The Strong Convergence Theorems for the Split Equality Fixed Point Problems in Hilbert Spaces

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Abstract
The purpose of this paper is to investigate the split equality fixed point problem for quasi-asymptotically pseudo-contractive mappings in Hilbert spaces. And without assumption of semi-compactness, the strong convergence of the sequence generated by the proposed iterative scheme is obtained. The results presented in this paper improve and extend some recent corresponding results announced.

Keywords: Split equality fixed point problems; Hilbert spaces; quasi-asymptotically pseudo-contractive mappings; Strong convergence

1 Introduction
The split feasibility problem (SFP) was first introduced in 1994 by Censor and Elfving [1] in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. It has been shown the it can be used in many areas of applications such as imagine restoration, computer tomograph and radiation therapy treatment planning [3-5].
Let $C$ and $Q$ be nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively, and $A : H_1 \rightarrow H_2$ be a bounded linear operator. The split feasibility problem (SFP) is formulated as:

\[
\text{finding } x^* \in C, \text{ such that } Ax^* \in Q. \tag{1.1}
\]

It is easy to see that $x^* \in C$ is a solution of (SFP) if and only if it solves the following fixed point equation

\[
x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*, \quad \forall x \in C,
\]

where $P_C$ (resp. $P_Q$) is the (orthogonal) projection from $H_1$ (resp. $H_2$) onto $C$ (resp. $Q$), $\gamma > 0$, and $A^*$ is the adjoint of $A$.

In 2013, Moudafi [7,8] proposed a new split feasibility problem which is also called split equality problem (SEP). Let $H_1$, $H_2$ and $H_3$ be real Hilbert spaces. $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be two bounded linear operators, $C \subset H_1$ and $Q \subset H_2$ be two nonempty closed convex sets. The split equality problem (SEP) is formulated as:

\[
\text{finding } x \in C, \ y \in Q \text{ such that } Ax = By. \tag{1.3}
\]

Obviously, if $B = I$ (identity mapping on $H_2$) and $H_3 = H_2$, then the split equality problem (SEP) reduces to the split feasibility problem (SFP).

In (1.3), when $C$ and $Q$ are the sets of fixed points of two nonlinear operator $T$ and $S$, and $C$ and $Q$ are nonempty closed convex, respectively, the split equality problem is called split equality fixed point problem (SEFPP). This is

\[
\text{finding } x \in C = F(T), \ y \in Q = F(S) \text{ such that } Ax = By. \tag{1.4}
\]

Since each closed and convex subset of a Hilbert space may be considered as a fixed point set of a projection on the subset, hence the split equality fixed point problem is a generalization of the split equality problem.

The split equality problem (SEP) and split equality fixed point problem (SEFPP) have been studied by many authors [9-17]. To solve the (SEFPP), Moudafi [9] presented the following simultaneous iterative method and obtained weak convergence theorem:

\[
\begin{align*}
{x_{n+1} = T(x_n - \gamma A^*(Ax_n - By_n)),} \\
{y_{n+1} = S(y_n + \beta B^*(Ax_n - By_n)),}
\end{align*} \tag{1.5}
\]

where $T$ and $S$ are two firmly quasi-nonexpansive operators.
In 2015, Che and Li [16] proposed the following iterative algorithm for finding a solution of the \((SEFPP)\) of strictly pseudo-nonexpansive mappings:

\[
\begin{aligned}
  u_n &= x_n - \gamma_n A^*(Ax_n - By_n), \\
  x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) Tu_n, \\
  v_n &= y_n + \gamma_n B^*(Ax_n - By_n), \\
  y_{n+1} &= \alpha_n y_n + (1 - \alpha_n) Sv_n. \\
\end{aligned}
\]

They also obtained the weak convergence of the iterative scheme (1.6).

In 2015, Chang et al. [17] proposed an iterative algorithm to establish the strong convergence and weak convergence results of the \((SEFPP)\) of \(L\)-Lipschitzian and quasi-pseudo-contractive mappings:

\[
\begin{aligned}
  u_n &= x_n - \gamma_n A^*(Ax_n - By_n), \\
  x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)((1 - \xi_n)I + \xi_n T((1 - \eta_n)I + \eta_n T)) u_n, \\
  v_n &= y_n + \gamma_n B^*(Ax_n - By_n), \\
  y_{n+1} &= \alpha_n y_n + (1 - \alpha_n)((1 - \xi_n)I + \xi_n S((1 - \eta_n)I + \eta_n S)) v_n. \\
\end{aligned}
\]

In the above research work, to prove the strong convergence, the semi-compactness on mappings is needed. In 2015, Zhang et al. [18] introduced a new iterative algorithm to solve split common fixed point problem of asymptotically nonexpansive mappings and proved its strong convergence without assumption of semi-compactness on mappings in Hilbert spaces:

\[
\begin{aligned}
  z_n &= x_n + \lambda A^*(T_2^n - I)Ax_n, \\
  y_n &= \alpha_n z_n + (1 - \alpha_n)T_1^n z_n, \\
  C_{n+1} &= \{ v \in C_n : \| y_n - v \| \leq k_n \| z_n - v \|, \| z_n - v \| \leq k_n \| x_n - v \| \}, \\
  x_{n+1} &= P_{C_{n+1}} x_1, n \geq 1. \\
\end{aligned}
\]

