Some Results on Infinite-Dimensional Demand Theory

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Abstract

In this article, we examine the existence of Hicksian Demand Correspondence and the duality between Marshallian and Hicksian Demand on reflexive Ordered Linear Spaces. We also prove an equilibrium theorem for Hilbert excess demand correspondences.

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1. Demand Correspondences in Infinite Dimensional Spaces

The present article is devoted to the approach on the demand theory problem in infinite-dimensional commodity spaces, which is equivalent to the one of the finite ones, through excess demand correspondence equilibrium theorems. Another approach is the one of the researchers, who study this problem through the existence of supporting prices of the better-off sets of the preferences of the individuals. This is the approach met for example in [12], [13], [2]. The essential obstacle in the maximization of the preferences of the individuals in the first approach, is that the budget sets are in most of the cases unbounded. Hence, the exact specification of the excess demand correspondence is not possible in most of the infinite-dimensional commodity spaces, even if the relative theorems exist in the literature for a long time. The key for specifying certain examples of commodity spaces and economies, under these theorems is the [14, Th.4]. In [1], a Riesz pair of commodity-price duality...
⟨E, E∗⟩ is proposed, where for the specific examples, AM-spaces with order unit are mentioned. This remark is also made in [17, Sect.4]. If e defines the price-simplex ∆, then ∆ is norm-bounded. This is true, since each of these spaces is a Banach space, hence order units are norm interior points, which rationally represent total endowment of the individuals participating in the exchange economy. In [10], the same assumption is used about on ∆. ∆ by its definition is a base in E∗. In [1], ∆ = {p ∈ L+∗||p|| = 1}, and if e is an order unit, the norm ∥p∥ of the space L∗ we may suppose that it is defined by the specific order unit e, hence p · e = 1. The same assumption is also mentioned in [17], while in [10] it is mentioned only as a base of the cone - in the main Proposition of the article. But if the space is reflexive, 0 /∈ ∆, since in this case weak star and weak topology on L∗ coincide. This is the reason for which reflexivity of L and hence of L∗, is selected. This may provides an excuse for the selection of a reflexive space in the application of the [8, Th.3].

About the problem of the non-empty values of the Hicksian Demand Correspondence, there exists the important [3]. Specifically, [3, Th.2(a)] proves that the existence of the Hicksian Demand Correspondence is assured if the commodity space E is reflexive. But the assumption posed in [3, Th.2(b)] is that the consumption set has non-empty interior. The assumption that we make in the present article is that E is a reflexive space with at least one bounded base of the consumption cone C defined by g ∈ E∗. This implies by [14, Th.4], that all bases defined by f ∈ E∗ on the cone C are bounded. About the existence of interior points in C, if the cone is a non-lattice cone (see [9, Th.4.4.4]), due to the existence of Bishop-Phelps cones, (see [9, p.127]), we may have cones that have interior points and provide bounded bases, as well. Hence, at this point the results of the present article are geometrically do not contradict [3] on Hicksian Demand, as well. By the Bishop-Phelps cones we denote the cones C = {x ∈ E∗|f(x) ≥ a||x||}, ∥f∥ = 1, a ∈ (0, 1). Also, the results of the present article provide families of cones on which the results of [1] hold. Furthermore, in order to make the reader initially familiar to the theory of cones and bases, there exists an Appendix in the end of the present article. The rest of the article is organized as follows: First, we remind the duality between the Marshallian and the Hicksian Demand optimization problem and we also recall from [14] the basic theorems that imply the existence of the solution to the optimization problem, which corresponds to the non-emptiness of the values of the Hicksian Demand. In this section, one of the minimax theorems in [?] is mainly used. On the other hand, this article entirely uses [14] in the existence of the Marshallian Demand Correspondence and its continuity properties, as well. We actually use the results of this article and the properties of bases in order to normalize the prices. We also use bounded bases as an alternative normalization of the prices in Hilbert spaces, instead of the positive part of the unit sphere of the space, seen in [7, Lem.1]. In the second part of the article, we deduce a result both on the non-empty values on the Hicks Correspondence and on Gale-Nikaido-Debreu Lemma for Hilbert commodity spaces. This Gale
-Nikaido -Debreu relies on Browder Selection Theorem. The use of Hilbert space is due to the fact that this Lemma uses the duality equality $C^{00} = C$, which is valid in reflexive spaces. The proof is in detail actually the same one to the one of [17, Th.3.1], also contained for finite -dimensional spaces from the book of [4, Ch.17, Ch.18], relying on [6]. What is new in the proof of this Theorem of [17, Th.3.1] is that the cone $C$ is well-based.

