Integrals with Respect to Gradual Number-valued Measures

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Abstract

In this paper, we firstly introduce the concept of gradual number-valued integrals of real-valued measurable functions with respect to a gradual number-valued measure. And then, we investigate some of its properties and obtain convergence theorems.

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1 Introduction

Motivated by the applications in several areas of applied science, such as mathematical economics, fuzzy optimal, process control and decision theory, much effort has been devoted to the generalization of different measure and integral concepts and classical results to the case when outcomes of a random experiment are represented by fuzzy sets, such as the concepts of fuzzy measures (see, e.g., [2,11,12,14]) and fuzzy integrals (see, e.g., [7,8]).

In above literatures, the most often proposed fuzzy sets are fuzzy numbers. It is well known that the term “fuzzy numbers” are often applied instead of “fuzzy intervals”, especially if the core of fuzzy interval is a point (like; triangular fuzzy number). But such fuzzy numbers generalize intervals, not
numbers. Furthermore, fuzzy arithmetics inherit algebraic properties of interval arithmetic, not of numbers. Hence the name “fuzzy number”, used by many authors is debatable. To avoid this confusion, the authors [4] introduced a new concept in fuzzy set theory as “gradual numbers”. A gradual number in general can not be considered as a fuzzy set of real numbers because the mapping from the unit interval to the real line is not necessarily one to one. However gradual numbers are equipped with the same algebraic structures as numbers (addition is a group, etc.). In the brief time since their introduction, gradual numbers have been employed as tools for computations on fuzzy intervals, with applications to combinatorial fuzzy optimization (see, e.g., [5,6,10]), and others (see, e.g., [1,3,4,9,13,15]). In particular, Zhou and Li [16] recently introduced new fuzzy measure concept based on gradual numbers. This paper is the continuation of former work [16] and it focuses on fuzzy integrals of real-valued measurable functions with respect to a gradual number-valued measure.

The organization of the paper is as follows. In Section 2, we state some basic results about gradual numbers. In Section 3, the concept of gradual number-valued integrals of real-valued measurable functions with respect to a gradual number-valued measure is introduced and some of its properties are investigated. Finally, convergence theorems for this kind of integral are obtained.

2 Preliminaries

In this section, we state some basic results about gradual numbers.

**Definition 2.1.** [4] A gradual number \( \tilde{r} \) is defined by an assignment function

\[
A_{\tilde{r}} : (0, 1] \rightarrow \mathbb{R}.
\]

Naturally a nonnegative gradual number is defined by its assignment function from \((0, 1]\) to \([0, +\infty)\).

In the sequel, \( \tilde{r}(\alpha) \) may be substituted for \( A_{\tilde{r}}(\alpha) \). The set of all gradual numbers (resp. nonnegative gradual numbers) is denoted by \( \mathbb{R}(I) \) (resp. \( \mathbb{R}^+(I) \)). A crisp element \( b \in \mathbb{R} \) has its own assignment function \( \tilde{b} : (0, 1] \rightarrow \mathbb{R} \) defined by \( \tilde{b}(\alpha) = b \) for each \( \alpha \in (0, 1] \). We call such elements in \( \mathbb{R}(I) \) constant gradual numbers. In particular, \( \tilde{0} \) (resp. \( \tilde{1} \)) denotes constant gradual number defined by \( \tilde{0}(\alpha) = 0 \) (resp. \( \tilde{1}(\alpha) = 1 \)) for all \( \alpha \in (0, 1] \).

**Definition 2.2.** [4] Let \( \tilde{r}, \tilde{s} \in \mathbb{R}(I) \). The operations are defined as follows:

1. \( (\tilde{r} + \tilde{s})(\alpha) = \tilde{r}(\alpha) + \tilde{s}(\alpha), \forall \alpha \in (0, 1] \);
2. \( (\tilde{r} - \tilde{s})(\alpha) = \tilde{r}(\alpha) - \tilde{s}(\alpha), \forall \alpha \in (0, 1] \);
Let $\tilde{r}, \tilde{s} \in \mathbb{R}(I)$. The relations between $\tilde{r}$ and $\tilde{s}$ are defined as follows:

