Duality in Infinite-Dimensional Production Economies

Christos E. Kountzakis
Dept. of Mathematics
University of the Aegean
Karlovassi, GR-83 200 Samos, Greece

Panagiotis G. Michaelides
School of Sciences
National Thechnical University of Athens
Zografou Campus, 15780 Athens, Greece

Abstract
In this paper, we present some useful mathematical propositions, which generalize previous results on duality between profit maximization and cost minimization in finite-dimensional production economies. These results are related to the so-called Shephard’s and Hotelling’s Lemmas and our main theorem is an extension of this duality under the frame of symmetric Riesz pairs of commodities and prices.

Mathematics Subject Classifications: 91B38; 52A41; 90C25

Keywords: Production -Cost functions’ Duality; Symmetric Riesz pairs

1 Introduction
The famous Hotelling and Shephard Lemmas -see in Hotelling (1932), Varian (1992), Par. 3.2 and 5.4 respectively, have always occupied a central position among the basic tools of mathematical economics and economic theory. Another reference for these Lemmas is in Mas-Colell et al. (1995) in Pr. 5.C.1 and 5.C.2, respectively. Let us recall the statements of these two Lemmas.
(i) (Hotelling’s Lemma) Let \( y_i(p) \) be the firm’s net supply function for the good \( i \). Then
\[
y_i(p) = \frac{\partial \pi(p)}{\partial p_i},
\]
if \( \pi(p) = p \cdot y(p) \) is the profit function.

(ii) (Shephard’s Lemma) Let \( x_i(w, y) \) the conditional factor demand of the firm for the \( i \)-input. If the cost function of the firm’s is differentiable at \( (w, y) \) and \( w_i > 0, i = 1, 2, ..., n \), then
\[
x_i(w, y) = \frac{\partial c(w, y)}{\partial w_i}, i = 1, 2, ..., n.
\]

In this paper, we extend these results for infinite-dimensional production economies. For this reason, we use the framework of Riesz spaces and moreover for Banach lattices. Let us give some notions about ordered linear spaces, which are also valid for Riesz spaces.

Let \( E \) be a (normed) linear space. A set \( C \subseteq E \) satisfying \( C + C \subseteq C \) and \( \lambda C \subseteq C \) for any \( \lambda \in \mathbb{R}_+ \) is called wedge. A wedge for which \( C \cap (-C) = \{ 0 \} \) is called cone. A pair \( (E, \geq) \) where \( E \) is a linear space and \( \geq \) is a binary relation on \( E \) satisfying the following properties:

(i) \( x \geq x \) for any \( x \in E \) (reflexive) item[(ii)] if \( x \geq y \), and \( y \geq x \), then \( x = y \), \( (x, y \in E) \) (antisymmetric)

(iii) If \( x \geq y \) and \( y \geq z \) then \( x \geq z \), where \( x, y, z \in E \) (transitive)

(iv) If \( x \geq y \) then \( \lambda x \geq \lambda y \) for any \( \lambda \in \mathbb{R}_+ \) and \( x + z \geq y + z \) for any \( z \in E \), where \( x, y \in E \) (compatible with the linear structure of \( E \)),

is called partially ordered linear space.

The binary relation \( \geq \) in this case is a partial ordering on \( E \). The set \( P = \{ x \in E | x \geq 0 \} \) is called (positive) wedge of the partial ordering \( \geq \) of \( E \). Given a wedge \( C \) in \( E \), the binary relation \( \geq C \) defined as follows:
\[
x \geq_C y \iff x - y \in C,
\]
is a partial ordering on \( E \), called partial ordering induced by \( C \) on \( E \). If the partial ordering \( \geq \) of the space \( E \) is antisymmetric, then the positive wedge \( P \) is a cone. If \( E \) is partially ordered by \( C \), then any set of the form \( [x, y] = \{ r \in E | y \geq_C r \geq_C x \} \) where \( x, y \in C \) is called order-interval of \( E \). \( E' \) denotes the linear space of all linear functionals of \( E \), while \( E^* \) is the norm dual of \( E^* \), in case where \( E \) is a normed linear space.

