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Simulation of the Dynamic Two-Dimensional Navier-Stokes Equation Applying the Galerkin Methodology

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Abstract

A study was carried out based on the discrete approximations of the model and the numerical integration; presenting the mixed formulation of the Navier-Stokes equations that consists of a linear system and a non-linear part of partial differential equations. To finally obtain the numerical solutions of the Navier-Stokes partial differential equations. In order to validate the mathematical development in the numerical solution applying the Galerkin methodology and proceeded to make two computer programs that allows solving and visualizing the solutions of flow equations in laminar regime between parallel planes and stationary incompressible laminar boundary layer on the wall, subject to arbitrary pressure gradients.

Keywords: Mathematical models, Navier-Stokes equations, Numerical simulation

Introduction

The Navier-Stokes equations model the behavior of fluids by conserving 3 quantities, namely: mass, amount of movement and energy [1]. At the beginning of the computer age these models could not be used in simulations given their high computational and mathematical complexity, instead potential flow models and later Eulerian descriptions were used [2].

As computer technology evolved, these models began to be used more and more frequently until it is now natural to simulate complex systems where fluids are resolved by a Navier-Stokes formulation [3]. The basic concepts of functional analysis are introduced, subsequently being used to develop what is proposed in this final report [4].

In the following work we analyze the results obtained through the visualization using the developed program, based on the application of the free Galerkin method, for fluid studies in laminar regime between two flat plates and laminar limit layer.

For the study, the distributions of the velocity profiles corresponding to each one of the applications are evaluated. The results obtained are graphically represented for a better interpretation, using the visualization program, which allows generating a graphic file by using a subroutine executed in Visual Scilab, thus facilitating the export of the necessary data for the realization of the profiles or distributions mentioned above. Among the improvements of the program with respect to similar works, it is worth highlighting:

• The mathematical development in each one of the operators giving as results expressions that can be easily codified. Similarly, non-linear differential equations by transforming them into non-linear algebraic equations by developing the trilinial matrices; It is one of the most significant contributions because this is a subject that is under investigation.

• The direct delivery of the integrals of each one of the operators that constitute the Navier Stokes equations, which allows to improve the performance of the program, since the necessary memory space is used. It was not necessary to use a supercomputer for the program to work very quickly.

• The organization of the program in functional structures that can be connected causally or directed to object.

Flow analysis in laminar regime between parallel planes

Figure 1 shows a visualization of the application of laminar flow between parallel planes, where the fluid in motion has the "sheets" well defined, each traveling at its speed bordering on those of the side, without mixing with each other [5]. The laminar regime is characterized by an ordered movement of the fluid particles, with well-defined current lines and trajectories.

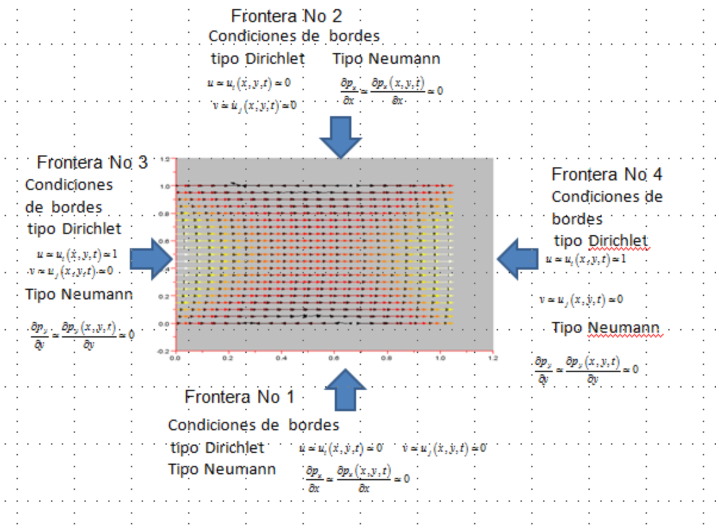

Figure 1 Laminar flow between parallel planes with edge conditions

Input effect

A phenomenon closely related to the development of the boundary layer is the entry (or embouchure) effect, due to which the asymptotic stationary profile that is established only at a certain distance from the embouchure of the parallel planes [6]. Figure 2 schematizes this process for the flow between two semi-infinite parallel faces separated by a distance h. At a small distance x from the embouchure, there is an almost uniform velocity profile whose modulus is equal to U, the velocity upstream. The transition with the null velocity condition on the surface of the plates takes place within a layer of (local) thickness Δx.