1. $\tilde{r} = \tilde{s}$ if and only if $\tilde{r}(\alpha) = \tilde{s}(\alpha), \forall \alpha \in (0, 1]$;
2. $\tilde{r} \geq \tilde{s}$ if and only if $\tilde{r}(\alpha) \geq \tilde{s}(\alpha), \forall \alpha \in (0, 1]$;
3. $\tilde{r} \leq \tilde{s}$ if and only if $\tilde{r}(\alpha) \leq \tilde{s}(\alpha), \forall \alpha \in (0, 1]$;
4. $\tilde{r} \succ \tilde{s}$ if and only if $\tilde{r}(\alpha) > \tilde{s}(\alpha), \forall \alpha \in (0, 1]$;
5. $\tilde{r} \prec \tilde{s}$ if and only if $\tilde{r}(\alpha) < \tilde{s}(\alpha), \forall \alpha \in (0, 1]$.

Definition 2.4. [16] Let $\{\tilde{r}_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}(I)$ and $\tilde{r} \in \mathbb{R}(I)$.

1. $\{\tilde{r}_n\}_{n \in \mathbb{N}}$ is said to converge to $\tilde{r}$ if for each $\alpha \in (0, 1]$, $\lim_{n \to \infty} \tilde{r}_n(\alpha) = \tilde{r}(\alpha)$ and it is denoted as $\lim_{n \to \infty} \tilde{r}_n = \tilde{r}$. If there is no such an $\tilde{r}$, the sequence $\{\tilde{r}_n\}_{n \in \mathbb{N}}$ is said to be divergent.

2. If $\lim_{n \to \infty} \sum_{i=1}^{n} \tilde{r}_i$ exists, then the infinite sum of sequence $\{\tilde{r}_n\}_{n \in \mathbb{N}}$ is defined by $\sum_{i=1}^{\infty} \tilde{r}_i = \lim_{n \to \infty} \sum_{i=1}^{n} \tilde{r}_i$.

Definition 2.5. [16] Let $(X, \mathcal{A})$ be a measurable space. A mapping $\tilde{m} : \mathcal{A} \to \mathbb{R}^+(I)$ is called a gradual number-valued measure if it satisfies the following two conditions:

1. $\tilde{m}(\emptyset) = 0$;
2. if $A_1, A_2, \ldots$ are in $\mathcal{A}$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, then $\tilde{m}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \tilde{m}(A_i)$.

The second condition is called countable additivity of the gradual number-valued measure $\tilde{m}$. We say that $(X, \mathcal{A}, \tilde{m})$ is a gradual number-valued measure space.

All concepts and results not given in this paper may be found in [4,10,16].

3 Main Results

In this section, we introduce the integral of real-valued measurable functions with respect to a gradual number-valued measure and investigate its properties.

Definition 3.1. Let $(X, \mathcal{A}, \tilde{m})$ be a gradual number-valued measure space, $f : X \to \mathbb{R}$ a measurable function and $A \in \mathcal{A}$. The gradual number-valued integral $\tilde{\int}_A f \, d\tilde{m} : (0, 1] \to \mathbb{R}(I)$ of $f$ with respect to $\tilde{m}$ on $A$ is defined as follows:

$$\left(\tilde{\int}_A f \, d\tilde{m}\right)(\alpha) = \int_A f \, d\tilde{m}_\alpha, \forall \alpha \in (0, 1].$$
Theorem 3.2. Let \( f, g \) and \( \{ f_n \}_{n \in \mathbb{N}} \) be measurable functions on the gradual number-valued measure space \((X, \mathcal{A}, \tilde{m})\).

1. If \( f \) is non-negative, then \( \int_X f \, d\tilde{m} \geq \tilde{0} \);
2. \( \int_X (f \pm g) \, d\tilde{m} = \int_X f \, d\tilde{m} \pm \int_X g \, d\tilde{m} \);
3. if \( f \leq g \), then \( \int_X f \, d\tilde{m} \leq \int_X g \, d\tilde{m} \);
4. if \( f = 0 \), then \( \int_X f \, d\tilde{m} = \tilde{0} \);
5. if \( \tilde{m}(A) = \tilde{0} \), then \( \int_A f \, d\tilde{m} = \tilde{0} \);
6. \( \int_X (f \lor g) \, d\tilde{m} = \int_X f \, d\tilde{m} \lor \int_X g \, d\tilde{m} \);
7. \( \int_X (f \land g) \, d\tilde{m} = \int_X f \, d\tilde{m} \land \int_X g \, d\tilde{m} \);
8. if \( A, B \in \mathcal{A} \), \( A \cup B = C \) and \( A \cap B = \emptyset \), then

\[
\int_C f \, d\tilde{m} = \int_A f \, d\tilde{m} + \int_B f \, d\tilde{m}.
\]

Proof. (1) If \( f \) is non-negative, then for each \( \alpha \in (0, 1] \), we have

\[
\int_X f \, d\tilde{m}_\alpha = \left( \int_X f \, d\tilde{m} \right)(\alpha) \geq 0.
\]

This implies that \( \int_X f \, d\tilde{m} \geq \tilde{0} \).