The partially ordered vector space \( E \) is a vector lattice or a Riesz space if for any \( x, y \in E \), the supremum and the infimum of \( \{ x, y \} \) with respect to the
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Partial ordering defined by $P$ exist in $E$. In this case $\sup \{x, y\}$ and $\inf \{x, y\}$ are denoted by $x \lor y$, $x \land y$ respectively. If so, $|x| = \sup \{x, -x\}$ is the absolute value of $x$. The $\sup \{x, y\}$ is the least element of $\{z \in E | z \geq x, y\}$, while $\inf \{x, y\}$ is the upper element of $\{w \in E | x, y \geq w\}$ in $E$. A Banach lattice is a Banach space, which is also a Riesz space having the property:

$$|x| \geq |y| \Rightarrow \|x\| \geq \|y\|.$$ 

In the paper below, whenever we refer to a Riesz space, we denote a Banach lattice. Hence, the weak topology of it, is the $\sigma(E, E^*)$-topology on $E$, where $E^*$ is the norm-dual of $E$, and the norm-topology is the topology implied by the norm $\|\cdot\|$ that makes $E$ a Banach space. As a reference for Riesz spaces, we mention Luxemburg and Zaanen (1971).

The first reference in which Riesz spaces are mentioned as a frame of study in the modern equilibrium theory is the work of Aliprantis and Brown (1983), while in modern equilibrium theory with production is the work of Zame (1987).

## 2 Framework of the study

Throughout this paper, we make the fairly standard assumptions that:

1. Assumption 1 (Price taking assumption-Competitive behaviour): There is a vector of prices $p$ and these prices are independent of the production plans of the economic units.

2. Assumption 2 (Profit maximization assumption): The economic units’ objective is to maximize profit, while the evaluation $p \cdot y = \langle y, p \rangle$ corresponds to the market price of the production plan $y$ under prices $p$.

3. Assumption 3 (Production Set): We assume that the economic unit’s production set $Y$ satisfies, at least, the so-called properties of non-emptiness, closedness and convexity.

We consider an infinite-dimensional production economy, in which we assume that the Banach lattice $E$ and its norm-dual formulate a symmetric Riesz pair $\langle E, E^* \rangle$. We also consider a convex, weakly closed production set $Y$ such that $Y \cap E_+ = \{0\}$, see also in p.179 in Aliprantis et al. (1990). Examples of symmetric Reisz pairs are (Aliprantis and Border (1999), p.296-97):

1. $\langle L^\infty(\mu), L^1(\mu) \rangle$, when $\mu$ is $\sigma$-finite,

2. $\langle L^1(\mu), L^\infty(\mu) \rangle$, when $\mu$ is $\sigma$-finite,

3. $\langle L^p(\mu), L^q(\mu) \rangle$, when $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, 

4. \((c_0, \ell^1)\)

We also suppose that there is some \(e \in E_+\) which defines a base in \(E^*_+\). We recall that the base \(B_e\) of \(E^*_+\) defined by \(e\) is the set \(\{f \in E^*_+ | f(e) = 1\}\). The reason for which Riesz spaces are selected for production plans’ modelling is that for a \(y \in Y\), \(y = y^+ - y^-\) holds, where \(y^-\) denotes the input of the production and \(y^+\) denotes the output of the production, respectively. The properties of a cost function are equivalent to the ones listed in Def.3 of Michaelides et al. (2015):

1. \(C(p, y) \geq 0, p \in E^*_+ \setminus \{0\}, y \in Y\).
2. \(C(p, y) \geq C(p', y), p > p'(p - p' \in E^*_+ \setminus \{0\}), y \in Y\).
3. \(C(\lambda \cdot p, y) = \lambda C(p, y), \lambda > 0\).
4. \(C(p, y)\) is concave and weakly continuous on \(y\).
5. \(y > y'(y - y' \in E_+ \neq 0), p \in E^*_+ \setminus \{0\}, \) implies \(C(p, y') \leq C(p, y)\).
6. \(p \in E^*_+ \neq 0\) and \(\{y_n\}_{n \in \mathbb{N}} \subset Y\) such that \(\|y_n\| \to \infty\) as \(n \to \infty\), imply \(C(p, y_n) \to \infty\), as \(n \to \infty\).

The commodity-price duality is taken to be given by a symmetric Riesz pair, such that order-intervals of the type \([-b, b]\) in which feasible production lie in, to be weakly compact sets, see also Th.4.2.4 in Aliprantis et al. (1990). We recall the definition of the feasible production set:

**Definition 2.1** We consider a private ownership production economy

\[ E = \{\langle E, E^*\rangle, ((\omega_i, u_i), i = 1, 2, ..., m), \]
\[ (Y_j, j = 1, 2, ..., k), (\theta_{ij}, i = 1, 2, ..., m, j = 1, 2, ..., k)\].

