Edge $r$-Irregularity Strength of Paths, Cycles, and Complete Graphs

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Abstract

Let $G = (V, E)$ be a graph and $f$ be a vertex $k$-labeling of $G$. The weight of an edge $uv \in E$ with respect to $f$, denoted by $w_f(uv)$, is given by $w_f(uv) = f(u) + f(v)$. A vertex $k$-labeling $f$ is said to be an edge $r$-irregular $k$-labeling if at most $r$ edges are permitted to have the same weight. The minimum $k$ for which the graph $G$ has an edge $r$-irregular $k$-labeling is called the edge $r$-irregularity strength of $G$, denoted by $es_r(G)$.

In this paper, we gave the edge $r$-irregularity strength of paths, present a tight upper-bound for the edge $r$-irregularity strength of cycles, and state some characterizations regarding the edge $r$-irregularity strength of complete graphs.

Mathematics Subject Classification: 05C15

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1 Introduction

Chartrand et al. [9] introduced the notion of irregularity strength. Since then many are studying the concept. Some of them were Bohman and Kravitz [9], Dinitz et al. [2], Frieze et al. [3], Jendrol et al. [31], Amar and Togni [1],

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Motivated by some investigations on irregularity strength, Baca et al. [13] introduced the notions of total vertex irregularity strength and total edge irregularity strength of a graph. Both concepts attracted a lot of attention. For example, the concept total edge irregularity strength was studied by Ahmad et al. [14], Al-Mushayt et al. [20], Baca and Siddiqui [15], Ivanco and Jendrol [4], Jendrol et al. [16], and Haque [25]. On the other hand, the concept total vertex irregularity strength was studied by Anholcer et al. [21], Majerski and Przybylo [24], Haque [25], and Nurdin et al. [17].

A pseudo-dual of the concept irregularity strength, called edge irregularity strength, was first found in [26] and was studied by a couple of researchers. Amad et al. [26] gave a lower bound of the edge irregularity strength of a simple graph in terms of its size and maximum degree, and another lower bound in terms of the its total irregularity strength. Moreover, they gave the irregularity strength of paths, cycles, stars, double stars and Cartesian product of paths.

Tarawneh et al. [29] gave the irregularity strength of the corona product $P_n \circ P_2$, $P_n \circ K_m$, and $P_n \circ S_m$. They also presented an open problem of finding the exact value of the edge irregularity strength of the corona product $P_n \circ P_m$ for $n, m \geq 3$.

Tarawneh et al. [30] gave the irregularity strength of the corona product of a cycle and an empty graph.

In this study, we extended the concept edge irregularity strength of graphs to a more general concept. We call it edge $r$-irregularity strength. It is anchored on the question: What if $r$ edges are allowed to have the same color? Thus, we define an edge $r$-irregular $k$-labeling as a vertex $k$-labeling $f$ of a graph $G$ with the property that at most $r$ edges may be permitted to have the same weight. The minimum $k$ for which the graph $G$ has an edge $r$-irregular $k$-labeling is called the edge $r$-irregularity strength of $G$.

The path $P_n = (v_1, v_2, \ldots, v_n)$ is the graph with distinct vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. The cycle $C_n = [v_1, v_2, \ldots, v_n]$, $n \geq 3$, is the graph with vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$. A complete graph of order $n$, denoted by $K_n$, is the graph in which every pair of distinct vertices are adjacent.

\section{Preliminary Results}

This section presents some basic properties of the edge $r$-irregularity strength of graphs. Observation 2.1 is a popular combinatorial problem and is found in [27].
Observation 2.1 Let \( n \) be a positive integer and \( A_n = \{1, 2, \ldots, n\} \). Then the number of distinct sums of the form \( i + j \) with \( i, j \in A_n \) is given by \( 2n - 1 \).

To see this, we look at the smallest and the largest number which can be formed. Observe that \( x = 1 + 1 = 2 \) and \( y = n + n = 2n \) is the smallest and the largest number which can formed, respectively. Also note that every number in between \( x \) and \( y \) is of the form \( i + j \) with \( i, j \in A_n \). Therefore, there are \( y - x + 1 = 2n - 2 + 1 = 2n - 1 \) distinct numbers created.

