Different Methods of Estimation for the Parameters of Half Logistic Lomax Distribution

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Abstract

The half logistic Lomax (HLL) is introduced as a modification of Lomax distribution for modeling lifetime data by Anwar and Zahoor [2]. Maximum likelihood (ML), least squares (LS), weighted least squares (WLS), percentiles (PC) and Cramer von Mises (CV) estimation techniques for the shape and scale parameters of HLL are regarded for complete sample. Performances of the proposed ML estimators are compared with their LS, WLS, PC and CV estimators on the basis of Monte Carlo study of simulated samples in terms of their risks.

Keywords: Half logistic Lomax distribution; Least squares; percentiles; Maximum likely hood estimation

1. Introduction

Lomax [11] introduced The Lomax (L) distribution. The L distribution has established wide applications such as income and wealth inequality, medical and biological sciences, engineering, lifetime and reliability modeling. The L distribution is used for reliability modelling and life testing by Hassan and Al-Ghamdi [8]. Corbelini et al. [3] proposed it to model firm size and queuing problems.
Many researchers introduced several generalizations of the L distribution. Ghitany et al. [7] investigated the Marshall–Olkin extended L distribution, Abdul-Moniem and Abdel-Hameed [1] studied the exponentiated L distribution, Lemonte and Cordeiro [10] proposed the McDonald L, Cordeiro et al. [4] investigated the gamma L distribution. The exponential L distribution is studied by ElBassiouny et al. [6], Tahir et al. [15] introduced Weibull L, Al-Weighted L introduced by Kilany [9] and Power Lomax (PL) distribution is studied by Rady et al. [14]. Recently The HLL distribution is studied by Anwar and Zahoor [2] and it has the following cumulative distribution function (cdf) and probability density function (pdf) as

\[ F(x; \alpha, \beta) = \frac{1 - (1 + \beta x)^{-\alpha}}{1 + (1 + \beta x)^{-\alpha}}, \quad x > 0, \alpha, \beta > 0, \quad (1) \]

and

\[ f(x; \alpha, \beta) = \frac{2\alpha\beta(1 + \beta x)^{-\alpha-1}}{(1 + (1 + \beta x)^{-\alpha})^2}, \quad x > 0, \alpha, \beta > 0, \quad (2) \]

Here \( \alpha \) is a shape parameter and \( \beta \) is a scale parameter.

The survival, hazard rate and quantile functions are given respectively by:

\[ R(x; \alpha, \beta) = \frac{2}{1 + (1 + \beta x)^{-\alpha}}, \]

\[ h(x; \alpha, \beta) = \frac{\alpha\beta}{(1 + (1 + \beta x)^{-\alpha})(1 + \beta x)'}, \]

and

\[ Q(u) = \frac{1}{\beta} \left[ \left( \frac{1 - u}{1 + u} \right)^{1/\alpha} - 1 \right]. \]

The remainder of the paper is arranged as follows In Section 2, ML method of estimation used to estimate the model parameters. In Section 3, LS and WLS methods of estimation used to estimate the model parameters. In Section 4, PC method of estimation used to estimate the parameters of HLL Model. The cramèr von mises method is used to estimate the model parameters in Sections 5. Numerical results are executed in Section 6. Finally, we give some concluding remarks in Section 7.

2. ML Estimators

To get the ML estimators (MLEs) of the HLL model with of parameters \( \alpha \) and \( \beta \) let \( X_1, \ldots, X_n \) be observed values from this distribution. Hence, the log-likelihood function say \( \ell \) can be written as
\[ \ell = n \log 2 \alpha + n \log \beta - (\alpha + 1) \sum_{i=0}^{\infty} \log (1 + \beta x_i) - 2 \sum_{i=0}^{\infty} \log (1 + (1 + \beta x_i)^{-\alpha}). \]

The ML equations of the HLL distribution are given by

\[
\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=0}^{\infty} \log (1 + \beta x_i) + 2 \sum_{i=0}^{\infty} \frac{(1 + \beta x_i)^{-\alpha} \ln(1 + \beta x_i)}{(1 + (1 + \beta x_i)^{-\alpha})},
\]

and,

\[
\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - (\alpha + 1) \sum_{i=0}^{\infty} \frac{x_i}{1 + \beta x_i} + 2 \alpha \sum_{i=0}^{\infty} \frac{x_i(1 + \beta x_i)^{-\alpha - 1}}{(1 + (1 + \beta x_i)^{-\alpha})}.
\]

Equating \( \partial \ell / \partial \alpha \) and \( \partial \ell / \partial \beta \) with zeros and solving simultaneously, we obtain the ML estimators of \( \alpha \) and \( \beta \).