In this economy the Riesz pair \(\langle E, E^*\rangle\) denotes the commodity-price duality, the pairs \((\omega_i, u_i)\) denote the initial endowments \(\omega \in E_+\) both with the utility functions \(u_i : E_+ \to \mathbb{R}_+\) of the individuals \(i = 1, 2, ..., m\), the sets \(Y_j, j = 1, 2, ..., k\) denote the production sets of the \(k\) firms, and \(\theta_{ij} \in [0, 1], i = 1, 2, ..., m, j = 1, 2, ..., k\) denotes the share of the profit of the \(j\)-firm that the \(i\)-consumer owns, according to the Assumptions 1,2,3.

**Definition 2.2** The set of allocations \(A\) of the economy \(E\) is defined as follows:

\[ A = \{(x_1, x_2, ..., x_m, y_1, y_2, ..., y_k) : x_i \in E_+, y_j \in Y_j, \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \omega_i + \sum_{j=1}^{k} y_j\}\]
**Definition 2.3** The convex feasible production set \( \hat{Y}_j \) for the \( j \)-firm is the convex set, such that all its \( j \)-firm’s feasible production plans, namely
\[
\hat{Y}_j = \{ y \in Y_j | \exists (x_1, x_2, ..., x_m, y_1, y_2, ..., y_k) \in A : y_j = y \},
\]
lie in.

In the rest of the paper, by \( Y \) we denote some \( \hat{Y}_j, j = 1, 2, ..., k \) in some economy \( \mathcal{E} \). Under Th.4.2.4 in Aliprantis et al. (1990), if any production set \( Y_j, j = 1, 2, ..., k \) is bounded from above, since \( \langle E, E^* \rangle \) is a symmetric Riesz pair, then \( \hat{Y}_j, j = 1, 2, ..., k, \) is weakly compact.

### 3 Duality between cost minimization - Profit maximization

Consider the profit function associated to a cost function \( C(p, y), p \in B_e, y \in Y \), where \( Y \) denotes a production set, to be the conjugate function with respect to \( y \in Y \):
\[
P(p) = \sup_{y \in Y} \{ p \cdot y - C(p, y) \}, p \in B_e.
\]

\( B_e = \{ p \in E_+^* | p \cdot e = 1 \} \). By \( C(p) \) we denote the optimal value of the problem of cost minimization with respect to \( y \in Y \), under a given price vector, namely:
\[
C(p) = \inf_{y \in Y} C(p, y), p \in B_e.
\]

A salient feature of economic units choosing a profit-maximizing behavior is that cost minimization is a necessary condition for profit maximization. This implies that there is no way of producing the same amount of outputs at a lower total expenditure.

The elements of \( Y \) which achieve \( C(p) \) are denoted by \( C(Y; p) \):
\[
C(Y; p) = \{ y \in Y | C(p, y) = C(p) \}, p \in B_e.
\]

If the economic unit is not a price taker in the market, we cannot use the notion of the profit function.

The optimal value of the profit maximization is denoted by \( P(p) \). The elements of \( Y \) which achieve \( P(p) \) are denoted by \( P(Y; p) \):
\[
P(Y; p) = \{ y \in Y | P(p, y) = P(p) \}, p \in B_e.
\]

**Proposition 3.1** \( C(Y; p) \neq \emptyset, p \in B_e. \)
Proof: Since from the properties of a production set $Y$, consider some $b \in E_+, b \neq 0$ exists, such that $y \leq b$ for any $y \in Y$. Also, from the property that $(-E_+) \subseteq Y$, we get that $-b \in Y$, from Th.4.2.4 (i) of Aliprantis et al. (1990). Hence the set $Y \cap [-b, b]$ is non-empty and weakly compact, since the order interval $[-b, b]$ is a weakly compact set, since $\langle E, E^* \rangle$ is a symmetric Riesz pair, see Theorem 8.56 in Aliprantis and Border (1999). Since the function $C(p, y)$ is weakly continuous with respect to $y$, it takes a minimum value on this subset of $Y$.