We consider the Marshallian Demand optimization problem in the commodity -price duality $⟨E, E^*⟩$:

$\text{Maximize } u(x) \text{ s.t. } x \in B(p, m),$

where

$B(p, m) = \{x \in C | p(x) \leq m\}, p \in C, m > 0,$

while $p$ is a strictly positive functional of $E_+$, corresponding to a price vector, $m$ denotes the monetary income of the consumer, and $u : C \to \mathbb{R}$ denotes the utility function of the consumer. The corresponding Hicksian Demand optimization problem in the same commodity -price dual pair $⟨E, E^*⟩$ is the following:

$\text{Minimize } p(x) \text{ s.t. } u(x) \geq u_0,$

while $p$ is a strictly positive functional of $E_+$, corresponding to a price vector as well, and $u_0 = u(p, m)$ is a minimum desirable level of utility. The solution correspondence of the first problem was extensively studied in [14] in partially ordered normed spaces. The essential Theorem of that article is [14, Th.12]: Suppose that in a competitive exchange economy the commodity-price duality is the dual system $⟨E, F⟩$, the consumption set is a cone $C$ of $E$ and suppose a preference relation $\succeq$ defined on $P$. (i) If $E$ is a normed space, $F \subseteq E^*$, and $\succeq$ is locally non-satiated, then $p(x) = m$, for any $x \in x(p, m)$, (ii) if $P$ is $\sigma(E, F)$-closed and $\succeq$ is $\sigma(E, F)$-upper semicontinuous, then the demand set $x(p, m)$ is $\sigma(E, F)$-closed, and (iii) if for some topology $\tau$ of $C$, $\succeq$ is $\tau$-upper semicontinuous and the budget set $B(p, m)$ is $\tau$-compact, then $x(p, m) \not= \emptyset$. In the statement of the above Theorem, $B(p, m) = \{x \in C | p(x) \leq m\}$ and we suppose that the utility $u : E_+ \to \mathbb{R}$ defines the preference relation $\succeq_u$, namely $u(x) \geq u(y) \iff x \succeq_u y$. We also suppose that the utility function has an extremely desirable bundle $e \in C$, and weakly semicontinuous, hence $\succeq_u$ is locally non-satiated and $\sigma(E, E^*)$-upper semicontinuous. The existence of an extremely desirable bundle $e \in C$ for $u$ is that for any $\varepsilon > 0$ and $x \in C$, $u(x + \varepsilon e) > u(x)$. The existence of extremely desirable bundle $e \in C$ for $u$, implies local non-satiation of $\succeq_u$. Hence, we suppose that the assumptions for the existence of the Marshallian demand sets $x(p, m)$ are satisfied. Finally, we assume that $u : C \to \mathbb{R}$ is concave, namely $u(tx + (1 - t)y) \geq tu(x) + (1 - t)u(y), t \in (0, 1).$
2. The Results of the Article

For the Hicksian Demand Correspondence, namely for the solution correspondence of the second previously mentioned optimization problem, we have the following

**Theorem 2.1.** In a commodity-price pair \((E, E^*)\) where \(B(p, m)\) are \(\sigma(E, E^*)\)-compact, then the Hicksian Demand Correspondence exists for the maximum utility \(v(p, m)\) of the corresponding Marshallian Demand, if the utility function \(u\) has extremely desirable bundles, is concave and upper weakly semicontinuous.