In 2016, Tang et al. [19] used the following hybrid projection algorithm to solve split equality fixed point problem \((SEFPP)\) for \(L\)-Lipschitzian and quasi-pseudo-contractive mappings in Hilbert spaces and proved its strong convergence theorem without assumption of semi-compactness on mappings:

\[
\begin{aligned}
  u_n &= x_n - \gamma_n A^*(Ax_n - By_n), \\
  w_n &= \alpha_n x_n + (1 - \alpha_n)((1 - \xi)I + \xi T((1 - \eta)I + \eta T)) u_n, \\
  v_n &= y_n + \gamma_n B^*(Ax_n - By_n), \\
  z_n &= \alpha_n y_n + (1 - \alpha_n)((1 - \xi)I + \xi S((1 - \eta)I + \eta S)) v_n, \\
  C_{n+1} \times Q_{n+1} &= \{ (p, q) \in C_n \times Q_n : \| w_n - p \|^2 + \| z_n - q \|^2 \leq \| x_n - p \|^2 + \| y_n - q \|^2 \}, \\
  x_{n+1} &= P_{C_{n+1}} x_1, \\
  y_{n+1} &= P_{Q_{n+1}} y_1. \\
\end{aligned}
\]
Motivated and inspired by the researches going on this direction, the purpose of this paper is to introduce a new iterative algorithm to solve split equality fixed point problem for quasi-asymptotically pseudo-contractive mappings in Hilbert spaces. And without assumption of semi-compactness on mappings, the strong convergence of the sequence generated by the proposed iterative scheme is obtained. The results presented in this paper improve and extend some recent corresponding results announced.

2 Preliminaries

Throughout this paper, we use the notations ”$\rightarrow$” to denote the strong convergence, and ”$\rightharpoonup$” to denote the weak convergence.

Let $C$ be a closed convex subset of $H$. For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, satisfying

$$\| x - P_C x \| \leq \| x - y \|, \forall y \in C. \quad (2.1)$$

The operator $P_C$ is called the metric projection mapping of $H$ onto $C$. The metric projection $P_C$ is characterized by the following inequalities:

$$\langle y - P_C x, x - P_C x \rangle \leq 0, \forall x \in H, y \in C, \quad (2.2)$$

and

$$\| y - P_C x \|^2 + \| x - P_C x \|^2 \leq \| x - y \|^2, \forall y \in C, \forall x \in H. \quad (2.3)$$

In a real Hilbert space $H$, it is also well known that for any $x, y \in H$

$$\| \lambda x + (1-\lambda) y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda(1-\lambda) \| x - y \|^2, \forall \lambda \in [0,1] \quad (2.4)$$

and

$$2\langle x, y \rangle = \| x \|^2 + \| y \|^2 - \| x - y \|^2.$$

Definition 2.1. An operator $T : C \to C$ is said to be

(I) pseudo-contractive if $\langle Tx - Ty, x - y \rangle \leq \| x - y \|^2$, for all $x, y \in C$, or equivalently,

$$\| Tx - Ty \|^2 \leq \| x - y \|^2 + \| (I - T)x - (I - T)y \|^2, \forall x, y \in C. \quad (I)$$

(II) quasi-pseudo-contractive if $F(T) \neq \emptyset$ and

$$\| Tx - p \|^2 \leq \| x - p \|^2 + \| Tx - x \|^2, \forall x, y \in C, \forall p \in F(T). \quad (II)$$

(III) asymptotically pseudo-contractive if there exists a sequence $\{l_n\} \subset [1, +\infty)$ with $\lim_{n \to \infty} l_n = 1$ such that

$$\| T^nx - T^ny \|^2 \leq l_n \| x - y \|^2 + \| (I - T^nx)(I - T^ny) \|^2, \forall x, y \in C, \text{ for each } n \geq 1, \quad (III)$$

where $T^n$ denotes the composition of $n$ times of $T$. If $\lim_{n \to \infty} l_n = 1$, then we call $T$ asymptotically nonexpansive.
or equivalently,

\[
(T^n x - T^n y, x - y) \leq \frac{l_n + 1}{2} \|x - y\|^2, \quad \forall x, y \in C, \text{ for each } n \geq 1.
\]

(IV) quasi-asymptotically pseudo-contractive if \( F(T) \neq \emptyset \) and there exists a sequence \( \{l_n\} \subset [1, +\infty) \) with \( \lim_{n \to \infty} l_n = 1 \) such that

\[
\|T^n x - p\|^2 \leq l_n \|x - p\|^2 + \|T^n x - x\|^2, \quad \forall x, y \in C, \quad \forall p \in F(T), \quad \text{for each } n \geq 1,
\]

Definition 2.2. An operator \( T : C \to C \) is said to be uniformly \( L \)-Lipschitzian, if there exists some \( L > 0 \) such that \( \|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall x, y \in C, \text{ for each } n \geq 1. \)

Definition 2.3. An operator \( T : C \to C \) is said to be demi-closed at zero, if for any sequence \( \{x_n\} \) with \( x_n \to x \) and \( \lim_{n \to \infty} \|x_n - T(x_n)\| = 0 \), then \( x = Tx \).