Theorem 2.2 presents a tight lower-bound for the edge \( r \)-irregularity strength of a graph. Equality is attained in paths.

**Theorem 2.2** Let \( G \) be a graph of size \( m \geq 2 \) and \( r \) be a positive integer. Then

\[
es_r(G) \geq \left\lceil \frac{m + r}{2r} \right\rceil.
\]

**Proof:** Let \( G \) be a graph of size \( m \geq 2 \) and \( r \) be a positive integer. We note that if \( f \) is a edge \( r \)-irregular \( k \)-labeling of \( G \), then \( f \) must have at least \( \lceil m/r \rceil \) colors in its co-domain. On the other hand, in Observation 2.1, the smallest \( k \) such that \( \{1, 2, \ldots, k\} \) produce distinct sums of the form \( i + j \) is given by \( 2k - 1 \). Thus, \( 2k - 1 = \lceil m/r \rceil \). Solving for \( k \), we have \( k = \lceil (m + r)/2r \rceil \). Therefore, if \( f \) is a edge \( r \)-irregular \( k \)-labeling of \( G \), then \( k \geq \lceil (m + r)/2r \rceil \), that is \( es_r(G) \geq \lceil (m + r)/2r \rceil \). ■

Theorem 2.3 characterizes graphs with edge \( r \)-irregularity strength equal to 1.

**Theorem 2.3** Let \( G \) be a graph of size \( m \) and \( r \) be a positive integer. Then \( es_r(G) = 1 \) if and only if \( r \geq m \).

**Proof:** Let \( G \) be a graph of size \( m \) and \( r \) be a positive integer. Assume that \( es_r(G) = 1 \) and let \( f \) is a edge \( r \)-irregular 1-labeling of \( G \). Then \( f \) must be given by \( f(v) = 1 \) for all \( v \). This would imply that \( r \geq m \).

Conversely, assume that \( r \geq m \). Define \( f : V(G) \to \{1\} \) by \( f(v) = 1 \) for all \( v \). Then \( f \) is a edge \( r \)-irregular 1-labeling of \( G \). Therefore, \( es_r(G) = 1 \). ■

Remark 2.4 follows immediately from Theorem 2.3.

**Remark 2.4** Let \( G \) be a connected graph of size \( m \geq 3 \) and \( r \) be a positive integer. If \( r = m - 1 \), then \( es_r(G) = 2 \).
To see this, let $G$ be a connected graph of size $m \geq 3$ and $r = m - 1$. Let $u \in V(G)$ and define $f : V(G) \to \{1, 2\}$ by $f(u) = 2$ and $f(v) = 1$ for all $v \in V(G) \setminus \{u\}$. Then $f$ is an edge $r$-irregular 2-labeling of $G$. Hence, $es_r(G) \leq 2$. By Theorem 2.3, $es_r(G) \leq 2$.

Observation 2.5 presents the monotonic property of the edge $r$-irregularity strength on the family of all subgraphs of a graph. If $H$ is a subgraph of $G$, then we note that every edge $r$-irregular $k$-labeling of $G$ restricted to the vertex set of $H$ is an edge $r$-irregular $k$-labeling of $H$. Hence, the following observation holds.

Observation 2.5 Let $G$ and $H$ be graphs. If $H$ is a subgraph of $G$, then $es_r(H) \leq es_r(G)$.

3 Edge $r$-Irregularity Strength of Complete Graphs

This section presents the edge $r$-irregularity strength of complete graphs. Corollary 3.1 follows from Observation 2.1 and Corollary 3.2 follows from Theorem 2.3 and Remark 3.3 follows from Remark 2.4.

Corollary 3.1 Let $K_n$ be a complete graph of order $n$. Then $es_1(K_n) \geq n^2 - n - 1$.

