positive constant. We shall see that in this case the trivial upper bounds obtained above are very close of better upper bounds. We obtain upper and lower bounds using only the distribution function \( \varphi(x) \sim c_1 \sqrt{x} \) and very elementary methods. In particular cases (see examples, below) asymptotic formulas have been obtained using not elementary methods (complex function theory).

Lemma 1.1 Let \( \sum_{i=1}^{\infty} a_i \) and \( \sum_{i=1}^{\infty} b_i \) be two series of positive terms such that \( \lim_{i \to \infty} \frac{a_i}{b_i} = 1 \). Then if \( \sum_{i=1}^{\infty} b_i \) is divergent, we have \( \sum_{i=1}^{n} a_i \sim \sum_{i=1}^{n} b_i \)

Proof. See [3, page 332]. The lemma is proved.

Lemma 1.2 Let \( s \) be an arbitrary but fixed positive integer. The following asymptotic formula holds
\[ 1^s + 2^s + \cdots + n^s \sim \frac{n^{s+1}}{s+1} = c_3 n^{s+1} \]

Proof. We have
\[ 1^s + 2^s + \cdots + n^s = \int_{0}^{n} x^s \, dx + O(n^s) \sim \frac{n^{s+1}}{s+1} = c_3 n^{s+1} \]

The lemma is proved.

Theorem 1.3 Let us consider a strictly increasing sequence \( A_i \) (\( i \geq 1 \)) of positive integers. Let
\[ \varphi(x) \sim c_1 \sqrt{x} \quad (1) \]

the number of these positive integers not exceeding \( x \), where \( c_1 \) is a positive constant and \( s \geq 1 \) is a positive integer. Let \( p(n) \) be the number of partitions of \( n \) into the numbers of the sequence \( A_i \). Then we have
\[ p(n) \leq e^{c_7 \log n \sqrt[3]{n}} \quad (2) \]

where \( c_7 \) is a positive constant.
Proof. In this proof the $c_i$ are constants. Substituting $x = A_n$ into (1) we obtain
\[ n = \varphi(A_n) \sim c_1 \sqrt[3]{A_n} \]
That is
\[ A_n \sim c_2 n^s \quad (3) \]
Let us consider the inequality (see Lemma 1.1, Lemma 1.2 and (3))
\[ \sum_{i=1}^{n'} A_i = g(n) \sum_{i=1}^{n'} c_2 i^s = h(n)c_3(n')^{s+1} > n \]
where $g(n) \to 1$ and $h(n) \to 1$. This inequality holds if we put
\[ n' = \left\lfloor c_4 n^{s+1} \right\rfloor \sim c_4 n^{s+1} \quad (4) \]
and choose the constant $c_4$ sufficiently large.

Note that (see (1))
\[ \log \varphi(n) \sim c_5 \log n \quad (5) \]
and (see (1) and (4))
\[ \lim_{n \to \infty} \frac{n'}{\varphi(n)} = 0 \quad (6) \]
Therefore, we have (see (4), (5) and (6))
\[
\begin{align*}
P(n) & \leq \sum_{i=1}^{n} p(i) \\
& \leq \sum_{1 \leq i_1 \leq \varphi(n)} \frac{n}{A_{i_1}} + \sum_{1 \leq i_1 < i_2 \leq \varphi(n)} \frac{n}{A_{i_1} A_{i_2}} \\
& \quad + \cdots + \sum_{1 \leq i_1 < i_2 < \cdots < i_{n'} \leq \varphi(n)} \frac{n}{A_{i_1} A_{i_2} \cdots A_{i_{n'}}} \leq \sum_{1 \leq i_1 \leq \varphi(n)} n + \sum_{1 \leq i_1 < i_2 \leq \varphi(n)} n^2 \\
& \quad + \cdots + \sum_{1 \leq i_1 < i_2 < \cdots < i_{n'} \leq \varphi(n)} n^{n'} = \left( \frac{\varphi(n)}{1} \right) n + \left( \frac{\varphi(n)}{2} \right) n^2 + \cdots + \left( \frac{\varphi(n)}{n'} \right) n^{n'} \\
& \leq n' \left( \frac{\varphi(n)}{n'} \right) n^{n'} \leq (\varphi(n))^{n'} n^{n'} = \exp(n' \log \varphi(n) + n' \log n) \\
& \leq \exp(c_6 n' \log n) \leq e^{c_7 \log n + \sqrt[3]{n}}
\end{align*}
\]
That is, inequality (2). The theorem is proved.
Theorem 1.4 Let us consider a strictly increasing sequence $A_i$ ($i \geq 1$) of positive integers such that $A_1 = 1$. Let