(2) For each \( \alpha \in (0, 1] \), we have

\[
\left( \int_X (f \pm g) \, d\tilde{m} \right)(\alpha) = \int_X (f \pm g) \, d\tilde{m}_\alpha
\]

\[
= \int_X f \, d\tilde{m}_\alpha \pm \int_X g \, d\tilde{m}_\alpha
\]

\[
= \left( \int_X f \, d\tilde{m} \right)(\alpha) \pm \left( \int_X g \, d\tilde{m} \right)(\alpha)
\]

\[
= \left[ \int_X f \, d\tilde{m} \pm \int_X g \, d\tilde{m} \right](\alpha).
\]

It follows that

\[
\int_X (f \pm g) \, d\tilde{m} = \int_X f \, d\tilde{m} \pm \int_X g \, d\tilde{m}.
\]

(3) In the same manner as in the proof of (2), we can obtain the conclusion.

(4) If \( f = 0 \), then for each \( \alpha \in (0, 1] \), we have

\[
\left( \int_X f \, d\tilde{m} \right)(\alpha) = \int_X f \, d\tilde{m}_\alpha = 0.
\]
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It follows that $\tilde{\int}_X f \, d\tilde{m} = 0$.

(5) If $\tilde{m}(A) = 0$, then for each $\alpha \in (0, 1]$, $\tilde{m}_\alpha(A) = 0$. It follows that

$$\left(\tilde{\int}_A f \, d\tilde{m}\right)(\alpha) = \int_A f \, d\tilde{m}_\alpha = 0$$

for each $\alpha \in (0, 1]$. This implies that $\tilde{\int}_A f \, d\tilde{m} = 0$.

(6) For each $\alpha \in (0, 1]$, we have

$$\left(\tilde{\int}_X (f \vee g) \, d\tilde{m}\right)(\alpha) = \int_X (f \vee g) \, d\tilde{m}_\alpha$$

$$= \int_X f \, d\tilde{m}_\alpha \vee \int_X g \, d\tilde{m}_\alpha$$

$$= \left[\left(\tilde{\int}_X f \, d\tilde{m}\right)(\alpha)\right] \vee \left[\left(\tilde{\int}_X g \, d\tilde{m}\right)(\alpha)\right]$$

$$= \left[\int_X f \, d\tilde{m} \vee \int_X g \, d\tilde{m}\right](\alpha).$$

It follows that

$$\tilde{\int}_X (f \vee g) \, d\tilde{m} = \tilde{\int}_X f \, d\tilde{m} \vee \tilde{\int}_X g \, d\tilde{m}.$$

(7) In the same manner as in the proof of (6), we can obtain the conclusion.

(8) Suppose that $A, B \in \mathcal{A}$, $A \cup B = C$ and $A \cap B = \emptyset$. Then for each $\alpha \in (0, 1]$, we have

$$\left(\tilde{\int}_C f \, d\tilde{m}\right)(\alpha) = \int_C f \, d\tilde{m}_\alpha$$

$$= \int_A f \, d\tilde{m}_\alpha + \int_B f \, d\tilde{m}_\alpha$$

$$= \left(\tilde{\int}_A f \, d\tilde{m} + \tilde{\int}_B f \, d\tilde{m}\right)(\alpha).$$

It follows that $\tilde{\int}_C f \, d\tilde{m} = \tilde{\int}_A f \, d\tilde{m} + \tilde{\int}_B f \, d\tilde{m}$. This completes the proof. \qed

In the following, we give several convergence theorems for this kind of integrals.

Theorem 3.3. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of non-negative measurable functions on the gradual number-valued measure space $(X, \mathcal{A}, \tilde{m})$. Suppose that

(1) $f_1 \leq f_2 \leq \cdots \leq f_n \leq \cdots$;
(2) \( \lim_{n \to \infty} f_n = f \).