**Proof.** For the study of the second optimization problem, we may introduce the function (see also [15]):

\[
K(x, y) = u(x)p(y), \quad x \in B(p, m), y \in U_u(p, m),
\]

where

\[
U_u = \{y \in E_+ | u(x) \geq u\}, u = u(p, m),
\]

and

\[
u(p, m) = \max\{u(x)|x \in B(p, m)\}.
\]

For each \(y \in U_u(p, m)\),

\[
K(x, y) \geq \inf_{y \in U_u(p, m)} K(x, y),
\]

\[
K(x, y) \leq \sup_{x \in B(p, m)} K(x, y).
\]

Hence, we get

\[
\sup_{x \in B(p, m)} K(x, y) \geq \sup_{x \in B(p, m)} \inf_{y \in U_u(p, m)} K(x, y),
\]

\[
\inf_{y \in U_u(p, m)} K(x, y) \leq \inf_{y \in U_u(p, m)} \sup_{x \in B(p, m)} K(x, y).
\]

If a saddle-point \((\bar{x}, \bar{y})\) of \(K\) exists in \(B(p, m) \times U_u(p, m)\), then

\[
K(\bar{x}, \bar{y}) \geq \inf_{y \in U_u(p, m)} K(\bar{x}, y), \quad (1)
\]

\[
K(\bar{x}, \bar{y}) \leq \sup_{x \in B(p, m)} K(x, \bar{y}), \quad (2)
\]

and

\[
\sup_{x \in B(p, m)} \inf_{y \in U_u(p, m)} K(x, y) = K(\bar{x}, \bar{y}) = \inf_{y \in U_u(p, m)} \sup_{x \in B(p, m)} K(x, y).
\]

Since by [14, Th.12] the existence of the maximum in the inequality (2) is assured, it suffices to prove the existence of the saddle-point \((\bar{x}, \bar{y})\), in order to assure the existence of the minimum, implied by the inequality (1). For this reason we recall the saddle-point theorem, [?], Cor.3.3: Let \(M, N\) be convex spaces on of which is compact, and \(f\) a function on \(M \times N\), quasi-concave-convex, and upper-semicontinuous-lower semicontinuous. Then \(\sup_{\mu \in M} \inf_{\nu \in N} f(\mu, \nu) = \inf_{\nu \in N} \sup_{\mu \in M} f(\mu, \nu)\). In the statement of this Theorem, a quasi-concave-convex function \(f\) is quasi-concave on \(\mu\) and quasi-convex on \(\nu\), which implies
that \( \{ \mu \in M : f(\mu, \nu) \geq c \} \) is convex for any \( \nu \in N \) and any \( c \in \mathbb{R} \), while \( \{ \nu \in N : f(\mu, \nu) \leq c \} \) is convex for any \( \mu \in M \) and any \( c \in \mathbb{R} \). Hence, we set \( M \) to be any budget set \( B(p, m) \) (which is convex and \( \sigma(E, E^*) \)-compact), \( N \) to be the set \( U_{u(p,m)} \) (which is convex, since \( u \) is concave). \( \mu = x, \nu = y \) in this case, and \( f : B(p, m) \times U_{u(p,m)} \to \mathbb{R} \) is the function \( f(x, y) = u(x)p(y) \). According to \([?, Cor.3.3]\), saddle-points of \( f \) exist, hence minimization points of \( p \) on \( U_{u(p,m)} \), as well. □

Under the assumptions of Theorem 2.1, the usual duality condition between expenditure function \( e(p, u(p,m)) \) and the indirect utility function \( u(p,m) \) (which are both well-defined in this case) holds.

**Corollary 2.2.** In a commodity-price pair \( \langle E, E^* \rangle \) where \( B(p, m) \) are \( \sigma(E, E^*) \)-compact, if the utility function \( u \) has extremely desirable bundles, is concave and upper weakly semicontinuous, then the expenditure function

\[
e(p, u(p,m)) = \min \{ p \cdot y | u(y) \geq u(p,m) \},
\]

is well-defined and \( e(p, u(p,m)) = m \).