$$\varphi(x) \sim c_1 \sqrt{x} \quad (7)$$

the number of these positive integers not exceeding $x$, where $c_1$ is a positive constant and $s \geq 1$ is a positive integer. Let $p(n)$ be the number of partitions of $n$ into the numbers of the sequence $A_i$. Then we have

$$p(n) \geq e^{c_8 s + \sqrt{n}} \quad (8)$$

where $c_8$ is a positive constant.

Proof. As in Theorem 1.3 we have the inequality (see Lemma 1.1, Lemma 1.2 and (3))

$$\sum_{i=2}^{n'} A_i = g(n) \sum_{i=1}^{n'} c_2 i^s = h(n) c_3 (n')^{s+1} < n$$

where $g(n) \to 1$ and $h(n) \to 1$. This inequality holds if we put

$$n' = \left[ c_4 n^{\frac{1}{s+1}} \right] \sim c_4 n^{\frac{1}{s+1}}$$

and choose the constant $c_4$ sufficiently small. Therefore

$$p(n) \geq \binom{n'-1}{1} + \binom{n'-1}{2} + \cdots + \binom{n'-1}{n'-1} = 2^{n'-1} - 1 \geq e^{c_8 s + \sqrt{n}}$$

Note that we add to each sum the number 1 all times is necessary up to to obtain $n$. The theorem is proved.

Let us consider the inequality, where $s \geq 1$ and $k \geq 2$ are arbitrary but fixed positive integers.

$$x_1^s + x_2^s + \cdots + x_k^s \leq x \quad (9)$$

The number of solutions $(x_1, \ldots, x_k)$, where the $x_i$ ($i = 1, \ldots, k$) are positive integers will be denoted $S_{k,s}(x)$. The following theorem is well-known (see [2]). We use it as a lemma.

Lemma 1.5 We have

$$S_{k,s}(x) \leq D_{k,s} x^\frac{k}{s} \quad (x \geq 0) \quad (10)$$

where

$$D_{k,s} \leq \frac{C^k}{(k!)^\frac{s}{s}} \quad (C > 1) \quad (11)$$
Proof. See [2]. The lemma is proved.

Let us consider a strictly increasing sequence $A_n$ of positive integers such that $A_1 = 1$ and the number of $A_n$ not exceeding $x$ is $\varphi(x) \sim c \sqrt{x}$ ($s \geq 1$). The number of partitions of the positive integer $n$ in parts (where the parts pertain to the sequence $A_n$) and only 1 can be repeated will be denoted $p_1(n)$. Recall the number of partitions in distinct parts is denoted (see the introduction) $p'(n)$. We have the following theorem.

**Theorem 1.6** The following inequality holds

$$e^{c_8 \sqrt[n]{s}} \leq p_1(n) \leq e^{c_9 \sqrt[n]{s}}$$  (12)

$$\sum_{i=1}^{n} p'(i) \leq e^{c_{10} \sqrt[n]{s}}$$  (13)

where $c_8$ and $c_9$ are positive constants.

