The Fifth-Moment Effect in Black-Scholes Model

and Its Performance at Market

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This article investigates the fifth moment (super skewness) adjusted Black-Scholes (BS) model of Corrado and Su (1996). The model in this paper is also inspired by the Gram-Charlier expansion to higher-moments adjustment on the Black-Scholes formula. Meanwhile, the approximation method which is used in this research is Hermite polynomial approach. The prominent finding of this research is that we have a general formula for European call option price using the fifth moment. Besides, based on case studies conducted in this research, it can be concluded that there is an evidence that super skewness BS model gives smaller error among methods if the log return data have significant fifth moment.

Keywords: Black Scholes, Super skewness, Option

1 Introduction

Black and Scholes (BS) (1973) developed a European-type option pricing model based on normal distributed stock returns. They gave their famous formula BS stated in equation (1) as follows

\[ C_{BS} = S_0 N(d_1) - Ke^{-rT} N(d_2) \]  

(1)

Here \( d_1 = \frac{\ln \frac{S_0}{K} + (r + \frac{1}{2} \sigma^2)T}{\sigma \sqrt{T}} \); \( d_2 = d_1 - \sigma \sqrt{T} \). Then, in their paper, Corrado and Su (1996) carried on research that succeeded to explain that the general formula for BS model can be generalized. This generalization is expressed by the following formula
\[ C_{BS} = e^{-rT} \int_{K}^{\infty} (S_T - K) g(S_T) dS_T \]
\[ = e^{-rT} \int_{-\infty}^{\infty} S_0 e^{(-0.5\sigma^2)t + \sigma \int_0^t r - K} n(z) dz \]  

(2)

where \( n(z) \) is normal standard density function.

According to several previous research, it has been recognized that asset return distributions tend to be leptokurtic. This condition was strengthened by some evidence in the market that the BS model’s option price valuation deviated in predicting market prices. A study conducted by Jarrow and Rudd (1982) showed that a given probability distribution could be approximated by the expansion of distributions of two or more moments. Another is that, in options trading, the exchanges often limit the movement of daily price changes of an asset. As a result, asset returns are not perfectly normal distributed.

Moreover, the inclusion of higher moments is considered as an essential aspect to get a better level of precision, in the condition that not all data is a perfectly normal distribution. Corrado and Su (1996) developed an option pricing formula using the Gram-Charlier expansion. Meanwhile, Hung et al. (2015) discussed options pricing with several models such as the Black-Scholes and the Gram-Charlier expansion. Based on the facts mentioned above, in this paper we focuses on the formula of option price European BS model using the fifth moment, super skewness. At the next section, the new proposed model will be symbolized by BSs.

2. Gram-Charlier Expansion

Hermite Polynomial was defined by Laplace in 1810 and studied in detail by Chebyshev in 1859, was ignored and only recognized after Charles Hermite wrote this polynomial in 1864. The form of this polynomial is written at equation (3) as follows

\[ H_n(z) = \sum_{k=0}^{n/2} \frac{(-1)^k n! 2k}{2^k k!} z^{n-2k} \]  

(3)

Then, from equation (3), we can get \( H_0(z) = 1, H_1(z) = z, H_2(z) = (z^2 - 1), ..., H_4(z) = (z^4 - 6z^2 + 3), ... \). This polynomial has orthogonal properties:

\[ \int_{-\infty}^{\infty} H_m(z) H_n(z) n(z) dz = \begin{cases} 0 & , m \neq n \\ m! & , m = n \end{cases} \]  

(4)

Gram-Charlier expansion is a very good to estimate normal standard density function. In the last two decades, the use of this expansion has been introduced in the field of finance for leptokurtic return model, skewness, group volatility and so on. On the other hand, Hermite polynomials can be used to obtain expansion of probability functions in a series of derivatives \( n(z) \).
Hermite polynomials has several advantages. One of the benefits is that there is the fact that the density function can be formally expanded as:

\[ g(z) = \sum_{n=0}^{\infty} c_n H_n(z) n(z) \quad (5) \]

Here \( H_n(z) \) in equation (5) is the Hermite-order nth polynomial of equation (3). Meanwhile, the coefficient \( c_n \) of (4) is derived from the Hermite Polynomial. If two sides of equation (5) are both multiplied by \( H_m(z) \), and it is integrated from \(-\infty \) to \( \infty \), it can be derived in the following results:

\[
\int_{-\infty}^{\infty} g(z) H_m(z) dz = \sum_{n=0}^{\infty} c_n \int_{-\infty}^{\infty} H_n(z) n(z) dz = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} c_n H_n(z) n(z) dz
\]

\[
= c_0 \int_{-\infty}^{\infty} H_0(z) n(z) dz + \sum_{n=1}^{\infty} c_n \left[ \int_{-\infty}^{\infty} H_n(z) n(z) dz \right] + \ldots + c_{m-1} \int_{-\infty}^{\infty} H_{m-1}(z) n(z) dz + c_m \int_{-\infty}^{\infty} H_m(z) n(z) dz + \ldots
\]

