Applied Mathematical Sciences, Vol. 13, 2019, no. 15, 733 - 741
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2019.9690

Optimal Control Governed by Stochastic Elliptic Equations with Regular States

Steeven Belvinos Affognon

Pan African University
Institute for Basic Sciences, Technology and Innovation, Kenya

Philip Ngare

School of Mathematics, University of Nairobi, Kenya

Guy Degla

Institut de Mathématiques et de Sciences Physiques
Université d'Abomey-Calavi, Benin

Abstract

This study deals with an optimal control problem subject to a stochastic elliptic equation with Dirichlet boundary condition and in which the state process is regular on a stochastic Hilbert space. We prove the existence and uniqueness of the optimal control and provide furthermore necessary and sufficient optimality conditions. The optimal solution is obtained in the case where there is no constraint. Our method is based on variational theory of elliptic boundary problems in Hilbert spaces.

Keywords: Optimal control, Stochastic elliptic equations, Stochastic Hilbert space, Stochastic fractional Sobolev space, Variational formulation

1 Introduction

Optimal control problems are currently among major research topics in applied mathematics, science engineering [1,2] and some related branches. Due to its

enormous applications in diverse fields such as economics, finance, robotics, engineering etc. [3], it is increasingly having a lot of impacts. Computational methods both deterministic and stochastic optimal control governed by partial differential equations(PDEs) have been investigated and are still challenging [1,2,4,5].
In the previous study, [6] focused on some particular problems governed by elliptic, parabolic and hyperbolic equations with boundary conditions. The formulation of the control problem for deterministic case was established and the conditions for obtaining a deterministic optimal control were given. The Sobolev space is commonly used to carry out the solution of PDEs due to its properties. Adopting the stochastic Sobolev space [2,7,8], the problems governed by a random or stochastic elliptic equations have been solved and it is still ongoing [1,2,4,9]. The motivations for solving these control problems subject to stochastic partial differential equations SPDEs are very huge ; and mostly we may find difficult systems with noises and Dirichlet, Neumann boundary condition [9,10].
The connection between the study of stochastic optimal control problems and the Hamiltonian Jacobi Bellman (HJB) discussed in [9] allows one to obtain the existence and uniqueness of the controllers but there is certain problem where the noise being present on the boundary can sometimes make the problem very complicated to be solved.
In [11], the existence and uniqueness for non cooperative stochastic elliptic equation with constraints were investigated where the set that characterizes the boundary control was given. For the system which involves higher order elliptic operator, the optimality conditions are derived in [12].

This paper is organized such that in section 2 we consider a stochastic optimal control problem governed by an elliptic equation with noise and Dirichlet condition with the state process being regular. We then proved the existence and uniqueness of stochastic optimal control and derived the necessary and sufficient condition in section 3. In section 4, a conclusion about this study is given.

2 Preliminaries and the control problem

2.1 Notations and functional spaces

Let K be an open, convex and bounded domain in $\mathbb{R}^n (n = 2, 3)$ with boundary $\partial K \subset \mathbb{R}^n$ and $\mathcal{B}(K)$ be the Borel σ -field generated by the open subset of K. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where Ω is the sample space, $\mathcal{F}$ is a σ -field and $\mathbb{P}$ the probability measure.
Let us introduce the stochastic Hilbert space $L^2(\Omega; L^2(K))$ equiped with the norm :

$$\|u\|_{L^2(\Omega;L^2(K))} = \left(\int_\Omega \int_K u^2 dx d\mathbb{P}\right)^{\frac{1}{2}} \tag{1}$$

where u is a stochastic function defined on $K \times \Omega$. In the same way, we consider the following stochastic Hilbert space : $L^2(\Omega; L^2(\partial K))$ and $L^2(\Omega; H^r(K))$ with $r \geq 0$ [2,7,8].

Let us consider the following stochastic elliptic equation:

$$\begin{cases} -\Delta y(x) & = qW(x) + \psi(x), \quad x \in K \\ \\ y(x) & = 0, \quad x \in \partial K \end{cases} \tag{2}$$

where $q \in \mathbb{R}^+$, $y(x) = y(x, \xi) \in L^2(\Omega; H_0^1(K))$ is the state of variable process, $(x, \xi) \in K \times \Omega$, and $W(x)$ is a Wiener process and ψ a given function.