Lemma 2.4. Let \( H \) be a real Hilbert space, \( T : H \to H \) be a uniformly \( L \)-Lipschitzian and \( \{l_n\} \)-quasi-asymptotically pseudo-contractive mapping with \( L \geq 1 \) and \( \{l_n\} \subset [1, \infty) \) and \( \lim_{n \to \infty} l_n = 1 \). Let \( \{K_n : H \to H\} \) be a sequence of mappings defined by:

\[
K_n := (1 - \xi)I + \xi T^n((1 - \eta)I + \eta T^n).
\]

If \( 0 < a < \xi < \eta < b < \frac{1}{M + \sqrt{(M + 1)^2 + L^2}} \) and \( M = \sup_{n \geq 1} l_n \), then the following conclusions hold:

1. \( F(T) = F(T^n((1 - \eta)I + \eta T^n)) = F(K_n) \) for each \( n \geq 1 \);
2. If \( T \) is demi-closed at zero, then \( K_1 \) is also demi-closed at zero;
3. For each \( n \geq 1 \) and \( x \in H, u \in F(T) = F(K_n) \),

\[
\|K_n x - u\| \leq k_n \|x - u\|,
\]

where \( k_n = 1 + \xi(l_n - 1)(1 + \eta l_n), \quad \{k_n\} \subset [1, \infty) \) and \( \lim_{n \to \infty} k_n = 1 \).

Proof. (1) If \( u \in F(T) \), i.e., \( u = Tu \), we have

\[
T^n((1 - \eta)I + \eta T^n)u = T^n((1 - \eta)u + \eta T^n u) = T^n u = u.
\]

This shows that \( u \in F(T^n((1 - \eta)I + \eta T^n)) \).

Conversely, if \( u \in F(T^n((1 - \eta)I + \eta T^n)) \) for all \( n \geq 1 \), i.e., \( u = T^n((1 - \eta)I + \eta T^n)u \). Put \( ((1 - \eta)I + \eta T^n)u = v \), then \( T^n v = u \). Now we prove that \( u = v \). In fact, we have

\[
\|u - v\| = \|u - ((1 - \eta)I + \eta T^n)u\| = \eta \|u - T^n u\| = \eta \|T^n v - T^n u\| \leq L\eta \|u - v\|.
\]
Since $0 < \eta < \frac{1}{M + \sqrt{(M + 1)^2 + L^2}} < \frac{1}{4}$, we have $0 < L\eta < 1$. Then we easily obtain $u = v$, i.e., $u \in F(T)$. This shows that $F(T) = F(T^n((1 - \eta)I + \eta T^n))$ for all $n \geq 1$.

It is obvious that $u \in F(K_n)$ if and only if $u \in F(T^n((1 - \eta)I + \eta T^n))$. Then the conclusion (1) is proved.

(2) For any sequence $\{u_n\}$ satisfying $u_n \rightharpoonup u$ and $\|u_n - K_1 u_n\| \to 0$. We show that $u \in F(K_1)$. From conclusion (1), it suffices to prove $u \in F(T)$.

In fact, since $T$ is $L$-Lipschitz, we have

$$
\|u_n - T u_n\| \leq \|u_n - T((1 - \eta)I + \eta T)u_n\| + \|T((1 - \eta)I + \eta T)u_n - T u_n\|
\leq \frac{1}{\xi}\|u_n - (1 - \xi)u_n - \xi(T((1 - \eta)I + \eta T)u_n\| + L\eta \|u_n - T u_n\|
= \frac{1}{\xi}\|u_n - K_1 u_n\| + L\eta \|u_n - T u_n\|.
$$

Simplifying it, we have

$$
\|u_n - T u_n\| \leq \frac{1}{\xi(1 - L\eta)}\|u_n - K_1 u_n\| \to 0.
$$

Since $T$ is demi-closed at 0, we have $u \in F(T) = F(K)$. The conclusion (2) is proved.

(3) For all $u \in F(T)$, it follows from Definition 2.1(IV) that

$$
\|T^n((1 - \eta)I + \eta T^n)x - u\|^2
\leq l_n\|((1 - \eta)x + \eta T^n x - u\|^2 + \|T^n((1 - \eta)I + \eta T^n)x - ((1 - \eta)I + \eta T^n)x\|^2
= l_n\|((1 - \eta)(x - u) + \eta(T^n x - u)\|^2
+\|T^n((1 - \eta)I + \eta T^n)x - ((1 - \eta)I + \eta T^n)x\|^2.
$$

Since $T$ is $L$-Lipschitz, we have

$$
\|T^n((1 - \eta)I + \eta T^n)x - T^n x\| \leq L\|(1 - \eta)x + \eta T^n x - x\| = L\eta\|T^n x - x\|. \tag{2.7}
$$

