Dual-Product Rollover Management with Consumer Memory

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Abstract

We study a monopolistic firm that introduces successive product generations under a dual-product rollover strategy, where different generations coexist in the market. We develop a joint pricing and product launch model to determine the optimal pricing and product launch policies in the presence of consumer memory through reference prices. We characterize the optimal pricing policies and develop closed-form approximations for the optimal pricing policies and for the optimal product launch time. Our results provide managers with easy-to-implement dual-product rollover policies under reference price effects in practice as well as an ability to conduct any sensitivity analysis.

Keywords: Dynamic Optimization, Reference Price, Dual-Product Rollover, Perturbation
1 Introduction

Effective management of product rollover decisions for successive product generations is an important challenge for many firms ([1]). The existing literature in product rollover management assume that there is no interaction between the firms and their consumers. However, there is a vast empirical support in the marketing literature indicating that past sales impact current sales through reference prices [6]. Consumers adaptively learn from observing posted prices over time and develop their price judgments (reference prices). They compare current prices with their formed reference prices before making their purchasing decisions. They perceive gains (losses) if prices are lower (higher) than their reference prices. Consumer demand increases (decreases) with perceived gains (losses) ([5]). Firms can not directly modify how consumers form their price perceptions. However, they can impact consumers’ price perceptions via changing price patterns that consumers experience dynamically. Firms thus need to be consider how their past pricing policies affect their current demand in forming more informed product rollover decisions. Empirical studies in operations management indicate that developing multiple generations of products, introducing and managing them effectively helps firms succeed ([7]). Thus, it is important to help firms develop more informed product rollover strategies when demand for each product generation depends on consumers’ price perceptions.

This paper develops a joint dynamic pricing and product introduction model to help a monopolistic firm develop effective dual-product rollover management policies for its successive product generations in the presence of consumer memory on past prices. We characterize the optimal pricing policies explicitly and show that they are fairly complex and difficult to implement in practice. We develop linear approximations through perturbation for the optimal pricing policies and product launch time. The resulting approximations are in closed-form and thus enable managers implement them easily in practice and develop any sensitivity analysis. The rest of the paper is organized as follows. Section 2 develops the model framework for a dual-product rollover strategy. In Section 3, we characterize the optimal pricing policies and the corresponding linear approximations through perturbation. In Section 4, we develop the linear approximation for the optimal launch time for the next generation. We conclude the paper with the suggested future research in Section 5.

2 Model Framework

We consider characterizing optimal dual-product rollover decisions for a single monopolistic firm when demand for each product generation depends on
consumers' memory on past prices. The two product generations are: current and new generation. The firm has been producing the current generation for some time. The firm has developed a new generation, that is ready to launch, in order to replace the current one eventually. Let $T$ denote the phase out time of the current generation and let $\tau \in [0,T]$ denote the launch time of the next generation. We assume the following demand functions for the two generations:

$$
D_0(t) = (\kappa(t)(a_0 - b_0 p_0(t)) - \epsilon(p_0(t) - \langle p_0 \rangle(t))) e^{-\gamma t}, t \in [0, T],
$$

$$
D_1(t) = (a_1 - b_1 p_1(t) - \epsilon(p_1(t) - \langle p_1 \rangle(t))) e^{-\gamma(t-\tau)}, t \in [\tau, T],
$$

where $a_i$ is the constant demand parameter, $p_i(t)$ is the dynamic price for each generation, $b_i$ represents the sensitivity of demand to dynamic price, $\langle p_i \rangle(t)$ is the dynamic reference price in the market, $\epsilon$ is the demand sensitivity of generation $i$ with respect to the deviation of its current posted price from the reference price. We assume that sales of each generation decays exponentially at at the same market determined rate $\gamma \in (0, 1)$. We assume that the launch of the new generation results in a drop in demand for the current generation by a factor $\kappa(t)$,

$$
\kappa(t) = \begin{cases} 
1, & 0 \leq t \leq \tau, \\
\kappa, & \tau < t \leq T,
\end{cases} 
\quad 0 \leq \kappa \leq 1.
$$

The impact of reference prices on consumer demand has been modeled similar to ours in [4] and [3] for a single-generation model.

We let $\langle p_0 \rangle(0) = r^*$ denote the initial reference price for the current generation. We assume that the firm holds perfect information on $\langle p_0 \rangle(0)$. Consumers adaptively learn and update their price perceptions over time through their repeat purchasing experience and dynamic price exposure. We use the following estimator of this dynamic reference price for the current generation as a weighted moving average of past prices,

$$
\langle p_0 \rangle(t) = e^{-\lambda t} \langle p_0 \rangle(0) + \lambda \int_0^t p_0(s)e^{-\lambda(t-s)} ds,
$$

where consumer memory parameter $\lambda$ measures the length of consumer memory for past prices. The reference price formation in (3) can be equivalently described by the following ordinary differential equation,

$$
\langle \dot{p}_0 \rangle(t) = \lambda (p_0(t) - \langle p_0 \rangle(t)).
$$

The firm launches the new generation at $\tau \in [0,T]$. Both generations are offered concurrently in the market during $[\tau,T]$. We assume that the two concurrent generations share the same reference price after the introduction of
the new generation at time $\tau$, which evolves according to (7), and contribute to the joint reference price with relative reference price contributions $\alpha_0$ and $\alpha_1$ for the current and the new generation, respectively. Only the joint reference price $r(t)$ exists after the launch of the new generation. The firm's objective is to maximize the total profit from sales of two generations:

$$\max_{0 \leq \tau \leq T, p_0(t), p_1(t)} \int_0^{\tau} (p_0(t) - c_0e^{-\delta t})(a_0 - b_0p_0(t) - \epsilon(p_0(t) - \langle p_0 \rangle(t)))e^{-\gamma t}dt + \int_\tau^{T} (p_0(t) - c_0e^{-\delta t})(\kappa(a_0 - b_0p_0(t)) - \epsilon(p_0(t) - r(t)))e^{-\gamma t}dt + \int_\tau^{T} (p_1(t) - c_1e^{-\delta t})(a_1 - b_1p_1(t) - \epsilon(p_1(t) - r(t)))e^{-\gamma(t-\tau)}dt$$

such that

$$\langle \dot{p}_0 \rangle(t) = \lambda(p_0(t) - \langle p_0 \rangle(t)), \ 0 \leq t \leq \tau, \ \langle p_0 \rangle(0) = r^*, \quad (6)$$

$$\dot{r}(t) = \alpha_0(p_0(t) - r(t)) + \alpha_1(p_1(t) - r(t)), \ \tau \leq t \leq T, \quad (7)$$

$$r(\tau^+) = \langle p_0 \rangle(\tau^-). \quad (8)$$

Next, we analyze the optimal pricing policies before and after the next generation launch.

### 3 Optimal Pricing Policies

We begin with analyzing the optimal pricing policy for a single generation: current generation over $[0, \tau]$. The firm has the objective of maximizing its profits $R[p_0(t)]$ from sales of the current generation over a selling horizon $[0, \tau]$,

$$\max_{p_0(t)} R[p_0(t)], \quad (9)$$

$$R[p_0(t)] = \int_0^{\tau} (p_0(t) - c_0e^{-\delta t})(a_0 - b_0p_0(t) - \epsilon(p_0(t) - \langle p_0 \rangle(t)))e^{-\gamma t}dt, \quad (10)$$

where we assume that unit production cost $c_0$ for the current product decreases as the firm gains production experience through learning-by-doing in manufacturing ([2]). We can interpret $\delta \geq 0$ as the production learning rate of the firm. We note that if the firm does not gain production experience through learning-by-doing in reducing the unit production cost i.e., $\delta = 0$, our one-generation model for the current product is equivalent to the one analyzed in [3].

We define an auxiliary variable $z(t) = \frac{1}{\lambda} \langle p_0 \rangle(t)$. This auxiliary variable satisfies

$$p_0(t) = \dot{z}(t) + \lambda z(t), \quad (11)$$
\[ z(t) = r^* \frac{1}{\lambda} e^{-\lambda t} + \int_0^t p_0(s) e^{-\lambda(t-s)} ds. \]  (12)

Thus, the profit function of the firm in terms of the new variable \( z(t) \) takes the form

\[ R[z(t)] = \int_0^\tau ( \dot{z}(t) + \lambda z(t) - c_0 e^{-\delta t} (a_0 - (b_0 + \epsilon) \dot{z}(t) - b_0 \lambda z(t)) e^{-\gamma t} dt. \]  (13)

Euler-Lagrange equation for the above profit function is the linear ODE

\[ \dddot{z}(t) - \gamma \ddot{z}(t) - \lambda b_0 + \epsilon \left( \frac{1}{2} \gamma \lambda + \frac{1}{2} \gamma \epsilon \right) z(t) + \frac{a_0}{2(b_0 + \epsilon)}(\lambda + \gamma) + \frac{1}{2} c_0 e^{-\delta t} \left( \frac{b_0}{b_0 + \epsilon} \lambda + \gamma + \delta \right) = 0, \]  (14)

Due to linearity, its general solution has the form of combination of exponential functions,

\[ z(t) = \phi + \psi e^{-\delta t} + C e^{\gamma t/2} e^{\omega t} + D e^{\gamma t/2} e^{-\omega t}, \]  (15)

\[ p_0(t) = \lambda \phi + (\lambda - \delta) \psi e^{-\delta t} + \left( \frac{1}{2} \gamma + \lambda + \omega \right) C e^{\gamma t/2} e^{\omega t} + \left( \frac{1}{2} \gamma + \lambda - \omega \right) D e^{\gamma t/2} e^{-\omega t}, \]  (16)

where coefficients \( \phi, \psi, \) and \( \omega \) have the following expressions:

\[ \phi = \left( 1 + \frac{\gamma}{\lambda} \right) \frac{a_0}{2 \gamma b_0 + 2 \lambda b_0 + \gamma \epsilon}, \]  (17)

\[ \psi = \frac{b_0 c_0 \lambda + c_0 (b_0 + \epsilon) (\delta + \gamma)}{\epsilon \gamma \lambda + 2 b_0 (\lambda - \delta) (\delta + \gamma + \lambda) - 2 \epsilon \delta (\delta + \gamma)}, \]  (18)

\[ \omega = \left( \frac{1}{4} \gamma^2 + \frac{\lambda}{b_0 + \epsilon} \left( \gamma b_0 + \lambda b_0 + \frac{1}{2} \gamma \epsilon \right) \right)^{1/2}. \]  (19)