Proof: Let $K_n$ be a complete graph of order $n$. Note that $|E(K_n)| = n(n - 1)/2$. If $f$ is an edge irregular labeling of $K_n$, then by Observation 2.1 the co-domain of $f$ must have at least $2|E(K_n)| - 1 = 2[n(n - 1)/2] - 1 = n^2 - n - 1$ colors. Therefore, $es_1(K_n) \geq n^2 - n - 1$. ■

Corollary 3.2 Let $K_n$ be a complete graph of order $n$. Then $es_r(K_n) = 1$ if and only if $r \geq n(n - 1)/2$.

Proof: Let $K_n$ be a complete graph of order $n$. Note that $|E(K_n)| = n(n - 1)/2$. Hence, by Theorem 2.3 the corollary follows. ■

Remark 3.3 Let $K_n$ be a complete graph of order $n$. If $r = [n(n - 1)/2] - 1$, then $es_r(K_n) = 2$.

To see this, let $K_n$ be a complete graph of order $n$. Note that $|E(K_n)| = n(n - 1)/2$. Hence, by Remark 2.4 $es_r(K_n) = 2$. 


4 Edge r-Irregularity Strength of Paths

In this section we gave the edge r-irregularity strength of paths. Remark 4.1 follows from Theorem 2.2.

**Remark 4.1** Let $P_n$ be a path of order $n$. Then $es_r(P_n) \geq \lceil (n + r - 1)/2r \rceil$.

**Theorem 4.2** Let $P_n$ be a path of order $n$. Then

$$es_r(P_n) = \left\lceil \frac{n + r - 1}{2r} \right\rceil.$$

**Proof:** Let $P_n = (v_1, v_2, \ldots, v_n)$ be a path of order $n$. Define $f : V(P_n) \to \{1, 2, \ldots, k\}$ (where $k = \lceil (n + r - 1)/2r \rceil$) by

$$f(v_i) = \begin{cases} 
\lceil i/(r+1) \rceil, & \text{if } i < n - (r-1)(k-1) \\
\lceil i/(r+1) \rceil, & \text{if } i \geq n - (r-1)n - (r-1)(k-1), \text{ and } i \equiv (n-k+j)(\text{mod } 2(k-1)) \\
k - j + 1, & \text{if } i \geq n - (r-1)n - (r-1)(k-1), \text{ and } i \equiv (n-k+j)(\text{mod } 2(k-1)) \text{ or } i \equiv (n-k-j)(\text{mod } 2(k-1)) \text{ with } j \in \{1, 2, \ldots, k\}.
\end{cases}$$

if $r$ is odd, and

$$f(v_i) = \begin{cases} 
\lceil i/(r+1) \rceil, & \text{if } i < n - (r-1)(k-1) \\
k - j + 1, & \text{if } i \geq n - (r-1)n - (r-1)(k-1), \text{ and } i \equiv (n-k+j)(\text{mod } 2(k-1)) \text{ or } i \equiv (n-k-j)(\text{mod } 2(k-1)) \text{ with } j \in \{1, 2, \ldots, k\}.
\end{cases}$$

if $r$ is even.

Then clearly $f$ is an edge r-irregular $k$-labeling of $P_n$. Hence, $es_r(P_n) \leq \lceil (n + r - 1)/2r \rceil$. By Remark 4.1 we must have $es_r(P_n) = \lceil (n + r - 1)/2r \rceil$.

\[\blacksquare\]

5 Edge r-Irregularity Strength of Cycles

In this section we gave the edge r-irregularity strength of cycles. Remark 5.1 follows from Theorem 2.2.