From (2.4) and (2.7) we have

$$
\|(1 - \eta)(x - u) + \eta(T^n x - u)\|^2
= (1 - \eta)\|x - u\|^2 + \eta\|T^n x - u\|^2 - \eta(1 - \eta)\|T^n x - x\|^2
\leq (1 - \eta)\|x - u\|^2 + \eta(l_n\|x - u\|^2 + \|T^n x - x\|^2) - \eta(1 - \eta)\|T^n x - x\|^2
= (1 + \eta(l_n - 1))\|x - u\|^2 + \eta^2\|T^n x - x\|^2. \tag{2.8}
$$
Using (2.4), we have
\[
\|T^n((1 - \eta)I + \eta T^n)x - ((1 - \eta)I + \eta T^n)x\| \leq (1 - \eta)\|T^n((1 - \eta)I + \eta T^n)x - x\| + \eta(\|T^n((1 - \eta)I + \eta T^n)x - T^n x\|) \\
\leq (1 - \eta)\|T^n((1 - \eta)I + \eta T^n)x - x\| + \eta\|T^n((1 - \eta)I + \eta T^n)x - T^n x\| \\
- \eta(1 - \eta)\|T^n x - x\|^2 \\
\leq (1 - \eta)\|T^n((1 - \eta)I + \eta T^n)x - x\|^2 - \eta(1 - \eta - \eta^2 L^2)\|T^n x - x\|^2. \quad (2.9)
\]

Substituting (2.8) and (2.9) into (2.6), we obtain
\[
\|T^n((1 - \eta)I + \eta T^n)x - u\|^2 \\
\leq l_n(1 + \eta(l_n - 1))\|x - u\|^2 + l_n\eta^2\|T^n x - x\|^2 \\
+ (1 - \eta)\|T^n((1 - \eta)I + \eta T^n)x - x\|^2 - \eta(1 - \eta - \eta^2 L^2)\|T^n x - x\|^2 \\
= l_n(1 + \eta(l_n - 1))\|x - u\|^2 + (1 - \eta)\|T^n((1 - \eta)I + \eta T^n)x - x\|^2 \\
- \eta(1 - \eta - \eta^2 L^2 - l_n\eta^2)\|T^n x - x\|^2. \quad (2.10)
\]

Since \( \eta < \frac{1}{M + 1 + \sqrt{(M + 1)^2 + L^2}} \), we deduce \( 1 - \eta - \eta^2 L^2 - l_n\eta^2 > 0 \). From (2.10) we get
\[
\|T^n((1 - \eta)I + \eta T^n)x - u\|^2 \leq l_n(1 + \eta(l_n - 1))\|x - u\|^2 \\
+ (1 - \eta)\|T^n((1 - \eta)I + \eta T^n)x - x\|^2. \quad (2.11)
\]

Combine (2.4) and (2.11) we have
\[
\|K_n x - u\|^2 = \|((1 - \xi)x + \xi T^n((1 - \eta)I + \eta T^n)x) - u\|^2 \\
= (1 - \xi)\|x - u\|^2 + \xi\|T^n((1 - \eta)I + \eta T^n)x - x\|^2 \\
- \xi(1 - \xi)\|T^n((1 - \eta)I + \eta T^n)x - x\|^2 \\
\leq (1 - \xi)\|x - u\|^2 + \xi l_n(1 + \eta(l_n - 1))\|x - u\|^2 \\
+ (\xi(1 - \eta) - \xi(1 - \xi))\|T^n((1 - \eta)I + \eta T^n)x - x\|^2 \\
= (1 + \xi (1 + \eta l_n)(l_n - 1))\|x - u\|^2 - \xi(\eta - \xi)\|T^n((1 - \eta)I + \eta T^n)x - x\|^2.
\]

This together with \( \xi < \eta \) implies that
\[
\|K_n x - u\|^2 \leq k_n\|x - u\|^2, \forall x \in H, \ u \in F(K_n), \ n \geq 1,
\]

where \( k_n = 1 + \xi (l_n - 1)(1 + \eta l_n) \).

In view of that \( \{l_n\} \subset [1, +\infty) \) and \( \lim_{n \to \infty} l_n = 1 \), we have \( \{k_n\} \subset [1, +\infty) \)
and \( \lim_{n \to \infty} k_n = 1 \). The conclusion (3) is proved.
3 Main Results

Theorem 3.1. Let $H_1, H_2$ and $H_3$ be three real Hilbert spaces, $A : H_1 \rightarrow H_3$ and $B : H_2 \rightarrow H_3$ be two bounded linear operators with adjoints $A^*$ and $B^*$, respectively. Let $T : H_1 \rightarrow H_1$ and $S : H_2 \rightarrow H_2$ be two uniformly $L$-Lipschitzian and $\{l_n\}$-quasi-asymptotically pseudo-contractive mappings with $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. For any given initial points $x_1 \in C_1 = H_1$, $y_1 \in Q_1 = H_2$, the sequence $\{(x_n, y_n)\}$ is defined as follows:

\[
\begin{aligned}
&u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\
v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\
z_n = \alpha_n x_n + (1 - \alpha_n) K_n u_n, \\
s_n = \alpha_n y_n + (1 - \alpha_n) G_n v_n, \\
C_{n+1} \times Q_{n+1} &= \{(x, y) \in C_n \times Q_n : \|z_n - x\|^2 + \|s_n - y\|^2 \\
&\leq (1 + (1 - \alpha_n)(k_n^2 - 1))(\|x_n - x\|^2 + \|y_n - y\|^2)\}, \\
x_{n+1} = P_{C_{n+1}} x_1, \quad y_{n+1} = P_{Q_{n+1}} y_1,
\end{aligned}
\]

where $K_n = (1 - \xi I + \xi T^n((1 - \eta I + \eta T^n))$, $G_n = (1 - \xi I + \xi S^n((1 - \eta I + \eta S^n))$, $k_n = 1 + \xi(l_n - 1)(1 + \eta l_n)$ and the following conditions are satisfied:

1. $L \geq 1$, and $\sum_{n=1}^{\infty} (l_n - 1) < \infty$;
2. $\{\alpha_n\} \subset (0, 1)$, $\liminf_{n \to \infty} \alpha_n > 0$;
3. $\gamma_n \in (0, \max(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}))$ with $\liminf_{n \to \infty} \gamma_n > 0$, $\forall n \geq 1$;
4. $0 < a < \xi < \eta < b < \frac{1}{M + 1 + \sqrt{\frac{M^2 + 1}{4} + L^2}}$, $M = \sup_{n \geq 1} l_n$, $\forall n \geq 1$.

If $\Omega = \{(p, q) \in F(T) \times F(S) \text{ such that } Ap = Bq \neq \emptyset$, $T$ and $S$ are demiclosed at zero, then the sequences $\{(x_n, y_n)\}$ converges strongly to a point $(x^*, y^*) \in \Omega$.

Proof. Since $k_n = 1 + \xi(l_n - 1)(1 + \eta l_n)$, $\xi < 1$, $\{l_n\} \subset [1, \infty)$ and $\sum_{n=1}^{\infty} (l_n - 1) < \infty$, we have $\sum_{n=1}^{\infty} (k_n^2 - 1) = \sum_{n=1}^{\infty} \xi(l_n - 1)(1 + \eta l_n) < \sum_{n=1}^{\infty} (l_n - 1)(l_n + 1) < \infty$. Further, due to $\{\alpha_n\} \subset (0, 1)$, we also have $\sum_{n=1}^{\infty} (1 - \alpha_n)(k_n^2 - 1) \leq \sum_{n=1}^{\infty} (k_n - 1)(k_n + 1) < \infty$.

We shall divide the proof into five steps.

Step 1. We show that $C_n \times Q_n$ is closed and convex for each $n \geq 1$.

Putting $\rho_n = 1 + (1 - \alpha_n)(k_n^2 - 1)$. Since $C_1 = H_1$ and $Q_1 = H_2$, so $C_1 \times Q_1$ is closed and convex. Suppose that $C_n \times Q_n$ is closed and convex. For any $(x, y) \in C_{n+1} \times Q_{n+1}$, we have

\[
\|z_n - x\|^2 + \|s_n - y\|^2 \leq (1 + (1 - \alpha_n)(k_n^2 - 1))(\|x_n - x\|^2 + \|y_n - y\|^2).
\]
which is equivalent to
\[
\langle 2\rho_n x_n - \rho_n x - 2z_n + x, x \rangle \leq \rho_n \|x_n\|^2 + \rho_n \|y_n - y\|^2 - \|z_n\|^2.
\]
So, we know that \( C_{n+1} \) is closed. Similarly, we can prove that \( Q_{n+1} \) is closed. Therefore \( C_{n+1} \times Q_{n+1} \) is closed. Besides, it is easy to prove that \( C_{n+1} \times Q_{n+1} \) also is a convex. Therefore \( C_{n+1} \times Q_{n+1} \) is a closed and convex for any \( n \geq 1 \).