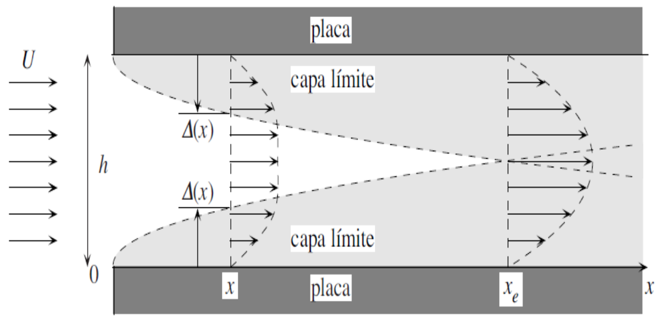

Figure 2 Embouchure effect for the flow between semi-infinite flat plates: evolution of the velocity profile with the distance from the entrance

As we move away from the embouchure, the thickness of the boundary layers of both plates grows until both ends meet at a distance xe. Therefore, x is the distance necessary for the stationary parabolic velocity profile that we already know to be established between the plates. Where Re (h) is the Reynolds number constructed with the velocity U and the distance h between plates [7].

Speed profile analysis

The analysis in Figure 3 shows that the simulation of the Navier Stokes equation with the developed algorithm is valid for the application of two-dimensional laminar flow between parallel planes.

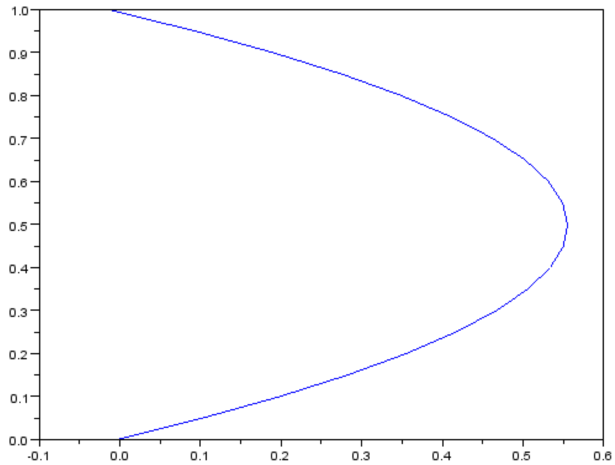

Figure 3 Laminar flow velocity profile between parallel planes

Validation of the results of the laminar limit layer

Then a brief description of the analysis of the results of the application of the laminar boundary layer is presented and the results obtained are presented.

Analysis of the laminar flow of the boundary layer

Continuation in Figure 4 shows the visualization of the application of laminar flow in the boundary layer, where the analysis is performed to validate the results obtained with the Galerkin method. We proceeded to run the algorithm for different simple boundary layer models whose results are known as there is an analytical solution when simplifying the Navier-Stokes equations [8]. According to the results of the simulations; it is observed that the velocity profile in the laminar limit layer regime is parabolic, being zero in the wall and parabolically increasing its speed as shown in the figures.