For the weak formulation of the stochastic elliptic equation we use the following notation in [8].

$$a(y, z) = \int_\Omega \left(\int_K \nabla y(x) \nabla z(x) dx\right) d\mathbb{P} \tag{3}$$

and

$$l(z) = \int_\Omega \left(\int_K qW(x)z(x)dx\right) d\mathbb{P} + \int_\Omega \left(\int_K \psi(x)z(x)dx\right) d\mathbb{P} \tag{4}$$

Proposition 2.1 *The solution of equation (2) exists and is unique.*

Proof

Take $z \in L^2(\Omega; H_0^1(K))$. By Green's formula for stochastic elliptic equations we get :

$$\mathbb{E}\left[\int_K \nabla y(x) \nabla z(x) dx\right] = \mathbb{E}\left[\int_K qW(x)z(x)dx\right] + \mathbb{E}\left[\int_K \psi(x)z(x)dx\right] \tag{5}$$

where $\mathbb{E}$ is the expected value.

The variational problem associated to the equation is to find $y \in L^2(\Omega; H_0^1(K))$ such that $a(y, z) = l(z)$. Then we have

$$a(y, z) = \mathbb{E}\left[\int_K \nabla y(x) \nabla z(x) dx\right] \tag{6}$$

$$L(z) = \mathbb{E}\left[\int_K qW(x)z(x)dx\right] + \mathbb{E}\left[\int_K \psi(x)z(x)dx\right] \tag{7}$$

We obtain :

$$\beta\|y\|^2_{L^2(\Omega;H_0^1(K)} \leq |a(y, y)| \quad \text{and} \quad |a(y, z)| \leq \|y\|_{L^2(\Omega;H_0^1(K))}\|z\|_{L^2(\Omega;H_0^1(K))} \tag{8}$$

$$|l(z)| \leq \alpha\|z\|_{L^2(\Omega;H_0^1(K))} \tag{9}$$

where α and β are positive constants. Using Lax-Milgram theorem, from inequalities (8) and (9), we deduce that the state process exists and is unique.

2.2 The stochastic optimal control problem

Let us consider the following control problem :

$$\inf_{u\in\mathcal{U}} J(u) = \inf_{u\in\mathcal{U}} \mathbb{E}\left[\int_{\partial K}(\frac{\partial y}{\partial \eta}(u) - \zeta_d)^2 d(\partial K) + \gamma \int_K u^2 dx\right] \tag{10}$$

subject to :

$$\begin{cases} -\Delta y(x,\xi) & = qW(x) + \psi(x) + \lambda u(x,\xi), \quad x \in K, \quad \xi \in \Omega \\ \\ y(x,\xi) & = 0, \quad x \in \partial K, \quad \xi \in \Omega \end{cases} \tag{11}$$

where $\frac{\partial}{\partial \eta}$ denotes the normal derivative, with η the outwards unit normal vector from K, ζ_d is a given element in $L^2(\Omega; H^{\frac{1}{2}}(\partial K))$, $\gamma \in \mathbb{R}_+^*$. The operator $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. J is the cost function, $y : K \times \Omega \longrightarrow \mathbb{R}$ is the state process with $\frac{\partial y}{\partial \eta} \in L^2(\Omega; H^{\frac{1}{2}}(\partial K))$, q is a positive constant, $W : K \longrightarrow \mathbb{R}$ is a wiener process in the probability space $(\Omega, \mathcal{F}, \mathbb{R})$, ψ is a given function, $\lambda \in \mathbb{R}_+^*$, $u : K \times \Omega \longrightarrow \mathbb{R}$ is a stochastic control, $\mathcal{U}$ is a convex set of admissible control given by :

$$\mathcal{U} = \left\{u \in L^2(\Omega; L^2(K)) : u_1(x) \leq u(x) \leq u_2(x) \quad \forall x \in K\right\} \tag{12}$$

where u_1 and u_2 belong to $L^2(\Omega; L^2(K))$

In the sequel, we assume that the state process y belongs to $L^2(\Omega; H^2(K))$ as it is for the deterministic case in [6] so that $\frac{\partial y}{\partial \eta} \in L^2(\Omega; H^{\frac{1}{2}}(\partial K))$.