**Proof.** The fact that \( e(p, u(p,m)) \) is well-defined is a consequence of the fact that \([14, Th.12]\) implies that the \( \max \{ u(x) | x \in B(p,m) \} \) is achieved, hence the indirect utility function \( u(p,m) \) is well-defined. Also, in Theorem 2.1, we proved that the \( \min \{ p(y) | y \in U_{u(p,m)} \} \) is achieved, too. Hence the expenditure function \( e(p, u(p,m)) \) is well-defined. □

### 3. A Gale-Nikaido-Debreu Lemma for Hilbert Spaces

It is well-known that the Gale-Nikaido-Debreu Lemma for finite commodity spaces’ excess demand correspondences is the following -see \([4, Th.1]\), where \( \Delta \) denotes the simplex of the \( \mathbb{R}^m_+ \), or else the base defined by the vector \( \mathbf{1} : \) Let \( \zeta : \Delta \to \mathbb{R}^m \) an upper hemicontinuous correspondence with nonempty compact convex values such that for all \( p \in \Delta, p \cdot z \) for each \( z \in \zeta(p) \). Let \( N = -\mathbb{R}^m_+ \). Then \( \{ p \in \Delta : N \cap \zeta(p) \neq \emptyset \} \) of free disposal equilibrium prices is nonempty and compact.

We recall the following

**Definition 3.1.** If \( \phi : E \to 2^F \) is a correspondence and \( E, Y, F \) are topological spaces, we call the set \( \phi^{-1}(y) = \{ x \in E : y \in \phi(x) \} \) lower section of \( \phi \). If these sets are open in \( E \) for any \( y \in F \), we say that \( \phi \) has open lower sections.

**Definition 3.2.** A selector from a correspondence \( \phi : E \to 2^F \), is a function \( f : E \to F \), such that \( f(x) \in \phi(x) \), for any \( x \in E \). If \( E, F \) are topological spaces, we say that \( f \) is a continuous selector if \( f \) is a selector and is also continuous.
We recall the Browder Selection Theorem (see [6]): A correspondence with nonempty, convex values and open lower sections from a compact, convex subset of a Hausdorff topological vector space into itself, admits a continuous selector.

We recall the definition of radial continuity from [14, Def.16]:

Suppose that $\succeq$ is a preference relation defined on a cone $P$ of a linear topological space $E$. If for each $x,y \in P$, the set $\{\lambda x | \lambda \in \mathbb{R}^+ : y \succeq \lambda x\}$ is closed, the preference relation $\succeq$ is radially lower semicontinuous. If for each $x,y \in P$ the set $\{\lambda x | \lambda \in \mathbb{R}^+ : \lambda x \succeq y\}$ is closed, then $\succeq$ is radially upper semicontinuous. If $\succeq$ is radially lower and radially upper semicontinuous, then $\succeq$ is radially continuous.

We also recall the statement of the [14, Th.17], regarding continuity of Marshallian demand correspondence, in case where its values are non-empty:

Suppose that in a competitive exchange economy the commodity-price duality is the dual system $\langle E,F \rangle$, where $E$ is a normed space and $F$ a subspace of $E^*$ and suppose also that the consumption set is a $\sigma(E,F)$-closed cone $C$ of $E$. If the positive part $U_+ = U \cap C$ of the unit ball $U_+$ of $E$ is $\sigma(E,F)$-compact and the cone $P$ has a norm-bounded budget set, then for any $\sigma(E,F)$-upper semicontinuous and $\sigma(E,F)$-radially lower semicontinuous preference relation $\succeq$ of $C$, the demand correspondence of $\succeq_u$ is norm-to- $\sigma(E,F)$ upper hemicontinuous.

Consider an exchange economy on some Hilbert space, whose total endowment is $e \in H^+$. We suppose that $e$ defines a bounded base.