### 3. Ordinary and Weighted LS Estimators

Suppose \( X_1, X_2, \ldots, X_n \) is a random sample (RS) of size \( n \) from HLL distribution and suppose \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) denotes the corresponding ordered sample. The expectation and the variance of distribution are independent of the unknown parameter and are given by

\[
E(F(X_{(i)})) = \frac{i}{n+1}, \quad \text{and} \quad \text{var}(F(X_{(i)})) = \frac{i(n-i+1)}{(n+1)^2(n+2)},
\]

where \( F(X_{(i)}) \) is cdf for any distribution and \( X_{(i)} \) is the \( i^{th} \) order statistic. Then LS estimators can be calculated by minimizing the sum of squares errors,

\[
\sum_{i=1}^{n} \left[ F_{(i)}(x) - \frac{i}{n+1} \right]^2,
\]

with respect to \( \alpha \) and \( \beta \). So, the LS estimators (LsEs) of the population parameters \( \alpha \) and \( \beta \) of the LS model are obtained by minimizing the following

\[
\sum_{i=1}^{n} \left[ \frac{1 - (1 + \beta x_{(i)})^{-\alpha}}{1 + (1 + \beta x_{(i)})^{-\alpha}} - \frac{i}{n+1} \right]^2.
\]
with respect to $\alpha$ and $\beta$. Also, the LSEs of $\alpha$ and $\beta$ can be obtained by solving the next two equations

\[
\sum_{i=1}^{n} \left[ 1 - \frac{(1 + \beta x(i))^{-\alpha}}{1 + (1 + \beta x(i))^{-\alpha}} \right] - \frac{i}{n+1} \left[ 2 \left( 1 + \beta x(i) \right)^{-\alpha} \ln \left( 1 + \beta x(i) \right) \right] = 0,
\]

\[
\sum_{i=1}^{n} \left[ 1 - \frac{(1 + \beta x(i))^{-\alpha}}{1 + (1 + \beta x(i))^{-\alpha}} \right] - \frac{i}{n+1} \left( \frac{2ax(i)(1 + \beta x(i))^{-\alpha-1}}{(1 + (1 + \beta x(i))^{-\alpha})^2} \right) = 0.
\]

The WLS estimators (WLSEs) of $\alpha$ and $\beta$ can be obtained by minimizing the next function

\[
\sum_{i=1}^{n} \left( n+1 \right)^2 \left( n+2 \right) \left[ 1 - \frac{(1 + \beta x(i))^{-\alpha}}{1 + (1 + \beta x(i))^{-\alpha}} - \frac{i}{n+1} \right]^2,
\]

with respect to $\alpha$ and $\beta$. Also, the WLSEs of $\alpha$ and $\beta$ can be obtained by solving the next two equations

\[
\sum_{i=1}^{n} \left( n+1 \right)^2 \left( n+2 \right) \frac{1 - \frac{(1 + \beta x(i))^{-\alpha}}{1 + (1 + \beta x(i))^{-\alpha}}}{i(n-i+1) - \frac{i}{n+1}} \left[ 2 \left( 1 + \beta x(i) \right)^{-\alpha} \ln \left( 1 + \beta x(i) \right) \right] = 0,
\]

\[
\sum_{i=1}^{n} \left( n+1 \right)^2 \left( n+2 \right) \frac{1 - \frac{(1 + \beta x(i))^{-\alpha}}{1 + (1 + \beta x(i))^{-\alpha}}}{i(n-i+1) - \frac{i}{n+1}} \left( \frac{2ax(i)(1 + \beta x(i))^{-\alpha-1}}{(1 + (1 + \beta x(i))^{-\alpha})^2} \right) = 0.
\]

4. **PC Estimator (PCEs)**

Let $X_1, \ldots, X_n$ be a RS from the HLL distribution and Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the corresponding order statistics. Based on PC method of estimation; the estimators of set of parameters $\alpha$ and $\beta$ are derived by minimizing the following

\[
\sum_{i=1}^{n} \left[ \ln \left( \frac{i}{n+1} \right) - \ln \left( \frac{1 - \left( 1 + \beta x(i) \right)^{-\alpha}}{1 + \left( 1 + \beta x(i) \right)^{-\alpha}} \right) \right]^2,
\]
5. The Cramer-von Mises Minimum Distance Estimators

The CV estimator is a type of minimum distance estimators which is based on the difference between the estimate of the cdf and the empirical cdf (see D’Agostino and Stephens [5] and Luceño [12]). The CV estimators are obtained by minimizing

\[
CV = \frac{1}{12n} + \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[ \frac{1 - (1 + \beta x(i))^\alpha}{1 + (1 + \beta x(i))^\alpha} - \frac{2i-1}{2n} \right]^2.
\]

MacDonald [13] mentioned that the choice of CV method type minimum distance estimators providing empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators.