**Proposition 3.2** $P(Y; p) \neq \emptyset, p \in B_e$.

Proof: $P(p, y)$ is weakly lower semicontinuous with respect to $y$, as the supremum of weakly continuous functions with respect to $y$. Hence $P(p, y)$ takes a maximum value on the weakly compact set.

**Theorem 3.3** $P(Y; p) = C(Y; p), p \in B_e$. (Duality Theorem)

Proof: We first prove the $' \subseteq'$ part. If $y^* \in P(Y; p)$, then

$$p \cdot y^* - C(p, y^*) = \sup_{y \in Y} \{p \cdot y - C(p, y)\} \leq \sup_{y \in Y} p \cdot y - \inf_{y \in Y} C(p, y).$$

The last inequality is equivalent to

$$p \cdot y^* - C(p, y^*) \leq p \cdot y^* - \inf_{y \in Y} C(p, y),$$

(from the weak compactness of $Y$) which is equivalent to

$$\inf_{y \in Y} C(p, y) \leq C(p, y^*).$$

The last inequality implies $y^* \in C(Y; p)$. By following the inverse way -if we assume that $y^* \in C(Y; p)$, since the inequalities are equivalent- we obtain that $y^* \in P(Y; p)$.

Relying on the same list of properties for the cost functions, the following theorem for the robust representation of cost functions in infinite-dimensional production economies arises:

**Theorem 3.4**

$$C(p, y) = \sup_{y \in Y} \{p \cdot y - P(p)\}, p \in B_e.$$
Proof: We notice that
\[ P(p) \geq p \cdot y - C(p, y), p \in B_e, y \in Y, \]
which implies
\[ C(p, y) \geq \sup_{y \in Y} \{ p \cdot y - P(p) \}, p \in B_e. \]
Suppose that some \( y_0 \in Y, p_0 \in B_e \) exist such that
\[ C(p_0, y_0) > \sup_{y \in Y} \{ p_0 \cdot y - P(p_0) \}. \]
By Duality Theorem
\[ C(p_0, y_0) > \sup_{y \in Y} \{ p_0 \cdot y - \sup_{y \in C(Y; p_0)} \{ p \cdot y - C(p, y) \} \}, \]
which implies
\[ C(y_0, p_0) > \sup_{y \in Y} \{ p_0 \cdot y - \sup_{y \in C(Y; p_0)} \{ p_0 \cdot y + \inf_{y \in C(Y; p_0)} C(p, y) \} \}, \]
namely
\[ C(y_0, p_0) > \sup_{y \in Y} \{ p_0 \cdot y - \sup_{y \in C(Y; p_0)} \{ p_0 \cdot y + C(p_0, y_0) \} \}, \]
a contradiction. Hence, the assumption of the existence of such a \( p_0 \) and such a \( y_0 \) is not valid. This implies that the equality is true for any \( p \in B_e \), hence
\[ C(p, y) = \sup_{y \in Y} \{ p \cdot y - P(p) \}, p \in B_e. \]

By the above theorem, we deduce the following duality theorem for the optimal values:

**Theorem 3.5** Given that \( P(p) = \sup_{y \in Y} \{ p \cdot y - C(p, y) \}, p \in B_e \) we have that
\[ P(p) = \sup_{y \in C(Y; p)} \{ p \cdot y - C(p) \} \iff C(p) = \sup_{y \in P(Y; p)} \{ p \cdot y - P(p) \}. \]

Proof: Suppose that \( C(p) = \rho \). Then \( C(p, y) \geq \rho \), which implies \( p \cdot y - C(p, y) \leq p \cdot y - \rho \), where the equality holds if \( y \in C(Y; p) \). Then \( P(p) = \sup_{y \in C(Y; p)} \{ p \cdot y - C(p) \} \), which implies as above that \( C(p) = \sup_{y \in P(Y; p)} \{ p \cdot y - P(p) \} \).
4 Concluding Remarks

In this paper, we studied the duality between profit and cost functions in completive production economies, under a Riesz commodity-price framework. We actually extended the validity of the well-known Shephard’s and Hotelling’s Lemmas in the case of symmetric Riesz pairs. This allows for both an infinite-dimensional space of commodities and prices. An interesting question of further research is whether the properties of the production set $Y$ may be also retrieved from the above duality results, either for the cost or for the profit function.

References


Received: March 1, 2017; Published: February 11, 2019