Independent 2-Domination in Graphs

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Abstract

An independent 2-dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex not in $S$ is dominated at least twice and every pair of vertices in $S$ are not adjacent. In this paper, we characterized the independent 2-dominating sets of the join of graphs and its independent 2-domination number is obtained. Also, a connected graph with a pre-assigned order, domination number, independent domination number, and independent 2-domination number is constructed.

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1 Introduction and Preliminary Results

Let $G = (V(G), E(G))$ be an undirected graph. A subset $S$ of $V(G)$ is an independent set of $G$ if for every $u, v \in S$, $uv \notin E(G)$.

A subset $S$ of $V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The domination number $\gamma(G)$ of $G$ is the smallest cardinality of a dominating set of $G$. A dominating set $S$ is an independent dominating set of $G$ if $S$ is an independent set of $G$. The independent domination number $i(G)$ of $G$ is the smallest cardinality of an independent dominating set of $G$. A subset $S$ of $V(G)$ is a 2-dominating set of
Let $G$ be a graph of order $n \geq 2$. If $S$ is an independent 2-dominating set of $G$, then $S$ must contain the leaves of $G$.

**Proof**: Let $v$ be a leaf of $G$. Suppose $v \notin S$. Since $S$ is a dominating set of $G$, there exists a unique $u \in S$ such that $uv \in E(G)$. This means that $N_G(v) = \{u\}$. Thus, $|S \cap N_G(v)| = 1$. This contradicts the hypothesis that $S$ is a 2-dominating set of $G$. Therefore, $v \in S$. \qed

If $S \subseteq V(G)$ is an independent 2-dominating set of $G$, then by the definition $2 \leq |S| \leq |V(G)|$. Hence, the following results follows.

**Remark 1.2** Let $G$ be a graph of order $n \geq 2$. Then $2 \leq i_2(G) \leq n$.

**Remark 1.3** Let $G$ be a graph of order $n \geq 2$. Then $i_2(G) = n$ if and only if $G = K_n$.

Every independent 2-dominating set is an independent dominating set. Thus, the next result follows.

**Remark 1.4** Let $G$ be a graph of order $n \geq 2$. Then $i(G) \leq i_2(G)$.
2 Main Results

Theorem 2.1 Given positive integers $a$, $b$, and $n$ with $4 \leq a \leq b$ and $n = 2a+b-2$, there exists a graph $G$ such that $i(G) = a$, $i_2(G) = b$, and $|V(G)| = n$.

Proof: Let $P_{2a-1} = [u_1, u_2, ..., u_{2a-1}]$. Consider the following cases:

Case 1. $a = b < n$.

Let $G$ be a graph obtained from $P_{2a-1} = [u_1, u_2, ..., u_{2a-1}]$ by adding the path $[u_1, v_1, u_3, v_2, ..., v_{a-1}, u_{2a-1}]$ (see Figure 1). Then $S = \{u_1, u_3, ..., u_{2a-1}\}$ is a minimum independent dominating set of $G$ and a minimum independent 2-dominating set of $G$. Hence, $i(G) = i_2(G) = |S| = a$ and $|V(G)| = (2a - 1) + (a - 1) = 3a - 2 = n$.

Case 2. $a < b < n$.

Let $H$ be the graph in Case 1 and let $G$ be a graph obtained from $H$ by adding the edges $u_{2a-2}w_i$, for $i = 1, 2, ..., b-a$ (see Figure 2). Then $S_1 = \{u_1, u_3, ..., u_{2a-5}, u_{2a-2}, v_{a-1}\}$ is a minimum independent dominating set of $G$ and $S_2 = \{u_1, u_3, ..., u_{2a-1}\} \cup \{w_1, w_2, ..., w_{b-a}\}$ is a minimum independent 2-dominating set of $G$. Hence, $i(G) = |S_1| = a$, $i_2(G) = |S_2| = b$, and $|V(G)| = (2a - 1) + (a - 1) + (b - a) = 2a + b - 2 = n$.

The next result follows from Theorem 2.1.