3 Results and Discussion

Let us consider the problem (10) subject to the system (11). We are going to prove the existence and uniqueness of the stochastic optimal control and to characterize the optimality conditions.
The cost function

$$J(u) = \mathbb{E}\left[\int_{\partial K}(\frac{\partial y}{\partial \eta}(u) - \zeta_d)^2 d(\partial K) + \gamma \int_K u^2 dx\right] \tag{13}$$

can be rewritten as follows :

$$J(v) = \Pi(v,v) - 2L(v) + \mathbb{E}\left[\int_{\partial K}(\phi(0) - \zeta_d)^2 d\sigma\right] \tag{14}$$

where $\sigma \in \partial K$, $\phi = \frac{\partial y}{\partial \eta}$ and

$$\Pi(u,v) = \mathbb{E}\left[\int_{\partial K}(\phi(u)-\phi(0))(\phi(v)-\phi(0))d\sigma + \int_K \gamma uv dx\right] \tag{15}$$

$$L(v) = \mathbb{E}\left[\int_{\partial K}(\zeta_d-\phi(0))(\phi(v)-\phi(0))d\sigma\right] \tag{16}$$

Remark 3.1 $\Pi(v,v)$ *is bilinear continuous on* $\mathcal{U}\times\mathcal{U}$ *and* $\mathcal{U}$*-coercive. Indeed,* $\Pi(v,v) \geq \gamma\|v\|^2_{L^2(\Omega;L^2(K))}$.
$L(v)$ *is linear continuous on* $\mathcal{U}$*. Indeed,* $L(v) \leq r_0\|v\|_{\mathcal{U}}$*, where* $r_0 \in \mathbb{R}^*_+$
J *is strictly convex. In fact* Π *is strictly convex.* J *is coercive that is*
$J(v) \geq \gamma\|v\|^2_{L^2(\Omega;L^2(K))} - r_0\|v\|_{\mathcal{U}} + \mathbb{E}\left[\int_{\partial K}(\zeta_d-\phi(0))^2 d\sigma\right]$

Theorem 3.2 *The stochastic control problem (10) has a unique solution.*

Proof
Consider the cost functional :

$$J(v) = \Pi(v,v) - 2L(v) + \mathbb{E}\left[\int_{\partial K}(\phi(0)-\zeta_d)^2 d\sigma\right] \tag{17}$$

Since $\mathbb{E}\left[\int_{\partial K}(\phi(0)-\zeta_d)^2 d\sigma\right] \geq 0$, from remark 3.1 and (11) there is a unique optimal control $u \in \mathcal{U}$ such that $J(u) = \inf_{v\in\mathcal{U}} J(v)$ according to Theorem 1.2 in [6].

Optimality conditions

We know that J is strictly convex. The map $v \mapsto \Pi(v,v)$ is continuous (Strong) then strongly lower semicontinuous (l.s.c.)[13]. Since it is convex it is also weakly l.s.c. Therefore J is l.s.c.

Lemma 3.3 *The cost function (13) is Gâteaux differentiable on* $\mathcal{U}$ *and* $\forall v \in \mathcal{U}$,

$$\nabla J(u).(v-u) = 2\mathbb{E}\left[\int_{\partial K}(\phi(v)-\phi(u))(\phi(u)-\zeta_d)d\sigma + \int_K \gamma u(v-u)dx\right] \tag{18}$$

Proof
It is well known that the operator $A = -\Delta$ is an isomorphism of $H_0^1(K)$ to $H^{-1}(K)$ [13,14], where $H^{-1}(K)$ is the dual space of $H_0^1(K)$. Therefore we have :

$$y(v) - y(u) = A^{-1}\lambda(v-u) \tag{19}$$

Now, let u and w be in $\mathcal{U}$. We compute the following :

$$lim_{h\to 0^+}\frac{J(u+hw)-J(u)}{h} \tag{20}$$

$$\begin{aligned}
J(u+hw) &= \mathbb{E}\left[\int_{\partial K}(\phi(u+hw)-\zeta_d)^2 d\sigma + \int_K \gamma(u+hw)^2 dx\right]\\
&= \mathbb{E}\left[\int_{\partial K}(\phi(u)+h\frac{\partial}{\partial\eta}(\lambda A^{-1}w)-\zeta_d)^2 d\sigma + \int_K \gamma(u^2+2huw+h^2w^2)dx\right]\\
&= \mathbb{E}\left[\int_{\partial K}(\phi(u)+h\frac{\partial}{\partial\eta}(\lambda A^{-1}w)-\zeta_d)^2 d\sigma\right] + \mathbb{E}\left[\int_K \gamma(u^2+2huw+h^2w^2)dx\right]
\end{aligned}$$

$$J(u) = \mathbb{E}\left[\int_{\partial K}(\phi(u)-\zeta_d)^2 d\sigma + \int_K \gamma u^2 dx\right] \tag{21}$$