**Step 2.** We prove that \( \Omega \subseteq C_n \times Q_n \) for any \( n \geq 1 \).

For any given \( (p, q) \in \Omega \), then \( p \in F(T) \), \( Q \in F(S) \) and \( Ap = Bq \). It follows from (3.1), we have
\[
\|u_n - p\|^2 = \|x_n - \gamma_n A^*(Ax_n - By_n) - p\|^2
\]
\[
= \|x_n - p\|^2 + \|\gamma_n A^*(Ax_n - By_n)\|^2 - 2\gamma_n \langle x_n - p, A^*(Ax_n - By_n) \rangle
\]
\[
\leq \|x_n - p\|^2 + \gamma_n^2 \|A\|^2 \|Ax_n - By_n\|^2 - 2\gamma_n \langle x_n - p, A^*(Ax_n - By_n) \rangle
\]
\[
= \|x_n - p\|^2 + \gamma_n^2 \|A\|^2 \|Ax_n - By_n\|^2 - 2\gamma_n \langle Ax_n - p, Ax_n - By_n \rangle.
\]
Similarly, from (3.1), we have
\[
\|v_n - q\|^2 \leq \|y_n - q\|^2 + \gamma_n^2 \|B\|^2 \|Ax_n - By_n\|^2 + 2\gamma_n \langle By_n - Bq, Ax_n - By_n \rangle.
\]
Adding up (3.4) and (3.5) and noting \( Ap = Bq \), we have that
\[
\|u_n - p\|^2 + \|v_n - q\|^2 \leq \|x_n - p\|^2 + \|y_n - q\|^2 + \gamma_n^2(\|A\|^2 + \|B\|^2) \|Ax_n - By_n\|^2
\]
\[
- 2\gamma_n \langle Ax_n - p, By_n + Bq, Ax_n - By_n \rangle
\]
\[
= \|x_n - p\|^2 + \|y_n - q\|^2 + \gamma_n(\|A\|^2 + \|B\|^2) - 2) \|Ax_n - By_n\|^2.
\]
Since \( \gamma_n \in (0, \max(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2})) \), \( \gamma_n \|A\|^2 < 1 \) and \( \gamma_n \|B\|^2 < 1 \). This implies that \( \gamma_n(\|A\|^2 + \|B\|^2) - 2 < 0 \). Therefore (3.6) can be written as
\[
\|u_n - p\|^2 + \|v_n - q\|^2 \leq \|x_n - p\|^2 + \|y_n - q\|^2.
\]
According to condition (4) and Lemma 2.4, we know
\[
F(T) = F(K_n), F(S) = F(G_n), \forall n \geq 1;
\]
\[
\|K_n u_n - p\| \leq k_n \|u_n - p\|, \|G_n v_n - q\| \leq k_n \|v_n - q\|, \forall n \geq 1.
\]
Then, it follows from (3.1), condition (2) and Lemma 2.4, we obtain that
\[
\|z_n - p\|^2 = \|\alpha_n x_n + (1 - \alpha_n)K_n u_n - p\|^2
\]
\[
= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \|K_n u_n - p\|^2 - \alpha_n (1 - \alpha_n) \|K_n u_n - x_n\|^2
\]
\[
\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) k_n^2 \|u_n - p\|^2 - \alpha_n (1 - \alpha_n) \|K_n u_n - x_n\|^2
\]
\[
\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) k_n^2 \|u_n - p\|^2.
\]
Similarly, it follows from (3.1), condition (2) and Lemma 2.4 that
\[
\|s_n - q\|^2 \leq \alpha_n\|y_n - q\|^2 + (1 - \alpha_n)k_n^2\|v_n - q\|^2 - \alpha_n(1 - \alpha_n)\|G_n v_n - y_n\|^2 \\
\leq \alpha_n\|y_n - q\|^2 + (1 - \alpha_n)k_n^2\|v_n - q\|^2.
\] (3.9)

Adding up (3.8) and (3.9), we have
\[
\|z_n - p\|^2 + \|s_n - q\|^2 \leq \alpha_n(\|x_n - p\|^2 + \|y_n - q\|^2) \\
+ (1 - \alpha_n)k_n^2(\|x_n - p\|^2 + \|y_n - q\|^2).
\] (3.10)

Substituting (3.7) into (3.10), we can obtain that
\[
\|z_n - p\|^2 + \|s_n - q\|^2 \leq \alpha_n(\|x_n - p\|^2 + \|y_n - q\|^2) \\
+ (1 - \alpha_n)k_n^2(\|x_n - p\|^2 + \|y_n - q\|^2) \\
= (\alpha_n + (1 - \alpha_n)k_n^2)(\|x_n - p\|^2 + \|y_n - q\|^2) \\
= (1 + (1 - \alpha_n)(k_n^2 - 1))(\|x_n - p\|^2 + \|y_n - q\|^2).
\]

Therefore, we know that \((p, q) \in \mathcal{C}_n \times Q_n\) and \(\Omega \subseteq \mathcal{C}_n \times Q_n\) for any \(n \geq 1\).

**Step 3.** We show that the sequences \(\{x_n\}\) and \(\{y_n\}\) are two Cauchy sequences.

Since \(\Omega \subseteq \mathcal{C}_{n+1} \times Q_{n+1} \subseteq \mathcal{C}_n \times Q_n\) and \((x_{n+1}, y_{n+1}) = (P_{C_n+1}x_1, P_{Q_n+1}y_1) \subseteq \mathcal{C}_{n+1} \times Q_{n+1} \subseteq \mathcal{C}_n \times Q_n\), then for any \(n \geq 1\) and \((p, q) \in \Omega\), we have
\[
\|x_{n+1} - x_1\| \leq \|p - x_1\|, \quad \forall n \geq 1, (3.11)
\]
\[
\|y_{n+1} - y_1\| \leq \|q - y_1\|, \quad \forall n \geq 1. (3.12)
\]

Hence, \(\{x_n\}\) and \(\{y_n\}\) are bounded. For any \(n \geq 1\), by using (2.3), we have
\[
\|x_{n+1} - x_n\|^2 + \|x_1 - x_n\|^2 = \|x_{n+1} - P_{C_n}x_1\|^2 + \|x_1 - P_{C_n}x_1\|^2 \leq \|x_{n+1} - x_1\|^2,
\]
and
\[
\|y_{n+1} - y_n\|^2 + \|y_1 - y_n\|^2 = \|y_{n+1} - P_{Q_n}y_1\|^2 + \|y_1 - P_{Q_n}y_1\|^2 \leq \|y_{n+1} - y_1\|^2,
\]
which imply that \(\{\|x_n - x_1\|\}\) and \(\{\|y_n - y_1\|\}\) are nondecreasing. By virtue of the boundedness of \(\{x_n\}\) and \(\{y_n\}\), \(\lim_{n \to \infty} \|x_n - x_1\|\) and \(\lim_{n \to \infty} \|y_n - y_1\|\) exist.