Then we have

\[
\lim_{n \to \infty} \int_X f_n \, d\tilde{m} = \int_X f \, d\tilde{m}.
\]

**Proof.** By classical Lebesgue’s Monotone Convergence Theorem, for each \( \alpha \in (0, 1] \), we have

\[
\lim_{n \to \infty} \int_X f_n \, d\tilde{m}_\alpha = \int_X f \, d\tilde{m}_\alpha.
\]

It follows that

\[
\left( \lim_{n \to \infty} \int_X f_n \, d\tilde{m} \right)(\alpha) = \lim_{n \to \infty} \int_X f_n \, d\tilde{m}_\alpha = \int_X f \, d\tilde{m}_\alpha = \left( \int_X f \, d\tilde{m} \right)(\alpha)
\]

for each \( \alpha \in (0, 1] \). This implies that

\[
\lim_{n \to \infty} \int_X f_n \, d\tilde{m} = \int_X f \, d\tilde{m}.
\]

This completes the proof. \( \Box \)

**Theorem 3.4.** Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of non-negative measurable functions on the gradual number-valued measure space \((X, \mathcal{A}, \tilde{m})\). If \( \liminf_{n \to \infty} f_n \) and \( \liminf_{n \to \infty} \int_X f_n \, d\tilde{m} \) exist, then

\[
\int_X \liminf_{n \to \infty} f_n \, d\tilde{m} \leq \liminf_{n \to \infty} \int_X f_n \, d\tilde{m}.
\]

**Proof.** If \( \liminf_{n \to \infty} f_n \) and \( \liminf_{n \to \infty} \int_X f_n \, d\tilde{m} \) exist, then by classical Fatou’s Lemma, for each \( \alpha \in (0, 1] \), we have

\[
\int_X \liminf_{n \to \infty} f_n \, d\tilde{m}_\alpha \leq \liminf_{n \to \infty} \int_X f_n \, d\tilde{m}_\alpha.
\]
It follows that
\[
\left( \int_X \liminf_{n \to \infty} f_n \, d\tilde{m} \right)(\alpha) = \int_X \lim\inf_{n \to \infty} f_n \, d\tilde{m}_\alpha \\
\leq \lim\inf_{n \to \infty} \int_X f_n \, d\tilde{m}_\alpha \\
= \lim\inf_{n \to \infty} \int_X f_n \, d\tilde{m}_\alpha \\
= \left( \lim\inf_{n \to \infty} \int_X f_n \, d\tilde{m} \right)(\alpha)
\]
for each \( \alpha \in (0, 1] \), which implies that
\[
\int_X \liminf_{n \to \infty} f_n \, d\tilde{m} \leq \lim\inf_{n \to \infty} \int_X f_n \, d\tilde{m}.
\]
This completes the proof.

**Theorem 3.5.** Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of measurable functions on the gradual number-valued measure space \((X, \mathcal{A}, \tilde{m})\) such that \( \lim_{n \to \infty} f_n = f \). If there exists a non-negative integrable function \( g(x) \) such that \( |f_n(x)| \leq g(x) \) for any \( n \in \mathbb{N} \) and \( x \in X \), then
\[
\lim_{n \to \infty} \int_X f_n \, d\tilde{m} = \int_X f \, d\tilde{m}.
\]

**Proof.** Since \( |f_n(x)| \leq g(x) \) for any \( n \in \mathbb{N} \) and \( x \in X \), then, by classical Lebesgue’s Dominated Convergence Theorem, for each \( \alpha \in (0, 1] \), we have
\[
\lim_{n \to \infty} \int_X f_n \, d\tilde{m}_\alpha = \int_X f \, d\tilde{m}_\alpha.
\]
It follows that
\[
\left( \int_X \liminf_{n \to \infty} f_n \, d\tilde{m} \right)(\alpha) = \lim_{n \to \infty} \int_X f_n \, d\tilde{m}_\alpha \\
= \int_X f \, d\tilde{m}_\alpha \\
= \left( \int_X f \, d\tilde{m} \right)(\alpha)
\]
for each \( \alpha \in (0, 1] \), which implies that
\[
\lim_{n \to \infty} \int_X f_n \, d\tilde{m} = \int_X f \, d\tilde{m}.
\]
This completes the proof.

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