Proof. Note that inequality $e^{c_8 \sqrt[n]{s}} \leq p_1(n)$ was proved in Theorem 1.4. Therefore we have to prove the other inequality in inequality (12). The proof of inequality (13) is the same as the proof of inequality (12). Since $\varphi(x) \sim c \sqrt{x}$ we have $A_n \sim \frac{1}{c} n^s$ and consequently there exist two positive constants $c_2$ and $c_1$ such that $c_2 \leq \frac{A_n}{n^s} \leq c_1$ ($n \geq 1$) ($c_2 \leq 1$). Let us consider the sums

$$A_{i_1} + A_{i_2} + \cdots + A_{i_k} \leq n \quad (2 \leq i_1 < i_2 < \cdots < i_k)$$

The number of these sums we denote $B_{k,s}(n)$. Note that we add $A_1 = 1$ (if necessary) up to $n$. That is, $1 + \cdots + 1 + A_{i_1} + A_{i_2} + \cdots + A_{i_k} = n$. Now, let us consider the inequality

$$c_2 i_1^s + c_2 i_2^s + \cdots + c_2 i_k^s \leq n \quad (2 \leq i_1 < i_2 < \cdots < i_k)$$

That is

$$i_1^s + i_2^s + \cdots + i_k^s \leq \frac{n}{c_2} \quad (2 \leq i_1 < i_2 < \cdots < i_k)$$

The number of these sums we denotes $C_{k,s}(n)$. Note that we have (Lemma 1.5)

$$B_{k,s}(n) \leq C_{k,s}(n) \leq \frac{1}{k!} S_{k,s} \left( \frac{n}{c_2} \right) \leq \frac{1}{k!} \frac{C^k}{(k!)^{\frac{1}{s}}} \left( \frac{n}{c_2} \right)^{\frac{1}{s}}$$

$$\leq \frac{1}{k!} \frac{C^k}{(k!)^{\frac{1}{s}}} \left( \frac{1}{c_2} \right)^k \left( \frac{C'}{k!} \right)^{\frac{k}{s}} n^{\frac{k}{s}} = \frac{(C')^k}{(k!)^{\frac{1}{s}}} n^{\frac{k}{s}} \quad (C' > 1)$$
and consequently we have

\[ p_1(n) \leq \sum_{k=0}^{\infty} \frac{(C')^k}{(k!) \frac{x}{n + 1}} = \sum_{k=0}^{\infty} \left( \frac{\left( (C') \frac{x}{n + 1} \right)^k}{k!} \right)^{\frac{x + 1}{n}} \]

\[ \leq \sum_{k=0}^{\infty} \left( \frac{\left( (C') \frac{x}{n + 1} \right)^{k^2}}{k!} \right) \leq \left( \frac{\sum_{k=0}^{\infty} \left( (C') \frac{x}{n + 1} \right)^{k^2}}{k!} \right)^2 = e^{e^{\frac{x}{n + 1}}} \]

where we have used the well-known power series for \( e^x \). The theorem is proved.

**Lemma 1.7** Let us consider the inequality

\[ r_1x_1 + \cdots + r_nx_n \leq x \quad (x \geq 0) \]

where the \( r_i \) (\( i = 1, \ldots, n \)) are fixed positive real numbers. The number of solutions \((x_1, \ldots, x_n)\) to this inequality, where the \( x_i \) (\( i = 1, \ldots, n \)) are positive integers, will be denoted \( S_n(x) \).

The following inequality holds

\[ S_n(x) \leq \frac{1}{n!} \frac{x^n}{r_1 \cdots r_n} \quad (x \geq 0) \]

Proof. If \( x \geq r_1 \) then the solutions to the inequality \( r_1x_1 \leq x \) are \( x_1 = 1, \ldots, \left\lfloor \frac{x}{r_1} \right\rfloor \) and consequently \( S_1(x) = \left\lfloor \frac{x}{r_1} \right\rfloor \leq \frac{x}{r_1} \). On the other hand, if \( 0 \leq x < r_1 \) we have \( S_1(x) = 0 \) and consequently also \( S_1(x) \leq \frac{x}{r_1} \). Therefore the lemma is true for \( n = 1 \). Suppose that the lemma is true for \( n - 1 \geq 1 \), we shall prove that the lemma is also true for \( n \). Suppose that \( x \geq r_1 + \cdots + r_n \) then