For positive integers \(m, n\) with \(m \leq n\), from \((x_n, y_n) = (P_{C_n}x_1, P_{Q_n}y_1) \subseteq \mathcal{C}_m \times Q_m\) and (2.3), we have
\[
\|x_m - x_n\|^2 + \|x_1 - x_n\|^2 = \|x_m - P_{C_n}x_1\|^2 + \|x_1 - P_{C_n}x_1\|^2 \leq \|x_m - x_1\|^2, (3.13)
\]
\[
\|y_m - y_n\|^2 + \|y_1 - y_n\|^2 = \|y_m - P_{Q_n}y_1\|^2 + \|y_1 - P_{Q_n}y_1\|^2 \leq \|y_m - y_1\|^2. (3.14)
\]
Since \( \lim_{n \to \infty} \|x_n - x_1\| \) and \( \lim_{n \to \infty} \|y_n - y_1\| \) exist, it follows from (3.13) and (3.14) that
\[
\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_1 - x_n\|^2 \to 0,
\]
\[
\|y_m - y_n\|^2 \leq \|y_m - y_1\|^2 - \|y_1 - y_n\|^2 \to 0.
\]

Therefore \( \{x_n\} \) and \( \{y_n\} \) are two Cauchy sequences.

**Step 4.** We show that \( \lim_{n \to \infty} \|Ax_n - By_n\| = 0 \), \( \lim_{n \to \infty} \|K_n u_n - x_n\| = 0 \) and \( \lim_{n \to \infty} \|G_n v_n - y_n\| = 0 \).

Since \( (x_{n+1}, y_{n+1}) = (P_{C_{n+1}} x_1, P_{Q_{n+1}} y_1) \subseteq C_{n+1} \times Q_{n+1} \subseteq C_n \times Q_n \), we have
\[
\|z_n - x_n\|^2 + \|s_n - y_n\|^2 \leq \big(\|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|\big)^2 + \|s_n - y_{n+1}\| + \|y_{n+1} - y_n\|\leq 2\|z_n - x_{n+1}\|^2 + 2\|x_{n+1} - x_n\|^2 + 2\|s_n - y_{n+1}\|^2 + 2\|y_{n+1} - y_n\|^2 \leq (2 + 2\rho_n) (\|x_{n+1} - x_n\|^2 + \|y_{n+1} - y_n\|^2) \to 0.
\]

So we know that \( \lim_{n \to \infty} \|z_n - x_n\| = 0 \) and \( \lim_{n \to \infty} \|s_n - y_n\| = 0 \). Again since \( \{x_n\} \) and \( \{y_n\} \) are bounded, we know that \( \{z_n\} \) and \( \{s_n\} \) are bounded.

From (3.6), (3.8) and (3.9), we have
\[
\|z_n - p\|^2 + \|s_n - q\|^2 \leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)k_n^2 \|u_n - p\|^2 - \alpha_n(1 - \alpha_n)k_n^2 \|K_n u_n - x_n\|^2 + \alpha_n \|y_n - q\|^2 + (1 - \alpha_n)k_n^2 \|v_n - q\|^2 - \alpha_n(1 - \alpha_n) \|G_n v_n - y_n\|^2 = \alpha_n \|x_n - p\|^2 + \|y_n - q\|^2 + (1 - \alpha_n)k_n^2 \|u_n - p\|^2 + \|v_n - q\|^2 - \alpha_n(1 - \alpha_n) \|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2 \leq (\alpha_n + (1 - \alpha_n)k_n^2) \|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2 + (1 - \alpha_n)k_n^2 \|u_n - p\|^2 + \|v_n - q\|^2 - \alpha_n(1 - \alpha_n) \|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2 = (1 + (1 - \alpha_n)(k_n^2 - 1)) \|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2 + (1 - \alpha_n)k_n^2 \|u_n - p\|^2 + \|v_n - q\|^2 - \alpha_n(1 - \alpha_n) \|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2.
\]
It follows from (3.15) that
\[
(1 - \alpha_n)k_n^2 \gamma_n(2 - \gamma_n(\|A\|^2 + \|B\|^2))\|Ax_n - By_n\|^2
+\alpha_n(1 - \alpha_n)(\|K_n u_n - x_n\|^2 + \|G_n v_n - y_n\|^2)
\leq (1 + (1 - \alpha_n)(k_n^2 - 1))(\|x_n - p\|^2 + \|y_n - q\|^2) - \|z_n - p\|^2 - \|s_n - q\|^2
\]
\[
= (1 - \alpha_n)(k_n^2 - 1)(\|x_n - p\|^2 + \|y_n - q\|^2)
+\|x_n - p\|^2 + \|y_n - q\|^2 - \|z_n - p\|^2 - \|s_n - q\|^2
\]
\[
= (1 - \alpha_n)(k_n^2 - 1)(\|x_n - p\|^2 + \|y_n - q\|^2)
+\|x_n - p\|^2 + \|y_n - q\|^2 - \|z_n - p\|^2 - \|s_n - q\|^2
\]
\[
\leq (1 - \alpha_n)(k_n^2 - 1)(\|x_n - p\|^2 + \|y_n - q\|^2) + (\|x_n - p\| + \|z_n - p\|)(\|x_n - p\| - \|z_n - p\|)
+\|y_n - q\| + \|s_n - q\|)(\|y_n - q\| - \|s_n - q\|)
\]