We can show using the Jacobi sufficient conditions that the solution of this ODE indeed maximizes the profit function in (10). Thus, we obtain the optimal pricing path for the current generation. The coefficients \( C \) and \( D \) in the optimal pricing path for the current generation have the following relations to initial reference price \( r^* = \langle p_0 \rangle(0) \) and initial quoted price \( p^* = p_0(0) \):

\[ r^* = \lambda (\phi + \psi + C + D), \]  (20)

\[ p^* = \lambda \phi + (\lambda - \delta) \psi + \left( \frac{1}{2} \gamma + \lambda + \omega \right) C + \left( \frac{1}{2} \gamma + \lambda - \omega \right) D. \]  (21)
where the initial reference price $r^*$ is a known constant, but initial posted price $p^*$ is a decision variable that the firm needs to set to maximize its total profits. Given the above conditions on initial reference price and posted price, the solutions for coefficients $C$ and $D$ in equations (20) and (21) become

$$C = \frac{1}{2\omega} \left( p^* - r^* + \delta \psi - \left( \frac{\gamma}{2} - \omega \right) \left( \frac{1}{\lambda} r^* - \phi - \psi \right) \right) = C^0 + \frac{1}{2\omega} p^*, \quad \text{(22)}$$

$$D = -\frac{1}{2\omega} \left( p^* - r^* + \delta \psi - \left( \frac{\gamma}{2} + \omega \right) \left( \frac{1}{\lambda} r^* - \phi - \psi \right) \right) = D^0 - \frac{1}{2\omega} p^*. \quad \text{(23)}$$

Properties of the solution are determined by the signs of coefficients $C$ and $D$, which are both functions of the initial quoted price $p^*$.

Note that, if the firm plans to keep the current generation in the market for a long period of time, it may choose to adopt an infinite horizon framework instead with $\gamma$ including appropriate cash flow discounting. In this case, we impose an infinity condition on the solution that the solution needs to stay finite (not blow up) at infinity. This gives us the condition on coefficient $C = 0$, and the following general solution for the pricing strategy

$$p_0(t) = \lambda \phi + (\lambda - \delta) \psi e^{-\delta t} + \left( \frac{1}{\lambda} r^* - \phi - \psi \right) \left( \frac{1}{2} \gamma + \lambda - \omega \right) e^{\gamma t/2} e^{-\omega t}, \quad \text{(24)}$$

$$p^* = \lambda \phi + (\lambda - \delta) \psi + \left( \frac{1}{\lambda} r^* - \phi - \psi \right) \left( \frac{1}{2} \gamma + \lambda - \omega \right). \quad \text{(25)}$$

Note also that, in the infinite horizon case, we are not at liberty to choose $p^*$, because the corresponding initial reference price $\langle p_0 \rangle(0)$ still needs to be matched with the known initial reference price $r^*$.

Since we reduce the optimal pricing problem for the current generation over $[0, \tau]$ to determining its optimal initial price $p^*$, we next state in Theorem 3.1 the corresponding result for the optimal initial price.

**Proposition 3.1.** The optimal initial price maximizing profits from the current generation over the sales period $[0, \tau]$ becomes

$$p^* = \frac{\omega}{\lambda (H_3 + H_4)} \left( H_2 - H_1 + \frac{1}{\omega} \left( -\Phi \left( \frac{\gamma}{2} + \omega \right) + r^* \left( \frac{\gamma}{2} + \omega + \lambda \right) \right) H_4 + \frac{1}{\omega} \left( -\Phi \left( \frac{\gamma}{2} - \omega \right) + r^* \left( \frac{\gamma}{2} - \omega + \lambda \right) \right) H_3 \right) \quad \text{(26)}$$

where

$$\Phi = \frac{a_0 (\lambda + \gamma)}{2\gamma b_0 + 2\lambda b_0 + \gamma \epsilon} + \lambda c_0 \frac{b_0 \lambda + (b_0 + \epsilon)(\delta + \gamma)}{e \gamma \lambda + 2b_0(\lambda - \delta)(\delta + \gamma + \lambda) - 2\delta \epsilon(\delta + \gamma)},$$
We define the functions $H_1(\tau), H_2(\tau), H_3(\tau), H_4(\tau)$ as follows

\[
H_1(\tau) = \frac{(a_0 + b_0c_0 - 2b_0\Phi)\left(\frac{\gamma}{2} + \omega + \lambda\right) + \epsilon\left(\frac{\gamma}{2} + \omega\right)(c_0 - \Phi)}{\lambda\left(\frac{\gamma}{2} - \omega\right)}\left(1 - e^{-\left(\frac{\gamma}{2} - \omega\right)\tau}\right),
\]

\[
H_2(\tau) = \frac{(a_0 + b_0c_0 - 2b_0\Phi)\left(\frac{\gamma}{2} - \omega + \lambda\right) + \epsilon\left(\frac{\gamma}{2} - \omega\right)(c_0 - \Phi)}{\lambda\left(\frac{\gamma}{2} + \omega\right)}\left(1 - e^{-\left(\frac{\gamma}{2} + \omega\right)\tau}\right),
\]

\[
H_3(\tau) = \frac{1}{2\lambda^2\omega}\left(\frac{\gamma}{2} + \omega + \lambda\right)\left(b_0\left(\frac{\gamma}{2} + \omega + \lambda\right) + \epsilon\left(\frac{\gamma}{2} + \omega\right)\right)(1 - e^{2\omega\tau}),
\]

\[
H_4(\tau) = -\frac{1}{2\lambda^2\omega}\left(\frac{\gamma}{2} - \omega + \lambda\right)\left(b_0\left(\frac{\gamma}{2} - \omega + \lambda\right) + \epsilon\left(\frac{\gamma}{2} - \omega\right)\right)(1 - e^{-2\omega\tau}).
\]
First-order derivative conditions on $R(p^*)$ are linear with respect to $p^*$, and thus lead to the following optimal initial price

$$p^* = \frac{\omega}{\lambda(B_3 + B_4)}(B_2 - B_1 + \frac{1}{\omega}\left(-\Phi\left(\frac{\gamma}{2} + \omega\right) + r^*\left(\frac{\gamma}{2} + \omega + \lambda\right)\right)B_4 + \frac{1}{\omega}\left(-\Phi\left(\frac{\gamma}{2} - \omega\right) + r^*\left(\frac{\gamma}{2} - \omega + \lambda\right)\right)B_3)$$

for a single generation: the current generation over $[0, \tau]$. \qed

The result in Proposition 3.1 provides us with the optimal pricing path for the current generation in closed-form until new generation is launched. This result is very useful in conducting any sensitivity analysis on how the optimal pricing path changes with different model parameters. We observe, however, that the optimal pricing policy for the current generation over $[0, \tau]$ results in a rather complex expression to extend our analysis to the concurrent generations over $[\tau, T]$ in order to determine the optimal launch time of the new generation in Section 4. Thus, we next develop linear approximations for the optimal pricing strategies.