6. Numerical Results

A simulation study is conducted to evaluate and compare the behavior of the estimates with respect to their root mean square errors (RMSEs). We generate 1000 random samples \(X_1, \ldots, X_n\) of sizes \(n=30, 50, 100\) from HLL model. Four different sets of parameters are considered as: \(set1 \equiv (\alpha = 0.5, \beta = 0.5)\), \(set2 \equiv (\alpha = 0.8, \beta = 0.5)\), \(set3 \equiv (\alpha = 0.5, \beta = 0.3)\) and \(set4 \equiv (\alpha = 0.8, \beta = 0.3)\).

The ML, LS, WLS, PC and CV estimates of \(\alpha\) and \(\beta\) are computed. Then, the RMSEs of the estimates of the unknown parameters are computed. Simulated outcomes are listed in Table 1 and the following observations are detected:

- The RMSEs decrease as sample sizes increase for all estimates.
- The RMSEs of the ML estimates of \(\alpha\) and \(\beta\) take the smallest value among the corresponding RMSEs for the other methods in almost all of the cases.

Table 1: Estimates and RMSEs of HLL distribution for ML, LS, WLS, PC and CV estimates

<table>
<thead>
<tr>
<th>(n)</th>
<th>MLEs Estimates</th>
<th>MLEs RMS Es</th>
<th>LSEs Estimates</th>
<th>LSEs RMS Es</th>
<th>WLSEs Estimates</th>
<th>WLSEs RMS Es</th>
<th>PCEs Estimates</th>
<th>PCEs RMS Es</th>
<th>CVEs Estimates</th>
<th>CVEs RMS Es</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.552</td>
<td>0.154</td>
<td>0.526</td>
<td>0.172</td>
<td>0.52</td>
<td>0.142</td>
<td>0.702</td>
<td>0.304</td>
<td>0.565</td>
<td>0.201</td>
</tr>
<tr>
<td></td>
<td>0.527</td>
<td>0.392</td>
<td>0.637</td>
<td>0.496</td>
<td>0.62</td>
<td>0.46</td>
<td>0.437</td>
<td>0.385</td>
<td>0.541</td>
<td>0.38</td>
</tr>
<tr>
<td>50</td>
<td>0.534</td>
<td>0.105</td>
<td>0.517</td>
<td>0.112</td>
<td>0.52</td>
<td>0.106</td>
<td>0.695</td>
<td>0.274</td>
<td>0.536</td>
<td>0.127</td>
</tr>
<tr>
<td></td>
<td>0.502</td>
<td>0.255</td>
<td>0.562</td>
<td>0.288</td>
<td>0.546</td>
<td>0.298</td>
<td>0.389</td>
<td>0.261</td>
<td>0.51</td>
<td>0.263</td>
</tr>
<tr>
<td>100</td>
<td>0.518</td>
<td>0.065</td>
<td>0.512</td>
<td>0.073</td>
<td>0.515</td>
<td>0.07</td>
<td>0.692</td>
<td>0.24</td>
<td>0.521</td>
<td>0.078</td>
</tr>
<tr>
<td></td>
<td>0.486</td>
<td>0.165</td>
<td>0.518</td>
<td>0.196</td>
<td>0.505</td>
<td>0.182</td>
<td>0.341</td>
<td>0.225</td>
<td>0.497</td>
<td>0.197</td>
</tr>
</tbody>
</table>
### Continued of Table 1