Corollary 2.2 The difference $i_2 - i$ and $i_2 - \gamma$ can be made arbitrarily large.

The next result characterizes the independent 2-dominating sets of the join $G + H$. 

Figure 1: A graph $G$ with $i(G) = i_2(G)$

Figure 2: A graph $G$ with $i(G) < i_2(G)$
Theorem 2.3  Let $G$ and $H$ be graphs. Then $S \subseteq V(G+H)$ is an independent 2-dominating set of $G+H$ if and only if $S$ is an independent 2-dominating set of $G$ or $S$ is an independent 2-dominating set of $H$.

Proof: Suppose $S \subseteq V(G+H)$ is an independent 2-dominating set of $G+H$. Then either $S \subseteq V(G)$ or $S \subseteq V(H)$ since $S$ is an independent set of $G+H$. Suppose $S \subseteq V(G)$. Clearly, $S$ is an independent set of $G$. Let $v \in V(G) \setminus S$. Suppose $S$ is not a 2-dominating set of $G$. Then $|S \cap N_G(v)| < 2$, which implies that $|S \cap N_{G+H}(v)| < 2$. This contradicts the assumption that $S$ is an independent 2-dominating set of $G+H$. Hence, $S$ is a 2-dominating set of $G$. Consequently, $S$ is an independent 2-dominating set of $G$. Similarly, if $S \subseteq V(H)$, then $S$ is an independent 2-dominating set of $G$.

Conversely, suppose $S$ is an independent 2-dominating set of $G$. Then $S$ is an independent set of $G+H$. Let $x \in V(G+H) \setminus S$. Then either $x \in V(G) \setminus S$ or $x \in V(H)$. If $x \in V(G) \setminus S$, then $S \cap N_G(x) \geq 2$ since $S$ is a 2-dominating set of $G$. This implies that $S \cap N_{G+H}(x) \geq 2$. Suppose $x \in V(H)$. Then $S \subseteq N_{G+H}(x)$, that is, $S \cap N_{G+H}(x) \geq 2$. Hence, $S$ is a 2-dominating set of $G+H$. Consequently, $S$ is an independent 2-dominating set of $G+H$. Similarly, if $S$ is an independent 2-dominating set of $H$, then $S$ is an independent 2-dominating set of $G+H$. □

The next result is a direct consequence of Theorem 2.3.

Corollary 2.4  Let $G$ and $H$ be graphs. Then $i_2(G+H) = \min\{i_2(G), i_2(H)\}$.

Proof: Suppose $i_2(G) \leq i_2(H)$. Let $S$ be a minimum independent 2-dominating set of $G$. Then $|S| = i_2(G)$. By Theorem 2.3, $S$ is an independent 2-dominating set of $G+H$. Thus, $i_2(G+H) \leq |S| = i_2(G)$. Next, suppose $S'$ is a minimum independent 2-dominating set of $G+H$. Then $i_2(G+H) = |S'|$. By Theorem 2.3, $S'$ is an independent 2-dominating set of $G$. Hence, $i_2(G+H) = |S'| \geq i_2(G)$. Therefore, $i_2(G+H) = i_2(G)$. Similarly, if we assume that $i_2(H) \leq i_2(G)$, then $i_2(G+H) = i_2(H)$. Consequently, $i_2(G+H) = \min\{i_2(G), i_2(H)\}$. □

Corollary 2.5  For positive integers $m \geq 2$ and $n \geq 2$, $i_2(K_{m,n}) = \min\{m, n\}$.

Proof: Since $K_{m,n} = \overline{K_m} + \overline{K_n}$, and by Remark 1.3, $i_2(\overline{K_m}) = m$ and $i_2(\overline{K_n}) = n$, by Corollary 2.4, $i_2(K_{m,n}) = \min\{m, n\}$. □

Corollary 2.6  Let $G$ be a graph and $n$ is a positive integer. Then $i_2(G+K_n) = i_2(G)$.

Proof: Since $i_2(K_n)$ does not exist, by Corollary 2.4, $i_2(G+K_n) = i_2(G)$. □
References


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