**Proposition 3.2.** The optimal pricing path for the current generation until the new generation is launched can be written as

$$p_0(t) = \hat{p}_0(t) - \frac{\epsilon}{2b_0}\left[\frac{a_0}{b_0} - \left(\hat{p}_0 + (r^* - \hat{p}_0)e^{-\lambda t}\right)\right] - \frac{1}{2}\left(\frac{\delta}{\lambda - \delta}\right)c_0e^{-\delta t}\left(1 - e^{-\lambda t}\right) - \frac{\lambda}{2(\lambda + \gamma)}\left(\frac{a_0}{b_0}\left(1 - e^{-(\tau-t)(\lambda+\gamma)}\right)\right) - \frac{\lambda}{2(\lambda + \gamma + \delta)}c_0\left(e^{-\delta t} - e^{-(\tau-t)(\lambda+\gamma)}e^{-\delta \tau}\right) + O(\epsilon^2), \quad (31)$$

where $O(\epsilon^2)$ represents the higher order terms in $\epsilon$ and

$$\hat{p}_0(t) = \frac{a_0}{2b_0} + \frac{1}{2}c_0e^{-\delta t}. \quad (32)$$

**Proof.** We use the Pontryagin maximum principle to find the optimal pricing policy. We define the current-value Hamiltonian for the problem of the current generation as

$$H = (p_0 - c_0e^{-\delta t})(a_0 - b_0p_0 - \epsilon(p_0 - \langle p_0 \rangle)) + q\lambda(p_0 - \langle p_0 \rangle), \quad (33)$$

where $q$ stands for the current-value costate variable corresponding to the reference price state equation in (4).
Solving for the first-order derivative $\frac{\partial H}{\partial p_0} = 0$, we obtain the following pricing policy maximizing $H$:

$$p_0 = \frac{c_0}{2} e^{-\delta t} + \frac{1}{2(b_0 + \epsilon)} (a_0 + \epsilon \langle p_0 \rangle + \lambda q).$$

(34)

The maximized Hamiltonian can be written as

$$H^* = \frac{1}{4(b_0 + \epsilon)} \left( \epsilon \langle p_0 \rangle + \lambda q + a_0 - (b_0 + \epsilon)c_0 e^{-\delta t} \right)$$

$$\left( \epsilon \langle p_0 \rangle - \lambda q + a_0 - (b_0 + \epsilon)c_0 e^{-\delta t} \right) + \frac{\lambda q}{2(b_0 + \epsilon)}$$

$$\left( \lambda q + a_0 + (b_0 + \epsilon)c_0 e^{-\delta t} - (2b_0 + \epsilon) \langle p_0 \rangle \right).$$

(35)

The reference price evolution equation (4) along with the costate evolution equation with its right-end boundary condition

$$\dot{q} = \gamma q - \frac{\partial H^*}{\partial \langle p_0 \rangle}, \quad q(\tau) = 0,$$

(36)

provide us with a complete system. After solving the boundary-value problem for this second-order linear ODE (Ordinary Differential Equation) system, we achieve the optimal pricing policy through substituting the costate variable $q$ into (34).

The current-value Hamiltonian $H$ is a concave function, thus the problem admits a unique solution, which indeed is a maximizer of the objective functional. The linear ODE system for $q$ and $\langle p_0 \rangle$ can thus be written as follows,

$$\dot{q} = \left( \gamma + \lambda \frac{2b_0 + \epsilon}{2b_0 + 2\epsilon} \right) q - \frac{\epsilon^2}{2b_0 + 2\epsilon} \langle p_0 \rangle - \frac{\epsilon}{2} \left( \frac{a_0}{b_0 + \epsilon} - c_0 e^{-\delta t} \right),$$

(37)

$$\langle \dot{p_0} \rangle = \frac{\lambda^2}{2b_0 + 2\epsilon} q - \lambda \frac{2b_0 + \epsilon}{2b_0 + 2\epsilon} \langle p_0 \rangle + \frac{\lambda}{2} \left( \frac{a_0}{b_0 + \epsilon} + c_0 e^{-\delta t} \right),$$

(38)

$$q(\tau) = 0, \quad \langle p_0 \rangle(0) = r^*. \quad \langle p_0 \rangle(0) = r^*.$$

(39)

We can expand the above ODE system up to higher order terms in reference price sensitivity parameter $\epsilon$ to obtain linear approximations for the optimal pricing policy and the corresponding optimal reference price path. Zeroth-order expansion around $\epsilon = 0$ results in

$$\dot{q} = (\gamma + \lambda) q + O(\epsilon),$$

(40)

$$\langle \dot{p_0} \rangle = \frac{\lambda^2}{2b_0} q - \lambda \langle p_0 \rangle + \frac{\lambda}{2} \left( \frac{a_0}{b_0} + c_0 e^{-\delta t} \right) + O(\epsilon),$$