\[ S_n(x) = \sum_{x_n=1}^{\left\lfloor \frac{x}{r_n} \right\rfloor} S_{n-1} \left( x - r_nx_n \right) \leq \frac{1}{(n-1)!} \frac{1}{r_1 \cdots r_{n-1}} \sum_{x_n=1}^{\left\lfloor \frac{x}{r_n} \right\rfloor} (x - r_nx_n)^{n-1} \]

\[ \leq \frac{1}{(n-1)!} \frac{1}{r_1 \cdots r_{n-1}} \int_0^{\frac{x}{r_n}} (x - r_nx_n)^{n-1} dx_n = \frac{1}{n!} \frac{x^n}{r_1 \cdots r_n} \]

Note that the function \( f(x_n) = (x - r_nx_n)^{n-1} \) is strictly decreasing in the interval \([0, \frac{x}{r_n}]\) and in this interval the area below the function is greater than the sum of the areas of the \( \left\lfloor \frac{x}{r_n} \right\rfloor \) rectangles of base 1 and height \( (x - r_nx_n)^{n-1} \), that is, the sum \( \sum_{x_n=1}^{\left\lfloor \frac{x}{r_n} \right\rfloor} (x - r_nx_n)^{n-1} \).

On the other hand, if \( 0 \leq x < r_1 + \cdots + r_n \) then \( S_n(x) = 0 \) and consequently the inequality also holds. The lemma is proved.

Now, we prove our main theorem.
Theorem 1.8 In equation (2) $\log n$ can be eliminated. Consequently we have $p(n) \leq e^{c_s n^{s+\sqrt{s}}}$. Consequently we have the inequality 

$$e^{c_s n^{s+\sqrt{s}}} \leq p(n) \leq e^{c_s n^{s+\sqrt{s}}} \quad (14)$$

Proof. There exists $c$ sufficiently large but fixed such that the following inequality holds (see the proof of Theorem 1.3)

$$1^s + 2^s + \cdots + [cn^i] > n$$

Let us consider the equation 

$$1^sx_1 + 2^sx_2 + \cdots + k^sx_k \leq n \quad (k = 1, 2, \ldots, [cn^i])$$

By Lemma 1.7 the number of solutions $S_k(n)$ to this equation satisfies the inequality 

$$S_k(n) \leq \frac{n^k}{(k!)^{s+1}}$$

If we consider the equation $a_1^sx_1 + a_2^sx_2 + \cdots + a_k^sx_k \leq n \quad (a_1^s < a_2^s < \cdots < a_k^s)$ the number of solutions to this equation does not exceed $S_k(n)$ and the number of these equations does not exceed the number of partitions of $n$ into $k$ distinct $s$-th powers, and by Theorem 1.6 (equation (13)) does not exceed $e^{c_{10} n^{s+\sqrt{s}}}$. On the other hand, note that the inequality 

$$\frac{n^{k-1}}{(k-1)!} \leq \frac{n}{(k!)^{s+\sqrt{s}}}$$

holds if $k = 2, 3, \ldots, [n^{1/(s+\sqrt{s})}]$ and the inequality 

$$\frac{n^{k-1}}{(k-1)!} > \frac{n}{(k!)^{s+\sqrt{s}}}$$

holds if $k = [n^{1/(s+\sqrt{s})}] + 1, \ldots, [cn^i]$. Therefore the greatest $S_k(n)$ is

$$S_{[n^{1/(s+\sqrt{s})}]}(n) \leq \frac{n}{([n^{1/(s+\sqrt{s})}]!^{s+1}} \leq e^{c_{12} n^{s+\sqrt{s}}}$$

where we have used the trivial well-known formula (whose elementary proof is an integral or an immediate consequence of the Stirling’s formula) $\log n! = \sum_{i=1}^{n} \log i = n \log n - n + o(n)$. Therefore