**Remark 5.1** Let $C_n$ be a cycle of order $n$. Then $es_r(C_n) \geq \lceil (n + r)/2r \rceil$. 
Theorem 5.2 Let $C_n$ be a cycle of order $n$. Then

$$es_r(C_n) \leq \begin{cases} 
[(n + 3r - 3)/(2r - 1)], & \text{if } r \text{ is odd} \\
[(n + 3r - 1)/2r], & \text{if } r \text{ is even.}
\end{cases}$$

Proof: Let $C_n = [v_1, v_2, \ldots, v_n]$ be a cycle of order $n$. Define $f : V(C_n) \to \{1, 2, \ldots, k\}$ (where $k = \lceil (n + 3r - 3)/(2r - 1) \rceil$ if $r$ is odd, and $k = \lfloor (n + 3r - 1)/2r \rfloor$ if $r$ is even) by

\[
\begin{align*}
f(v_i) = & \begin{cases} 
[i/(r + 1)], & \text{if } i < n - (r - 2)(k - 1) + 1 \\
1, & \text{if } i \geq n - (r - 2)(k - 1) + 1 \text{ and } i \equiv n + 1(\text{mod } 2(k - 1)) \\
2, & \text{if } i \geq n - (r - 2)(k - 1) + 1 \\
& \quad \text{and } i \equiv n(\text{mod } 2(k - 1)) \\
& \quad \text{or } i \equiv (n + 2)(\text{mod } 2(k - 1)) \\
& \quad \quad \vdots \\
k - 1, & \text{if } i \geq n - (r - 2)(k - 1) + 1, \\
& \quad \text{and } i \equiv (n - k + 3)(\text{mod } 2(k - 1)) \\
& \quad \text{or } i \equiv (n + k - 1)(\text{mod } 2(k - 1)) \\
k, & \text{if } i \geq n - (r - 2)(k - 1) + 1 \\
& \quad \text{and } i \equiv (n - k + 2)(\text{mod } 2(k - 1)).
\end{cases}
\end{align*}
\]

if $r$ is odd, and

\[
\begin{align*}
f(v_i) = & \begin{cases} 
[i/(r + 1)], & \text{if } i < n - (r - 1)(k - 1) + 1 \\
1, & \text{if } i \geq n - (r - 1)(k - 1) + 1 \text{ and } i \equiv n + 1(\text{mod } 2(k - 1)) \\
2, & \text{if } i \geq n - (r - 1)(k - 1) + 1 \\
& \quad \text{and } i \equiv n(\text{mod } 2(k - 1)) \\
& \quad \text{or } i \equiv (n + 2)(\text{mod } 2(k - 1)) \\
& \quad \quad \vdots \\
k - 1, & \text{if } i \geq n - (r - 1)(k - 1) + 1, \\
& \quad \text{and } i \equiv (n - k + 3)(\text{mod } 2(k - 1)) \\
& \quad \text{or } i \equiv (n + k - 1)(\text{mod } 2(k - 1)) \\
k, & \text{if } i \geq n - (r - 1)(k - 1) + 1 \\
& \quad \text{and } i \equiv (n - k + 2)(\text{mod } 2(k - 1)).
\end{cases}
\end{align*}
\]
if $r$ is even.

Then $f$ is an edge $r$-irregular $k$-labeling of $C_n$. Hence,

$$es_r(C_n) \leq \begin{cases} 
(n + 3r - 3)/(2r - 1), & \text{if } r \text{ is odd} \\
\lceil (n + 3r - 1)/2r \rceil, & \text{if } r \text{ is even.}
\end{cases}$$

The bound in Theorem 5.2 is sharp. To see this, let $r = 2$ and $n = 7$. Then by Theorem 4.2, $es_2(C_7) \leq \lceil (7 + 3(2) - 1)/2(2) \rceil = \lceil 12/4 \rceil = 3$. And by Remark 5.1, $es_2(C_7) \geq \lceil (7 + 2)/2(2) \rceil = \lceil 9/4 \rceil = 3$. Hence, $es_2(C_7) = 3$.

It is observed that for small values of $r$ and $n$, equality in Theorem 4.2 holds. This pointed us to the next conjecture.

**Conjecture 5.3** Let $C_n$ be a cycle of order $n$. Then

$$es_r(C_n) = \begin{cases} 
(n + 3r - 3)/(2r - 1), & \text{if } r \text{ is odd} \\
\lceil (n + 3r - 1)/2r \rceil, & \text{if } r \text{ is even.}
\end{cases}$$

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