(41)
where \( O(\epsilon) \) represents the higher order terms in \( \epsilon \). The solution to the perturbed system gives us the following linear approximation for the optimal reference price path

\[
\langle p_0 \rangle(t) = \hat{p}_0(t) + (r^* - \hat{p}_0(t))e^{-\lambda t} + \frac{1}{2} \left( \frac{\delta}{\lambda - \delta} \right) c_0 e^{-\delta t} \left( 1 - e^{-\lambda t} \right) + O(\epsilon),
\]

where

\[
\hat{p}_0(t) = \frac{a_0}{2b_0} + \frac{1}{2} c_0 e^{-\delta t}.
\]

First-order expansion of the expression in (37) around \( \epsilon = 0 \) in linear ODE system gives us

\[
\dot{q} = (\gamma + \lambda - \frac{\lambda}{2b_0} \epsilon)q - \frac{1}{2} \left( \frac{a_0}{b_0} - c_0 e^{-\delta t} \right) \epsilon + O(\epsilon^2).
\]

We can solve the above differential equation along with the boundary condition \( q(\tau) = 0 \) to obtain the following linear approximation for \( q(t) \)

\[
q(t) = \frac{1}{2(\lambda + \gamma)} \left( \frac{a_0}{b_0} \right) \left( 1 - e^{-(\tau-t)(\lambda+\gamma)} \right) \epsilon + \frac{1}{2(\lambda + \gamma + \delta)} c_0 \left[ e^{-\delta t} - e^{-(\tau-t)(\lambda+\gamma)} e^{-\delta \tau} \right] \epsilon + O(\epsilon^2).
\]

We can plug in the linear approximations in (42) and (44) for \( \langle p_0 \rangle \) and \( q(t) \) into the optimal pricing policy obtained in (34) and then expand it around \( \epsilon = 0 \). This results in the following linear approximation for the optimal pricing policy for the current generation over \([0, \tau]\)

\[
p_0(t) = \hat{p}_0(t) - \frac{\epsilon}{2b_0} \left[ \frac{a_0}{b_0} - \left( \hat{p}_0(t) + (r^* - \hat{p}_0(t))e^{-\lambda t} \right) - \frac{1}{2} \left( \frac{\delta}{\lambda - \delta} \right) c_0 e^{-\delta t} \left( 1 - e^{-\lambda t} \right) - \frac{\lambda}{2(\lambda + \gamma)} \left( \frac{a_0}{b_0} \right) \left( 1 - e^{-(\tau-t)(\lambda+\gamma)} \right) - \frac{\lambda}{2(\lambda + \gamma + \delta)} c_0 \left( e^{-\delta t} - e^{-(\tau-t)(\lambda+\gamma)} e^{-\delta \tau} \right) \right] + O(\epsilon^2).
\]
We note that \( \hat{p}_0(t) \) refers to the optimal pricing policy when there is no reference price effects. We can plug in the optimal pricing policy we obtained in Proposition 3.2 to the objective function of the firm in (10). This provides us with the following expression for the optimal total profits for the firm (up to \( O(\epsilon^2) \)) over the selling period of \([0, \tau]\) for a given initial reference price \( r^* \)

\[
R(\tau, r^*) = \int_0^\tau \left( p_0(t) - c_0 e^{-\delta t} \right) \left( a_0 - b_0 p_0(t) - \epsilon p_0(t) + \epsilon \langle p_0 \rangle(t) \right) e^{-\gamma t} dt
\]

\[
= \int_0^\tau \left( \hat{L}_0(t) + \frac{\epsilon}{2} \left( \frac{a_0}{b_0} - c_0 e^{-\delta t} \right) \left( (r^* - \hat{p}_0(t)) e^{-\lambda t} + \frac{1}{2} \left( \frac{\delta}{\lambda - \delta} \right) c_0 e^{-\delta t} (1 - e^{-\lambda t}) \right) \right) e^{-\gamma t} dt + O(\epsilon^2)
\]

\[
= \int_0^\tau \hat{L}_0(t) dt + \frac{\epsilon}{2} \left( r^* - \hat{p}_0(t) \right) \left[ \frac{1}{\gamma + \lambda} \left( 1 - e^{-(\gamma + \lambda)\tau} \right) \left( \frac{a_0}{b_0} \right) - \frac{1}{\gamma + \lambda + \delta} \left( 1 - e^{-(\gamma + \lambda + \delta)\tau} \right) c_0 \right] +
\]

\[
\frac{\epsilon}{4} \left( \frac{\delta}{\lambda - \delta} \right) c_0 \int_0^\tau \left( \frac{a_0}{b_0} - c_0 e^{-\delta t} \right) \left( 1 - e^{-\lambda t} \right) e^{-(\delta + \gamma)t} dt + O(\epsilon^2),
\]

where

\[
\hat{L}_0(t) = \frac{b}{4} \left( \frac{a_0}{b_0} - c_0 e^{-\delta t} \right)^2
\]

is the profit per unit of time of the firm when there are no reference-price effects.

We have developed a linear approximation for the optimal pricing policy of the current generation over \([0, \tau]\) before the new generation is introduced. We have also formed a linear approximation for the optimal profits collected from the sales of the current generation until a new generation is launched at time \( \tau \). We next develop linear approximations for the optimal pricing policies of the two concurrent generations after the launch of the new generation. This concurrent-generations problem leads to the extension of the variational problem of the independent generations, and can be approached using the standard control theory tools. For brevity, we do not include \( \kappa \) in the formulas in the rest of this section (\( \kappa = 1 \)); \( a_0 \) and \( b_0 \) refer to the demand parameter values after the new product introduction.