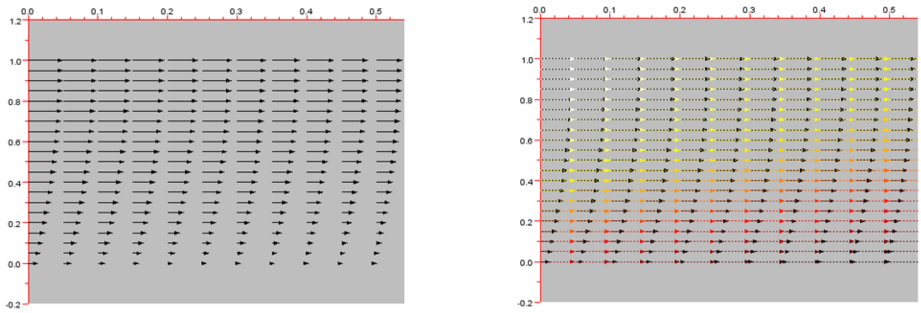

Figure 4 Limit layer laminar flow sequence

In this case, the fluid in motion has the well-defined "sheets", each one traveling at its speed rubbing one next to the other, without mixing with each other. The laminar regime is characterized by an ordered movement of the fluid particles, with well-defined current lines and trajectories. In this case the forces of inertia prevail inside the fluid, while the effects of viscosity here almost disappear [9]. Consider the flat stream of a fluid with low viscosity around a sharp body. The table of current lines and even the distribution of speeds widely coincide with those of a current without friction. However, next to the surface the fluid does not slide like the potential current, but stops.

The transition of the null speed, next to the wall, at full speed at a certain distance from the body, takes place within a very thin layer, called the boundary layer. In this it distinguishes two regions:

• A very thin layer in immediate contact with the body, inside which the velocity gradient in the direction perpendicular to the wall is very large.

• The rest of the field, outside this layer, where a large velocity gradient does not appear, and where, therefore, the effect of viscosity is unimportant.

In general, it can be said that the boundary layer is so much thinner when the viscosity is lower, or in other words, the higher the Reynolds number. Basing on some exact solutions of the Navier-Stokes equations, it can be said that the thickness of the boundary layer is proportional to the square root of the kinematic viscosity.

Speed profile analysis in laminar limit layer

i) A solution is obtained by means of a mesh-free polynomial Galerkin scheme for the viscous, non-linear Navier-Stokes equation, two-dimensional stationary case, by means of a least squares minimization scheme. The scheme is innovative in that it does not depend on an underlying triangulation, neither for the evaluation of the weight functions nor for the evaluation of the edge conditions. This new

methodology will allow the application of this same methodology in conditions of mobile frontiers or in shape optimization schemes.
ii) The solutions obtained are validated in simple cases against the values given in the literature and against known solutions for viscous flow in the laminar boundary layer, as is the case with the Blasius equation. The coincidence of the numerical results obtained with the proposed methodology is satisfactory.
iii) The Galerkin weighted residual scheme applied to the complete Navier-Stokes equation with its non-linear component generally leads to computationally demanding models. The weakening of the solutions is not
imposes, but prefers to work with a procedure that does not resort to this process. In return, the use of polynomial weight functions allows the direct evaluation of the integrals involved reducing consumption
.computational
iv) The optimization process follows a standard least squares scheme that uses the gradient of the trilinear term previously calculated as a matrix.

Analytical solution for laminar flow in boundary layer

Initially we solved the Navier Stokes equation in an analytically simplified way to find the Blasius Laminar Boundary Layer solution as shown in Figure 5, where the blue color velocity profile as shown in the figure reaches its maximum value when the value of y = 1.0 and the value of x = 5; corresponding to the maximum value of Reynolds for laminar flow [10].