**Theorem 3.3.** Consider a closed cone $C$ in some Hilbert space $H$ and a bounded base $B_e$ of it. Any norm-to-weak upper hemicontinuous correspondence $z : B_e \to 2^H$ with non-empty, convex, compact values satifies the weak Walras law:

$$\forall p \in B_e, \exists z \in z(p) : p \cdot z \leq 0.$$ 

Then the set of equilibrium prices $\{p \in B_e | z(p) \cap C^0\}$ is weakly compact and non-empty.

**Proof.** We may follow the lines of the proof of the equilibrium theorem in [7], as it is also presented in [4, Th.18.17], in the case of a Hilbert space, since $H \cong H^*$. The set of equilibrium prices is a closed subset of the base $B_e$, from the upper -hemicontinuity of $z$. But since $B_e$ is a weakly compact set of $H$, the set of equilibrium prices is also a weakly compact set. In order to proove that it is a non-empty set, we suppose that the set is empty, hence for any $p \in B_e$, there exists a functional $k \in H^* \cong H, k \neq 0$, such that

$$k \cdot z > 0 \geq k \cdot g, z \in z(p), g \in C^0,$$

since $C^0$ is a wedge. Also, since $k \neq 0$, it may taken to lie on the base $B_e$. Hence, a new correspondence is defined by $K : B_e \to 2^H$, such that

$$p \mapsto K(p) = \{h \in B_e | h \cdot z > 0 > h \cdot g, z \in z(p), g \in C^0\}.$$
$K$ has non-empty, convex values for any $p$. Also, if $p \in K^{-1}(h)$, this implies $h \cdot z > 0 > h \cdot g, z \in z(p), g \in C^0$, where also we may define another correspondence $\phi : B_e \rightarrow 2^H$, with $\phi(p) = C^0, p \in B_e$. Since $z$ is upper hemicontinuous and the correspondence $\phi(p) = C^0, p \in B_e$ is also upper hemicontinuous, the set $k^u(z : h \cdot z > 0) \cap \phi^u(g : h \cdot g < 0)$ is a neighborhood of $p$, contained in $K^{-1}(p)$. Hence, by the Browder’s Selection Theorem, a continuous selection $k(p) \in K(p)$ exists, such that

$$k(p) \cdot z > 0 > k(p) \cdot g, z \in z(p), g \in C^0, p \in B_e.$$ 

From Brouwer Fixed Point Theorem, there exists some fixed point of the weakly continuous function $k : B_e \rightarrow B_e$, since $B_e$ is weakly compact. For this fixed point $p_0$ such that $p_0 = k(p_0), p_0 \cdot z > 0$ for any $z \in z(p_0)$, a contradiction, because weak Walras law is satisfied by $z$ for any $p \in B_e$. Hence the set of equilibrium prices is non-empty. $\square$

Hence, according to the [14, Th.17] we take the following:

**Corollary 3.4.** If in a competitive exchange economy, the commodity-price duality is the dual system $\langle H, H \rangle$, where $H$ is an infinite-dimensional Hilbert space, and the consumption set is a weakly closed cone $C$ of $H$, and the total endowment $e$ of the economy defines a norm-bounded budget set, while the preferences of the consumers are concave, weakly upper semicontinuous and weakly radially lower semicontinuous, then the set of equilibrium prices $\{p \in B_e | z(p) \cap C^0\}$ of the excess demand correspondence $z : B_e \rightarrow 2^H$ is weakly compact and non-empty.

### 4. Appendix- Ordered Linear Spaces and Bases of Cones

Let $E$ be a (normed) linear space. A set $C \subseteq E$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbb{R}_+$ is called wedge. A wedge for which $C \cap (-C) = \{0\}$ is called cone. A pair $(E, \geq)$ where $E$ is a linear space and $\geq$ is a binary relation on $E$ satisfying the following properties:

(i) $x \geq x$ for any $x \in E$ (reflexive),

(ii) If $x \geq y$ and $y \geq z$ then $x \geq z$, where $x, y, z \in E$ (transitive),