**Proposition 3.3.** Given a launch time for the new generation, the optimal pricing paths for the two concurrent generations over the sales period \([\tau, T]\) are
The costate evolution equation with its right-end boundary condition governed by the following linear Ordinary Differential Equation (ODE) system,

\[ \dot{q} = \left( \gamma + \alpha_0 \frac{2b_0 + \epsilon}{2b_0 + 2\epsilon} + \alpha_1 \frac{2b_1 + \epsilon}{2b_1 + 2\epsilon} \right) q - \frac{1}{2} \epsilon^2 \left( \frac{1}{b_0 + \epsilon} + \frac{1}{b_1 + \epsilon} \right) r 
- \frac{1}{2} \epsilon \left( \frac{a_0}{b_0 + \epsilon} + \frac{a_1}{b_1 + \epsilon} \right) + \frac{1}{2} \epsilon (c_0 + c_1) e^{-\delta(t-\tau)}, \]  

(47)

\[ \dot{r} = \left( \frac{\alpha_0^2}{2b_0 + 2\epsilon} + \frac{\alpha_1^2}{2b_1 + 2\epsilon} \right) q - \left( \alpha_0 \frac{2b_0 + \epsilon}{2b_0 + 2\epsilon} + \alpha_1 \frac{2b_1 + \epsilon}{2b_1 + 2\epsilon} \right) r 
+ \frac{\alpha_0}{2} \frac{a_0}{b_0 + \epsilon} + \frac{\alpha_1}{2} \frac{a_1}{b_1 + \epsilon} + \frac{1}{2} (\alpha_0 c_0 + \alpha_1 c_1) e^{-\delta(t-\tau)}, \]  

(48)

\[ \tau \leq t \leq T, \quad q(T) = 0, \quad r(\tau) = (p_0)(\tau) = r^{**}. \]  

(49)

Proof. We utilize the Pontryagin maximum principle to find optimal pricing policies. We define the current-value hamiltonian for the problem as

\[ H = (p_0 - c_0 e^{-\delta(t-\tau)}) (a_0 - b_0 - \epsilon (p_0 - r)) + 
(p_1 - c_1 e^{-\delta(t-\tau)}) (a_1 - b_1 - \epsilon (p_1 - r)) + q \left( \alpha_0 (p_0 - r) + \alpha_1 (p_1 - r) \right), \]  

(50)

where \( q \) is the current-value costate variable corresponding to the reference price state equation. As we indicated earlier, only the joint reference price \( r(t) \) exists after the introduction of the new generation. Equating the first-order derivative \( \frac{\partial H}{\partial p_i} \), \( i = 0, 1 \), to zero, we arrive to the following pricing policy that maximizes \( H \):

\[ p_i = \frac{1}{2(b_i + \epsilon)} \left( \epsilon r + \alpha_i q + a_i + (b_i + \epsilon) c_i e^{-\delta(t-\tau)} \right). \]  

(51)

The maximized hamiltonian then is

\[ H^* = \frac{1}{4(b_0 + \epsilon)} \left( \epsilon r + \alpha_0 q + a_0 - (b_0 + \epsilon) c_0 e^{-\delta(t-\tau)} \right) 
+ \frac{1}{4(b_1 + \epsilon)} \left( \epsilon r + \alpha_1 q + a_1 - (b_1 + \epsilon) c_1 e^{-\delta(t-\tau)} \right) 
+ \frac{\alpha_0 q}{2(b_0 + \epsilon)} \left( \alpha_0 q + a_0 + (b_0 + \epsilon) c_0 e^{-\delta(t-\tau)} - (2b_0 + \epsilon) r \right) 
+ \frac{\alpha_1 q}{2(b_1 + \epsilon)} \left( \alpha_1 q + a_1 + (b_1 + \epsilon) c_1 e^{-\delta(t-\tau)} - (2b_1 + \epsilon) r \right). \]  

(52)

The costate evolution equation with its right-end boundary condition

\[ \dot{q} = \gamma q - \frac{\partial H^*}{\partial r}, \quad q(T) = 0, \]  

(53)
along with the reference price evolution equations (7) and (8) form a complete system. Having solved the boundary-value problem for this second-order linear ODE system, we obtain the optimal pricing policies by substituting the costate variable $q$ into (51). Due to the concavity of the current-value Hamiltonian $H$, the problem admits a unique solution, which indeed is a maximizer of the objective functional. The linear ODE problem for $q$ and $r$ can be rewritten to arrive at the statement of the proposition.

The solution of this boundary-value problem is a linear combination of a constant vector, a constant vector times $e^{-\delta(t-\tau)}$, and two more exponentials with exponents $\Lambda_+$ and $\Lambda_-$. The exponents are defined as

$$
\Lambda_{\pm} = \gamma \frac{1}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 - \det A}, 
$$

where $A$ is the matrix defining the ODE system above. One can see that the determinant is always non-positive, therefore the exponents are always real and have opposing signs. The coefficients of the exponentials are chosen to satisfy the boundary value problem (49). These solutions for the optimal prices lead to a quite complex expression for the optimal total profit as a function of launch time $\tau$. In order to observe how reference price effects impact optimal pricing and introduction policies, we apply linear approximations to obtain closed-form expressions. The following proposition characterizes our linear approximations. For the clarity of our presentation, we assume a constant unit production cost for each generation, i.e., $\delta = 0$, for our results during concurrent generations period.