Then from above equalities, as $h \to 0^+$, $\forall u, w \in \mathcal{U}$

$$J'(u).w = 2\mathbb{E}\left[\int_{\partial K}\frac{\partial}{\partial\eta}(\lambda A^{-1}w)(\phi(u)-\zeta_d)dx + \int_K \gamma(uw)dx\right] \tag{22}$$

By setting $w = v - u$, we get :

$$J'(u).(v-u) = 2\mathbb{E}\left[\int_{\partial K}(\phi(v)-\phi(u))(\phi(u)-\zeta_d)d\sigma + \int_K \gamma u(v-u)dx\right] \tag{23}$$

Thus, J is Gâteaux differentiable and then we have :

$$\mathbb{E}\left[\int_{\partial K}(\phi(v)-\phi(u))(\phi(u)-\zeta_d)d\sigma + \int_K \gamma u(v-u)dx\right] \geq 0 \quad \forall u \in \mathcal{U} \tag{24}$$

Remark 3.4 *u is an optimal control if and only if*

$$\mathbb{E}\left[\int_{\partial K}(\phi(v)-\phi(u))(\phi(u)-\zeta_d)d\sigma + \int_K \gamma u(v-u)dx\right] \geq 0 \quad \forall u \in \mathcal{U} \tag{25}$$

Theorem 3.5 *Consider the state process defined in (11) and the cost function defined in (13), the following is a necessary and sufficient condition for u to be an optimal control :*

$$\begin{cases}
-\Delta y(u) = qW + \psi + \lambda u\\
\Delta\rho(u) = 0, \qquad \rho|_{\partial K} = -(\phi(u)-\zeta_d)\\
\mathbb{E}\left[\int_K(\lambda\rho+\gamma u)(v-u)dx\right] \geq 0, \quad \forall v \in \mathcal{U}
\end{cases} \tag{26}$$

Proof
By introducing the adjoint state ρ such that :

$$\begin{cases} \Delta\rho = 0 \\ \\ \rho|_{\partial K} = -(\phi(u) - \zeta_d) \end{cases} \tag{27}$$

as it is well justified in [6]. By using Green's Formula, we get the following equality:

$$\mathbb{E}\left[\int_K \rho(u)\Delta\kappa dx\right] = \mathbb{E}\left[\int_{\partial K} (\phi(u) - \zeta_d)\frac{\partial\kappa}{\partial\eta}d\sigma\right] \quad \forall\kappa \in L^2(\Omega; H^2(K) \cap H_0^1(K)) \tag{28}$$

Therefore, replacing κ by $y(v) - y(u)$, we obtain:

$$\mathbb{E}\left[\int_K \lambda\rho(u)(v - u)dx\right] = \mathbb{E}\left[\int_{\partial K} (\phi(v) - \phi(u))(\phi(u) - \zeta_d)d\sigma\right] \tag{29}$$

Accordingly, from (25) we obtain the following inequality :

$$\mathbb{E}\left[\int_K (\lambda\rho(u) + \gamma u)(v - u)dx\right] \geq 0 \quad \forall u \in \mathcal{U} \tag{30}$$

Remark 3.6 *In the case where there is no constraint on the space control, we obtain :*

$$\lambda\rho + \gamma u = 0 \tag{31}$$

Then we get :

$$u = -\lambda\frac{\rho}{\gamma} \tag{32}$$

Therefore, we obtain the following system:

$$\begin{cases} -\Delta y + \lambda^2\dfrac{\rho}{\gamma} = qW + \psi \quad \textbf{\textit{in}} \quad K, \quad y = 0 \quad \textbf{\textit{on}} \quad \partial K \\ \\ \Delta\rho = 0, \qquad \rho|_{\partial K} = -(\phi(u) - \zeta_d) \end{cases} \tag{33}$$

Discussion

The optimal control is determined in the case where there is no constraint but it can be also determined in the case of positive cone as well. The fact that the state process is regular enables to choose the observation function on the stochastic fractional Sobolev space. However it is not generally true that the solution of the control problem is regular when the state is regular. It must have in addition other conditions for it.