(iii) If $x \geq y$ then $\lambda x \geq \lambda y$ for any $\lambda \in \mathbb{R}_+$ and $x + z \geq y + z$ for any $z \in E$, where $x, y \in E$ (compatible with the linear structure of $E$),

is called partially ordered linear space. The binary relation $\geq$ in this case is a partial ordering on $E$. The set $P = \{x \in E | x \geq 0\}$ is called (positive) wedge of the partial ordering $\geq$ of $E$. Given a wedge $C$ in $E$, the binary relation $\geq_C$ defined as follows:

$$x \geq_C y \iff x - y \in C,$$

is a partial ordering on $E$, called partial ordering induced by $C$ on $E$. If the partial ordering $\geq$ of the space $E$ is antisymmetric, namely if $x \geq y$ and $y \geq x$ implies $x = y$, where $x, y \in E$, then $P$ is a cone. $E'$ denotes the linear space of all linear functionals of $E$, while $E^*$ is the norm dual of $E$. 

...
$E^*$, in case where $E$ is a normed linear space. Suppose that $C$ is a wedge of $E$ A functional $f \in E'$ is called positive functional of $C$ if $f(x) \geq 0$ for any $x \in C$. $f \in E'$ is a strictly positive functional of $C$ if $f(x) > 0$ for any $x \in C \setminus \{0\}$. A linear functional $f \in E'$ where $E$ is a normed linear space, is called uniformly monotonic functional of $C$ if there is some real number $a > 0$ such that $f(x) \geq a\|x\|$ for any $x \in C$. In case where a uniformly monotonic functional of $C$ exists, $C$ is a cone. $C^0 = \{f \in E^*|f(x) \geq 0 \text{ for any } x \in C\}$ is the dual wedge of $C$ in $E^*$. Also, by $C^{00}$ we denote the subset $(C^0)^0$ of $E^{**}$.

It can be easily proved that if $C$ is a closed wedge of a reflexive space, then $C^{00} = C$. If $C$ is a wedge of $E^*$, then the set $C_0 = \{x \in E|x(f) \geq 0 \text{ for any } f \in C\}$ is the dual wedge of $C$ in $E$, where $\hat{\cdot} : E \to E^{**}$ denotes the natural embedding map from $E$ to the second dual space $E^{**}$ of $E$. Note that if for two wedges $K, C$ of $E K \subseteq C$ holds, then $C^0 \subseteq K^0$. If $C$ is a cone, then a set $B \subseteq C$ is called base of $C$ if for any $x \in C \setminus \{0\}$ there exists a unique $\lambda_x > 0$ such that $\lambda_xx \in B$. The set $B_f = \{x \in C|f(x) = 1\}$ where $f$ is a strictly positive functional of $C$ is the base of $C$ defined by $f$. $B_f$ is bounded if and only if $f$ is uniformly monotonic (for the proof of this, see [14, Pr.2]). If $B$ is a bounded base of $C$ such that $0 \notin \overline{B}$ then $C$ is called well-based. If $C$ is well-based, then a bounded base of $C$ defined by a $g \in E^*$ exists (for the proof of it, see [11, Pr.3]). If $E = C - C$ then the wedge $C$ is called generating, while if $E = C - C$ it is called almost generating. If $C$ is generating, then $C^0$ is a cone of $E^*$ in case where $E$ is a normed linear space. Also, $f \in E^*$ is a uniformly monotonic functional of $C$ if and only if $f \in intC^0$, where $intC^0$ denotes the norm-interior of $C^0$ (for the proof of it see [9, Pr.3.8.12]).

If $E$ is partially ordered by $C$, then any set of the form $[x,y) = \{r \in E|y \geq_C r \geq_C x\}$ where $x,y \in C$ is called order-interval of $E$. If $E$ is partially ordered by $C$ and for some $e \in E$, $E = \bigcup_{n=1}^{\infty}[-ne,ne]$ holds, then $e$ is called order unit of $E$. If $E$ is a normed linear space then if every interior point of $C$ is an order-unit of $E$. If $E$ is moreover a Banach space and $C$ is closed, then every order unit of $E$ is an interior point of $C$.

References


Some results on infinite-dimensional demand theory


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