**Proposition 3.4.** Given a launch time for the new generation, the optimal pricing paths for the two concurrent generations over $[\tau, T]$ follow

$$
p_i(t) = \hat{p}_i - \frac{\epsilon a_i}{2b_i^2} + \frac{\epsilon}{2b_i} r(t) + \frac{\epsilon a_i}{4b_i(\gamma + \alpha_0 + \alpha_1)} \left( \frac{a_0}{b_0} - c_0 + \frac{a_1}{b_1} - c_1 \right) \left( 1 - e^{-(\gamma + \alpha_0 + \alpha_1)(T-t)} \right) + O(\epsilon^2),
$$

where the joint reference price is

$$
r(t) = \left( \frac{\alpha_0}{\alpha_0 + \alpha_1} \hat{p}_0 + \frac{\alpha_1}{\alpha_0 + \alpha_1} \hat{p}_1 \right) \left( 1 - e^{-(\alpha_0 + \alpha_1)(T-t)} \right) + r^{**} e^{-(\alpha_0 + \alpha_1)(t-\tau)} + O(\epsilon)
$$

where

$$
\hat{p}_i = \frac{1}{2} \left( \frac{a_i}{b_i} + c_i \right), i \in \{0, 1\}.
$$
Proof. We observe that optimal price is governed by the expression in (51). In the linear approximation, we are concerned with the zeroth order in epsilon of the ODE system (47)-(48), which for $\epsilon = 0$ becomes

\[
\dot{q} = (\gamma + \alpha_0 + \alpha_1)q + O(\epsilon),
\]

\[
\dot{r} = -(\alpha_0 + \alpha_1)r + (\alpha_0 \hat{p}_0 + \alpha_1 \hat{p}_1) + O(\epsilon),
\]

\[
q(T) = 0, r(\tau^+) = r^*,
\]

which admits a unique explicit solution

\[
q(t) = O(\epsilon),
\]

\[
r(t) = \left(\frac{\alpha_0}{\alpha_0 + \alpha_1} \hat{p}_0 + \frac{\alpha_1}{\alpha_0 + \alpha_1} \hat{p}_1\right) \left(1 - e^{-(\alpha_0 + \alpha_1)(t - \tau^+)}\right) + r^* e^{-(\alpha_0 + \alpha_1)(t - \tau^+)} + O(\epsilon),
\]

where $r^*$ is the zeroth-order approximation for the reference price of the current generation at time $\tau^-$. Using our earlier result in (42), we have

\[
r^* = \hat{p}_0 + (r^* - \hat{p}_0) e^{-\lambda T} + O(\epsilon).
\]

The zeroth order term in the costate expansion is zero on the whole time interval. In the expansion of ODE system (47), the next order (linear) terms yield (due to the trivial zeroth term)

\[
\dot{q} = (\gamma + \alpha_0 + \alpha_1)q - \frac{\epsilon}{2} \left(\frac{a_0}{b_0} - c_0 + \frac{a_1}{b_1} - c_1\right) + O(\epsilon^2),
\]

the solution for which is given by

\[
q(t) = \frac{\epsilon}{2(\gamma + \alpha_0 + \alpha_1)} \left(\frac{a_0}{b_0} - c_0 + \frac{a_1}{b_1} - c_1\right) \left(1 - e^{(\gamma + \alpha_0 + \alpha_1)(t - T)}\right) + O(\epsilon^2).
\]

Expanding (51) in terms of $\epsilon$, we obtain

\[
p_i(t) = \hat{p}_i - \frac{\epsilon a_i}{2b_i^2} + \frac{\epsilon}{2b_i} r(t) + \frac{\alpha_i}{2b_i} q(t) + O(\epsilon^2).
\]

Substituting the previous expression for $q(t)$ into the one above, we arrive at the defining formula for the optimal prices (55) for the concurrent generations at the statement of the proposition. □

We next use the linear approximations we have derived for the optimal pricing policies for both generations in Proposition 3.2 and Proposition 3.4 to characterize the optimal launch time.
4 Optimal Launch Time

Similar to the earlier derivation for the current generation over \([0, \tau]\), the optimal profit rate function for concurrent generations when optimal prices in Theorem 3.4 are followed becomes as follows,

\[
L_i(t, \rho) = \hat{L}_i + \epsilon \left( \hat{p}_i - c_i \right) \left( r(t) - \hat{p}_i \right) + O(\epsilon^2), \tag{66}
\]

where \(\rho\) stands for the initial reference price of the concurrent generations.

Substituting the expression we derived for the joint reference price in (62) and integrating the optimal profit rates, we arrive at the following expression for the optimal profit earned by generation \(i \in \{0, 1\}\) during a sales interval of \([0, t]\),

\[
R_i(t, \rho) = \hat{R}_i(t) + \epsilon (\hat{p}_i - c_i) \left[ \left( \frac{\alpha_0}{\alpha_0 + \alpha_1} \hat{p}_0 + \frac{\alpha_1}{\alpha_0 + \alpha_1} \hat{p}_1 \right) t + \left( \rho - \frac{\alpha_0}{\alpha_0 + \alpha_1} \hat{p}_0 - \frac{\alpha_1}{\alpha_0 + \alpha_1} \hat{p}_1 \right) \frac{1}{\alpha_0 + \alpha_1} \left( 1 - e^{-\left(\alpha_0 + \alpha_1\right)T} \right) \right] + O(\epsilon^2). \tag{67}
\]

Using the expressions we derived in (45) for profits from current generation over \([0, \tau]\) and in (67) for profits from concurrent generations over \([\tau, T]\), we can write down the total profits for the firm as a function of \(\tau\) for a given initial reference price \(r^*\) for the current generation. We characterize the corresponding optimal launch time maximizing the firm’s total profits from both generations in the following proposition.