(6)

Then, using the orthogonal properties on equation (4) to equation (6), we have:

\[
c_m = \frac{1}{m!} \int_{-\infty}^{\infty} g(z) H_m(z) dz
\]

(7)

Next, from equation (7) we get the values : \( c_0 = 1 \), \( c_1 = \mu_1 \), \( c_2 = (1/2!) (\mu_2 - 1) \), \( c_3 = (1/3!) (\mu_3 - 3\mu_1) \), \( c_4 = (1/4!) (\mu_4 - 6\mu_2 + 3) \), \( c_5 = (1/5!) (\mu_5 - 10\mu_3 + 15\mu_1) \) with \( \mu_3 \), \( \mu_4 \), \( \mu_5 \) are skewness, kurtosis, and super skewness respectively. Furthermore, the expansion for normal standard pdf until fifth moment is:

\[
g(z) = \sum_{n=0}^{5} c_n H_n(z) n(z)
\]

\[
g(z) = n(z) \left[ \begin{array}{c}
H_0(z) + \mu_1 H_1(z) + \frac{1}{2!} (\mu_2 - 1) H_2(z) + \\
\frac{1}{3!} (\mu_3 - 3\mu_1) H_3(z) + \frac{1}{4!} (\mu_4 - 6\mu_2 + 3) H_4(z) + \\
\frac{1}{5!} (\mu_5 - 10\mu_3 + 15\mu_1) H_5(z)
\end{array} \right]_{g(z)}^{s(z)}
\]

(8)

Then, after substituting equation (8) to equation (2) give the option pricing formula based on super skewness moment will be obtained as follows
\[ C_{BS} = e^{-rT} \int_{-\infty}^{\infty} \left( S_T - K \right) g(z) dz \]
\[ = e^{-rT} \int_{-\infty}^{\infty} \left( S_0 e^{(r-0.5\sigma^2)T + \sigma \sqrt{T} - K} \right) \left[ \phi(z) + \frac{S_0}{K} \Phi(z) \right] dz \]
\[ = C_{GC4} + C_{GC5} \]  

\( C_{GC5} \) on equation (9) is the formula of Corrado and Su (1996),
\[ C_{GC5} = C_{BS} + \mu_3 Q_3 + (\mu_4 - 3)Q_3 + (\mu_5 - 10\mu_3)Q_5 \]  

where
\[ Q_3 = \frac{1}{3!} S_0 \sigma \sqrt{T} \left( n(d_1) \left( \frac{\mu_3}{\sigma} \sqrt{T} - d_1 \right) + \sigma^2 T N(d_1) \right) \]
\[ Q_5 = \frac{1}{5!} S_0 \sigma \sqrt{T} \left( n(d_1) \left( d_1^3 - 3\sigma \sqrt{T} \left( d_1 - \sigma \sqrt{T} \right) - 1 \right) + \left( \sigma \sqrt{T} \right)^5 N(d_1) \right) \]  

Next, we focus on the second part of equation (9), \( C_{GC5} \), the main of our work. Use partial integral \( u = S_0 e^{(r-0.5\sigma^2)T + \sigma \sqrt{T} - K} \) and \( v = \int \frac{d}{dz} H_4(z) n(z) dz \), give
\[ C_{GC5} = e^{-rT} \int_{-\infty}^{\infty} \left( S_0 e^{(r-0.5\sigma^2)T + \sigma \sqrt{T} - K} \right) \left[ \left( \frac{\mu_5 - 10\mu_3}{5!} \right) H_5(z) n(z) dz \right] \]
\[ = \left( \frac{\mu_5 - 10\mu_3}{5!} \right) \left[ S_0 \left( \sigma \sqrt{T} \right)^5 N(d_1) + S_0 \sum_{j=2}^{5} \left( \sigma \sqrt{T} \right)^{j-1} H_5,-j \left( -d_2 \right) n(d_1) \right] \]  

Finally, after several steps conducted previously, the formula for call option price using the fifth moment is
\[ C_{BS} = C_{BS} + \mu_3 Q_3 + (\mu_4 - 3)Q_3 + (\mu_5 - 10\mu_3)Q_5 \]  

where
\[ Q_5 = \frac{1}{120} S_0 \sigma \sqrt{T} \]
\[ \left( \left( \sigma^4 T^2 N(d_1) + n(d_1) \left[ 4 \sigma^5 T^{3/2} - 6d_1 \sigma^3 T + 3d_1^3 \sigma \sqrt{T} + d_1 \sigma \sqrt{T} - 3\sigma \sqrt{T} - d_1^3 + 3d_1 \right] \right) \right) \]  

Furthermore, from the option pricing formula in equation (11), we can conclude two noticeable results as follows:

1. The option price formula in equation (11) is an extension of the BS option pricing formula up to fifth moment.

2. Theoretically, if the data is normally distributed (skewness = 0, kurtosis = 3 and the super skewness = 0), then the option price formula in equation (11) equals the B-S’s formula.
The fifth-moment effect in Black-Scholes model

3. Case Study

The data used for the case study is the daily close price of Google shares, Inc. (GOOG) daily from December 19, 2015 to December 16, 2016. The data were obtained through the website http://finance.yahoo.com/. The summaries of data utilized in this research is shown in Table 1 as follows:

<table>
<thead>
<tr>
<th>Table 1: Summary of Google Inc. stock data (GOOG)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S₀</td>
</tr>
<tr>
<td>Volatility</td>
</tr>
<tr>
<td>Variance</td>
</tr>
<tr>
<td>Mean</td>
</tr>
<tr>
<td>Standard Deviation</td>
</tr>
<tr>
<td>Skewness</td>
</tr>
<tr>
<td>Kurtosis</td>
</tr>
<tr>
<td>Super skewness</td>
</tr>
<tr>
<td>Interest rate (r)</td>
</tr>
<tr>
<td>Time to maturity T</td>
</tr>
</tbody>
</table>

According to information in Table 1, the prices of call option from the three different models can be calculated by using different equation written previously. The price of call option BS could be obtained by using formulas in equation (1) while Gram-Charlier expansion model could be derived by utilization of equation (8). On the other hand, based on Table 1, super skewness BS model could be obtained by using equation (10).

The call option prices of the three different models at several contract prices to data Inc. shares (GOOG) is shown in Table 2:

<p>| Table 2: European call option price of B-S model and Gram-Charlier Expansion |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|</p>
<table>
<thead>
<tr>
<th>K</th>
<th>C-Market (Price)</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
<th>C-BS (Price)</th>
<th>C-GC (Price)</th>
<th>C-BSs (Price)</th>
</tr>
</thead>
<tbody>
<tr>
<td>650</td>
<td>140.94</td>
<td>1.72E-03</td>
<td>1.01E-05</td>
<td>4.75E-08</td>
<td>144.376</td>
<td>144.372</td>
<td>144.372</td>
</tr>
<tr>
<td>690</td>
<td>92.28</td>
<td>1.72E-03</td>
<td>1.03E-05</td>
<td>-1.29E-07</td>
<td>104.398</td>
<td>104.394</td>
<td>104.394</td>
</tr>
<tr>
<td>700</td>
<td>90</td>
<td>1.72E-03</td>
<td>1.42E-05</td>
<td>-4.05E-06</td>
<td>94.403</td>
<td>94.399</td>
<td>94.400</td>
</tr>
<tr>
<td>730</td>
<td>55.93</td>
<td>-4.80E-03</td>
<td>5.40E-03</td>
<td>-3.23E-03</td>
<td>64.421</td>
<td>64.532</td>
<td>65.134</td>
</tr>
<tr>
<td>742.5</td>
<td>50.3</td>
<td>-5.26E-02</td>
<td>3.42E-02</td>
<td>-1.42E-02</td>
<td>51.937</td>
<td>52.704</td>
<td>55.350</td>
</tr>
<tr>
<td>750</td>
<td>41.01</td>
<td>-7.55E-02</td>
<td>4.54E-02</td>
<td>-1.71E-02</td>
<td>49.445</td>
<td>50.479</td>
<td>53.661</td>
</tr>
<tr>
<td>745</td>
<td>40.55</td>
<td>-1.43E-01</td>
<td>7.33E-02</td>
<td>-2.14E-02</td>
<td>44.473</td>
<td>46.199</td>
<td>50.177</td>
</tr>
<tr>
<td>767.5</td>
<td>23.56</td>
<td>-5.93E-01</td>
<td>1.13E-01</td>
<td>2.73E-02</td>
<td>27.518</td>
<td>31.177</td>
<td>26.101</td>
</tr>
<tr>
<td>770</td>
<td>20.5</td>
<td>-6.52E-01</td>
<td>8.92E-02</td>
<td>4.27E-02</td>
<td>25.226</td>
<td>28.610</td>
<td>20.649</td>
</tr>
<tr>
<td>795</td>
<td>2.42</td>
<td>8.08E-02</td>
<td>-3.09E-01</td>
<td>-7.84E-03</td>
<td>7.180</td>
<td>1.307</td>
<td>2.768</td>
</tr>
</tbody>
</table>

SRPE (Squared Relative Price Error) | 0.05683 | 0.095148 | 0.016292 |
Based on Table 2, it can be easy to compare the error performance of the three models. It can be seen from table 2, it can be seen that the SRPE (Squared Relative Price Error) of super skewness BS is smaller than Gram-Charlier and BS models. So there is an evidence that super skewness BS model is better among the others when the log return data have a significant fifth moment.

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References


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