$$\sum_{i=2}^{n} p_s(i) \leq [cn^i] e^{c_{10} n^{s+\sqrt{s}}} e^{c_{12} n^{s+\sqrt{s}}} \leq e^{c_{11} n^{s+\sqrt{s}}} \quad (15)$$

where $p_s(i)$ denotes the number of partitions of $i$ into $s$-th powers. We consider only partitions without 1. Now, as in Theorem 1.6 we have the partitions (now, $k$ is variable)

$$A_{i_1} + A_{i_2} + \cdots + A_{i_k} \leq n \quad (2 \leq i_1 \leq i_2 \leq \cdots \leq i_k)$$
The number of these sums we denote $B_s(n)$. Note that we add $A_1 = 1$ (if necessary) up to $n$. That is, $1 + \cdots + 1 + A_{i_1} + A_{i_2} + \cdots + A_{i_k} = n$. Now, let us consider the inequality

$$c_2i_1^s + c_2i_2^s + \cdots + c_2i_k^s \leq n \quad (2 \leq i_1 \leq i_2 \leq \cdots \leq i_k)$$

That is

$$i_1^s + i_2^s + \cdots + i_k^s \leq \frac{n}{c_2} \quad (2 \leq i_1 \leq i_2 \leq \cdots \leq i_k)$$

The number of these sums we denote $C_s(n)$. Now, the proof is an immediate consequence of inequality (15), since $B_s(n) \leq C_s(n) \leq \sum_{i=2}^{\left\lfloor \frac{n}{c_2} \right\rfloor} p_s(i)$. The theorem is proved.

**Example 1.9** Theorem 1.8 is applicable to the number of partitions of $n$ into positive integers since in this case $\varphi(x) \sim x$ ($k = 1$). Here is well-known the asymptotic formula obtained by Hardy and Ramanujan using not elementary methods (complex function theory)

$$p(n) \sim \frac{1}{4\sqrt{3n}} e^{\pi \sqrt{n}}$$

**Example 1.10** Theorem 1.8 is also applicable to the number of partitions $p_k(n)$ of $n$ into $k$-th powers, since in this case $\varphi(x) \sim \sqrt[k]{x}$ ($k \geq 2$). This problem was studied by various authors. Hardy and Ramanujan obtained using not elementary methods the following asymptotic formula

$$p_k(n) \sim \frac{b_k e^{c_k \sqrt[k]{n}}}{n^{\frac{4k+1}{2k+2}}}$$

where $b_k$ and $c_k$ are constants depending of $k$.

**Example 1.11** Theorem 1.8 is also applicable to the number of partitions of $n$ into perfect powers of the form $a^s$, where $s \geq k \geq 2$ and $k$ is an arbitrary but fixed positive integer. Since in this case we have

$$\left\lfloor \sqrt[k]{x} \right\rfloor \leq \varphi(x) \leq \sqrt[k]{x} + \sqrt[k+1]{x} + \cdots + x \left\lfloor \frac{\log x}{\log 2} \right\rfloor \leq \sqrt[k]{x} + \frac{\log x}{\log 2}$$

and consequently $\varphi(x) \sim \sqrt[k]{x}$. In particular, if $k = 2$ then we obtain all perfect powers.

**Example 1.12** Theorem 1.8 is also applicable to the number of partitions of $n$ into $k$-full numbers. That is, the numbers such that in their prime factorization all exponents are greater than or equal to $k$, since it is well known [1] that $\varphi(x) \sim c_k \sqrt[k]{x}$, where $c_k$ is a constant depending of $k$. 
Example 1.13 Since \( \varphi(x) \sim c_1 \sqrt{x} \) is equivalent to the establishment \( A_n \sim \frac{1}{c_1} n^k \), Theorem 1.8 is applicable to the polynomial sequence \( A_n = a_k n^k + \cdots + a_1 n + a_0 \), where the \( a_i \) are nonnegative integers and \( a_k \) is a positive integer. etc.

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