**Proposition 4.1.** The optimal launch time, when the two generations are offered simultaneously after the launch and share a joint reference price is given by

\[
\tau = \hat{\tau} - \frac{\epsilon}{\gamma} \left( \hat{L} e^{-\gamma \hat{\tau}} + (\hat{L}_0 + \hat{L}_1) e^{-\gamma (T - \hat{\tau})} \right)^{-1} \left[ -\left( \hat{p} - c \right) (r^* - \hat{p}) e^{-\gamma (T - \hat{\tau})} + \sum_{i=0, 1} e^{-\gamma (T - \hat{\tau})} (\hat{p}_i - c_i) \right] \left[ (\hat{p} + (r^* - \hat{p}) e^{-\gamma \hat{\tau}}) e^{-\left(\alpha_0 + \alpha_1\right) \left( T - \hat{\tau} \right)} - \hat{p}_i + \left( \frac{\alpha_0}{\alpha_0 + \alpha_1} \hat{p}_0 + \frac{\alpha_1}{\alpha_0 + \alpha_1} \hat{p}_1 \right) \left( 1 - e^{-\left(\alpha_0 + \alpha_1\right) \left( T - \hat{\tau} \right)} \right) \right] + \frac{\lambda}{\gamma + \alpha_0 + \alpha_1} \left( \hat{p}_i - c_i \right) (r^* - \hat{p}) \left( 1 - e^{-\gamma \left( T - \hat{\tau} \right)} \right) \right] + O(\epsilon^2), \tag{68}
\]

where

\[
\hat{\tau} = \frac{T}{2} - \frac{1}{2\gamma} \log \frac{\hat{L}_0 + \hat{L}_1}{\hat{L}}, \tag{69}
\]
Proof. Considering profits from sales of the current generation over \([0, \tau]\) and concurrent generations over \([\tau, T]\), we can show that the optimal introduction time \(\tau\) satisfies the following first-order optimality condition,

\[
L(\tau, r^*) e^{-\gamma \tau} - \sum_{i=0,1} L_i(T - \tau, \rho) e^{-\gamma(T - \tau)} - \\
\lambda (r^* - \hat{p}_0) \int_0^{T - \tau} e^{-\gamma t} dt \sum_{i=0,1} \frac{\partial L_i}{\partial \rho} + O(\epsilon^2),
\]

where it is assumed that the reference price of the concurrent generations at time \(\tau\) (denoted by \(\rho\)) is given by the reference price of the previous generation:

\[
\rho = \hat{p}_0 + (r^* - \hat{p}_0) e^{-\gamma \tau} + O(\epsilon^2).
\]

Here, \(L(\tau, r^*)\) (without an index) is the profit rate function of the current generation before the launch time \(\tau\). Using the expression in (66), we obtain

\[
\frac{\partial L_i}{\partial r^*} = \epsilon (\hat{p}_i - c_i) e^{-(\alpha_0 + \alpha_1) t} + O(\epsilon^2).
\]

Substituting the expression in (66) to the one in (70) and expanding around the reference sensitivity parameter \(\epsilon\) results in the optimal launch time in the statement of the proposition.

We carry out numerical experiments with various parameters to study how our linear approximation performs by comparing optimal launch time and the corresponding linear approximation in Proposition 4.1. We use the following parameters in our experiments: consumer memory parameter \(\lambda = 3\); reference price contributions for the concurrent generations \(\alpha_0 = 1, \alpha_1 = 2\); constant demand parameter for the current generation \(a_0 = 150\); constant demand parameter for the new generation \(a_1 = \{150, 160, 170\}\); constant demand parameter \(b_0 = b_1 = 1\); unit production costs \(c_0 = 30, c_1 = 50\); reference price sensitivity parameter \(\epsilon = \{0.3, 0.5, 0.7, 0.9\}\); initial reference price for the current generation \(r^* = \{40, 60, 80\}\); drop factor for the demand for the current generation \(\kappa = 0.5\); sales length \(T = 1\); and demand decay rate \(\gamma = 0.7\). This results in 36 combinations. For each of the 36 instances generated, we observe that total profits from both generations is quasi-polynomial in launch time. We find the optimal launch time for each instance numerically. In all the instances, optimal launch times result in non-trivial expressions. Our numerical investigation, however, show us that bang-bang type launch policies may become optimal for some other combinations of model parameters, in which it is optimal either to launch the new generation immediately or to delay its launch until the current generation is off the market, depending on which generation is more profitable. We find that our perturbation approximations
for the optimal launch times under a dual-product rollover strategy result in an average gap of 3.43% from the corresponding optimal values. The gap ranges between 0.03% and 11.90%. 28 instances result in a gap that is less than 5%. Only 3 instances result in a gap that is higher than 8%. We also find that the gap of our perturbation approximations for the optimal total profits ranges between 0.02% and 5.54% with an average value of 1.23%. We also observe that the optimal launch time and the corresponding linear approximation shift towards the less profitable generation as consumers become more sensitive to reference prices through higher $\epsilon$ values.

5 Conclusion

In this paper, we develop optimal dual-product rollover strategies at a firm when consumers build price perceptions in the form of reference prices through their past price exposures. The reference price of the first (current) generation evolves as a moving average of past prices with an exponentially decaying weight. Upon launch of the second (new) generation, consumers start forming a joint reference price for both generations through their price experience from each generation while both generations coexist in the market until the first one is phased out.

We first derive explicitly the optimal pricing policy for the current generation before the next generation is introduced at a given launch time. We observe that the corresponding optimal price path is fairly cumbersome to extend our analysis to characterize the optimal launch time and quite difficult to implement in practice. As an alternative, we develop linear approximations through perturbation for the optimal dual-product rollover strategies. Our approximation approach provides us with closed-form solutions for the dual-product rollover policies when consumer demand is affected by consumer-memory through reference prices. Our results become useful for managers in conducting any sensitivity analysis as well as implementing the dual-product rollover policies in practice. Our numerical investigations illustrate that such perturbation approximations result in good performance. Our model focuses on a monopolistic firm. We intend to extend our dual-product rollover model to a duopoly game between two competing firms to determine the impact of competition on product launch times.

References


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