#### Set 2: $\alpha = 0.8, \beta = 0.5$

<table>
<thead>
<tr>
<th>$n$</th>
<th>MLEs RMS Estim.</th>
<th>LSEs RMS Estim.</th>
<th>WLSEs RMS Estim.</th>
<th>PCEs RMS Estim.</th>
<th>CVEs RMS Estim.</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.924 0.378</td>
<td>0.898 0.407</td>
<td>0.877 0.35</td>
<td>1.347 3.455</td>
<td>0.992 0.502</td>
</tr>
<tr>
<td></td>
<td>0.525 0.345</td>
<td>0.609 0.441</td>
<td>0.595 0.391</td>
<td>0.446 0.666</td>
<td>0.516 0.358</td>
</tr>
<tr>
<td>50</td>
<td>0.872 0.228</td>
<td>0.866 0.289</td>
<td>0.849 0.23</td>
<td>1.148 0.435</td>
<td>0.891 0.294</td>
</tr>
<tr>
<td></td>
<td>0.514 0.249</td>
<td>0.562 0.311</td>
<td>0.552 0.289</td>
<td>0.384 0.245</td>
<td>0.516 0.272</td>
</tr>
<tr>
<td>100</td>
<td>0.824 0.13</td>
<td>0.836 0.206</td>
<td>0.82 0.142</td>
<td>1.16 0.414</td>
<td>0.866 0.204</td>
</tr>
<tr>
<td></td>
<td>0.513 0.179</td>
<td>0.533 0.223</td>
<td>0.528 0.195</td>
<td>0.344 0.227</td>
<td>0.49 0.193</td>
</tr>
</tbody>
</table>

#### Set 3: $\alpha = 0.5, \beta = 0.3$

<table>
<thead>
<tr>
<th>$n$</th>
<th>MLEs RMS Estim.</th>
<th>LSEs RMS Estim.</th>
<th>WLSEs RMS Estim.</th>
<th>PCEs RMS Estim.</th>
<th>CVEs RMS Estim.</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.542 0.155</td>
<td>0.512 0.165</td>
<td>0.515 0.138</td>
<td>0.599 0.246</td>
<td>0.565 0.191</td>
</tr>
<tr>
<td></td>
<td>0.33 0.219</td>
<td>0.409 0.298</td>
<td>0.384 0.287</td>
<td>0.372 0.426</td>
<td>0.329 0.239</td>
</tr>
<tr>
<td>50</td>
<td>0.532 0.106</td>
<td>0.513 0.11</td>
<td>0.516 0.103</td>
<td>0.586 0.182</td>
<td>0.537 0.126</td>
</tr>
<tr>
<td></td>
<td>0.305 0.158</td>
<td>0.35 0.204</td>
<td>0.337 0.189</td>
<td>0.322 0.268</td>
<td>0.314 0.183</td>
</tr>
<tr>
<td>100</td>
<td>0.511 0.063</td>
<td>0.505 0.071</td>
<td>0.51 0.068</td>
<td>0.584 0.157</td>
<td>0.519 0.075</td>
</tr>
<tr>
<td></td>
<td>0.306 0.108</td>
<td>0.325 0.13</td>
<td>0.314 0.118</td>
<td>0.28 0.153</td>
<td>0.301 0.117</td>
</tr>
</tbody>
</table>

#### Set 4: $\alpha = 0.8, \beta = 0.3$

<table>
<thead>
<tr>
<th>$n$</th>
<th>MLEs RMS Estim.</th>
<th>LSEs RMS Estim.</th>
<th>WLSEs RMS Estim.</th>
<th>PCEs RMS Estim.</th>
<th>CVEs RMS Estim.</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.907 0.329</td>
<td>0.848 0.371</td>
<td>0.867 0.365</td>
<td>0.983 0.868</td>
<td>0.981 0.502</td>
</tr>
<tr>
<td></td>
<td>0.319 0.217</td>
<td>0.39 0.28</td>
<td>0.373 0.263</td>
<td>0.344 0.295</td>
<td>0.316 0.23</td>
</tr>
<tr>
<td>50</td>
<td>0.869 0.229</td>
<td>0.835 0.25</td>
<td>0.842 0.241</td>
<td>0.962 0.298</td>
<td>0.877 0.273</td>
</tr>
<tr>
<td></td>
<td>0.309 0.156</td>
<td>0.355 0.202</td>
<td>0.337 0.18</td>
<td>0.309 0.203</td>
<td>0.328 0.188</td>
</tr>
<tr>
<td>100</td>
<td>0.83 0.131</td>
<td>0.812 0.15</td>
<td>0.82 0.141</td>
<td>0.976 0.249</td>
<td>0.833 0.164</td>
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<tr>
<td></td>
<td>0.303 0.108</td>
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<td>0.315 0.118</td>
<td>0.254 0.104</td>
<td>0.314 0.121</td>
</tr>
</tbody>
</table>

### 7. Concluding Remarks

The two-parameter HLL distribution is considered for deriving ML, LS, WLS, PC and CV estimations for parameters based on simple random sample. The ML, LS,
WLS, PC and CV is obtained by using Mathcad 14.0. The performances of the estimates are conducted by using the RMSE.

References


**Received: February 5, 2019; Published: February 20, 2019**