4 Conclusion

In this paper, we treated a control problem governed by stochastic elliptic equations which involves stochastic Hilbert spaces. The solution of the considered control problem exists and is unique. It is expressed in terms of the adjoint state in the case without any constraint.

Acknowledgements. This research is supported by the African Union Commission under the Pan African University Program (PAUISTI).

References

[1] Ge, Liang and Sun, Tongjun, A sparse grid stochastic collocation and finite volume element method for constrained optimal control problem governed by random elliptic equations, *Journal of Computational Mathematics,* **36** (2018), 310 - 330.
https://doi.org/10.4208/jcm.1703-m2016-0692

[2] Du, Ning and Shen, Wanfang, A fast stochastic galerkin method for a constrained optimal control problem governed by a random fractional diffusion equation, *Journal of Computational Mathematics,* **36** (2018), 259-275. https://doi.org/10.4208/jcm.1612-m2016-0696

[3] Yong, Jiongmin and Zhou, Xun Yu, *Stochastic controls: Hamiltonian systems and HJB equations*, Springer Science & Business Media, Vol. 43, 1999.

[4] Shen, Wanfang and Ge, Liang, On effective stochastic galerkin finite element method for stochastic optimal control governed by integral-differential equations with random coefficients, *Journal of Computational Mathematics,* **36** (2018), 183 - 201.
https://doi.org/10.4208/jcm.1611-m2016-0676

[5] Hou, L.S., Lee, J. and Manouzi, H., Finite element approximations of stochastic optimal control problems constrained by stochastic elliptic PDEs, *Journal of Mathematical Analysis and Applications,* **384** (2011), 87 - 103. https://doi.org/10.1016/j.jmaa.2010.07.036

[6] Lions, Jacques Louis, *Optimal Control of Systems Governed by Partial Differential Equations (Grundlehren der Mathematischen Wissenschaften)*, Springer, Berlin, Vol. 170, 1971.
https://doi.org/10.1007/978-3-642-65024-6

[7] Lee, Hyung-Chun and Lee, Jangwoon, A stochastic Galerkin method for stochastic control problems, *Communications in Computational Physics,* **14** (2013), 77 - 106.

[8] Chen, Peng and Quarteroni, Alfio and Rozza, Gianluigi, Stochastic optimal Robin boundary control problems of advection-dominated elliptic equations, *SIAM Journal on Numerical Analysis,* **51** (2013), 2700 - 2722. https://doi.org/10.1137/120884158

[9] Fabbri, Giorgio and Gozzi, Fausto and Swiech, Andrzej, *Stochastic Optimal Control in Infinite Dimension, Probability and Stochastic Modelling*, Springer, 2017. https://doi.org/10.1007/978-3-319-53067-3

[10] Boutselis, George I and Pereira, Marcus A and Evans, Ethan N and Theodorou, Evangelos A, Variational Optimization for Distributed and Boundary Control of Stochastic Fields, *arXiv preprint arXiv:1904.02274,* 2019. https://arxiv.org/abs/1904.02274

[11] El Bab, AS Okb and Hyder, Abd-Allah and Abdallah, AM, Boundary Control of Stochastic Elliptic Systems with Constraints, *MAYFEB Journal of Mathematics,* **3** (2017).

[12] Abdallah, Ahmed Mohammed Abdel Aal, On Stochastic Elliptic System Involving Higher Order Operator with Dirichlet Conditions, *AMO-Advanced Modeling and Optimization,* **20** (2018), 81 - 88.

[13] Barbu, Viorel, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer Science & Business Media, 2010. https://doi.org/10.1007/978-1-4419-5542-5

[14] Agranovich, Mikhail S, Sobolev Spaces, *Sobolev Spaces, Their Generalizations and Elliptic Problems in Smooth and Lipschitz Domains*, Springer, 2015. https://doi.org/10.1007/978-3-319-14648-5

Received: July 9, 2019; Published: August 17, 2019