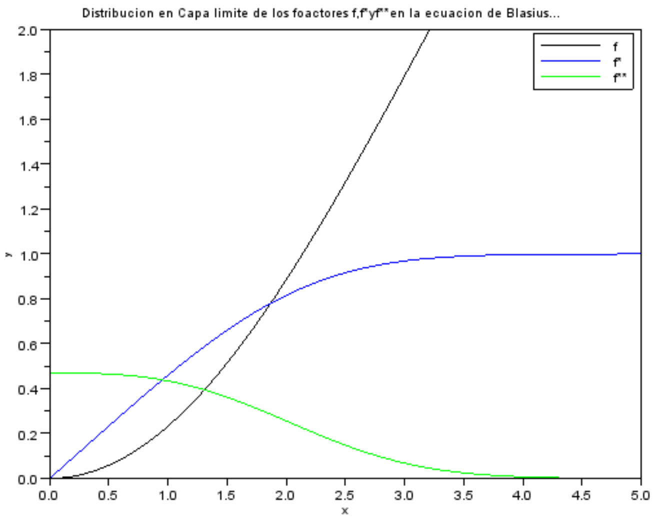

Figure 5 Representation of the Blasius equation

Next, the results of the Blasius solution are compared with the final work software. Where the profile of limit layer velocities with the solution of the Blasius equation for the case of the laminar limit layer on a flat plate coincides with one of the velocity profiles of the Navier-Stokes equation. In

Figure 6 we show the graph of the Blasius equation with its normalized values on the different simulated profiles with the Navier-Stokes equation with the Galerkin methodology. In the same way, different velocity profiles are shown for polynomial functions of different order; we found that the blue line corresponds to the Blasius solution with polynomials of order3 and the green one corresponds to the Blasius solution with polynomials of order two (2). The triangles correspond to the analytical solution of the Blasius solution and the complete numerical solution. The numerical solutions of the complete Navier Stokes equations corresponding to order three (3) and four (4) present abrupt variations of the pressure gradient.

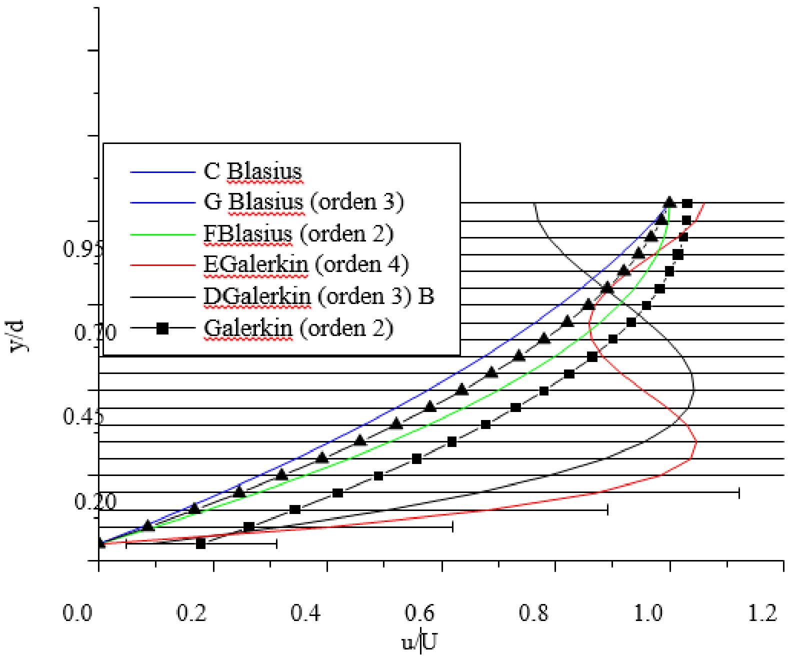

Figure 6 Graphs of velocity profiles for different polynomial order

Conclusion

From the results obtained, the following conclusions can be presented: The problem presented by the fact that the convective term is much greater than the viscous or diffuse is well known in the field of numerical methods. This fact is produced for high numbers of Reynolds and causes that the classic numerical solution of the method of the finite elements (scheme developed from Galerkin's method) fails and oscillations appear throughout the domain. The aforementioned oscillations disappear as the mesh size decreases, something that from the practical-computational point of view is unthinkable due to its slowness and enormous memory expenditure. From the physical point of view, it is known that when applying the Galerkin method, negative viscosities proportional to the Reynolds number are added, causing a bad stability to the problem and giving rise to oscillations. These oscillations tend to be localized and do not tend to propagate in linear systems, but since the Navier Stokes equations have non-linear terms that at high Reynolds numbers have great importance, global instabilities are reached. In addition, the behavior of the boundary layer was calculated with the simplified analytical solution corresponding to the Blasius equation, and no discrepancy was found.

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