Since \(\lim_{n \to \infty} k_n = 1\) and \(\{\alpha_n\} \subset (0, 1)\), we have \(\lim_{n \to \infty} (1 - \alpha_n)(k_n^2 - 1) = 0\). By virtue of the boundedness of \(\{z_n\}, \{s_n\}, \{x_n\}\) and \(\{y_n\}\), \(\lim_{n \to \infty} \|z_n - x_n\| = 0\) and \(\lim_{n \to \infty} \|s_n - y_n\| = 0\), we get \((1 - \alpha_n)(k_n^2 - 1)(\|x_n - p\|^2 + \|y_n - q\|^2) + (\|x_n - p\| + \|z_n - p\|)(\|x_n - p\| - \|z_n - p\|)(\|x_n - p\| - \|z_n - p\|) \to 0\). In addition, it follows from Conditions (2) and (3) that \((1 - \alpha_n)k_n^2 \gamma_n(2 - \gamma_n(\|A\|^2 + \|B\|^2)) > 0\) and \(\alpha_n(1 - \alpha_n) > 0\). Thus we may get

\[
\lim_{n \to \infty} \|Ax_n - By_n\| = 0; \quad (3.16)
\]
\[
\lim_{n \to \infty} \|K_n u_n - x_n\| = 0, \text{ for each } n = 1, 2, 3, \ldots; \quad (3.17)
\]
\[
\lim_{n \to \infty} \|G_n v_n - y_n\| = 0, \text{ for each } n = 1, 2, 3, \ldots. \quad (3.18)
\]

**Step 5.** We show that \(\{(x_n, y_n)\}\) converges strongly to an element of \(\Omega\). By (3.16), we have
\[
\|u_n - x_n\| + \|v_n - y_n\| = \|z_n - p\| + \|s_n - q\| \leq \gamma_n(\|A\| + \|B\|)\|Ax_n - By_n\| \to 0.
\]

So, we know
\[
\lim_{n \to \infty} \|u_n - x_n\| = 0, \lim_{n \to \infty} \|v_n - y_n\| = 0. \quad (3.19)
\]

From (3.17), (3.18) and (3.19), we have
\[
\|K_n u_n - u_n\| \leq \|K_n u_n - x_n\| + \|u_n - x_n\| \to 0; \quad (3.20)
\]
\[
\|G_n v_n - v_n\| \leq \|G_n v_n - y_n\| + \|v_n - y_n\| \to 0. \quad (3.21)
\]

Since \(\{x_n\}\) and \(\{y_n\}\) are two Cauchy sequences, there exists \(x^* \in H_1\) and \(y^* \in H_2\) such that \(x_n \to x^*\) and \(y_n \to y^*\). From (3.19) we also have \(u_n \to x^*\)
and \( v_n \to y^* \). So it follows from (3.20), (2.21) and Lemma 2.4 that \( x^* \in F(K_1) = F(T) \) and \( y^* \in F(G_1) = F(S) \).

On the other hand, since \( A \) and \( B \) are two bounded linear operators, we have that \( Ax_n - By_n \to Ax^* - By^* \). By using the weakly lower semi-continuity of squared norm, we have

\[
\| Ax^* - By^* \|^2 \leq \liminf_{n \to \infty} \| Ax_n - By_n \|^2 = \lim_{n \to \infty} \| Ax_n - By_n \|^2 = 0,
\]

thus \( Ax^* = By^* \).

Therefore, \( \{(x_n, y_n)\} \) converges strongly to \((x^*, y^*) \in \Omega \). The proof is completed.

**Remark 3.2.** Since a quasi-pseudo-contractive mapping is a quasi-asymptotically pseudo-contractive mapping, the Theorem 3.1 extends the main results in [17] and [19] from quasi-pseudo-contractive mappings to quasi-asymptotically pseudo-contractive mappings.

The following corollary may be directly concluded from Theorem 3.1.

**Corollary 3.3.** Let \( H_1, H_2 \) and \( H_3 \) be three real Hilbert spaces, \( A : H_1 \to H_3 \) and \( B : H_2 \to H_3 \) be two bounded linear operators with their adjoints \( A^* \) and \( B^* \), respectively. Let \( T : H_1 \to H_1 \) and \( S : H_2 \to H_2 \) be two uniformly \( L \)-Lipschitzian and quasi-pseudo-contractive mappings, \( F(T) \neq \emptyset \), and \( F(S) \neq \emptyset \). For given initial value \( x_1 \in C_1 = H_1, y_1 \in Q_1 = H_2 \), and let \( \{(x_n, y_n)\} \) be defined as follows:

\[
\begin{align*}
   u_n &= x_n - \gamma_n A^*(Ax_n - By_n), \\
   v_n &= y_n + \gamma_n B^*(Ax_n - By_n), \\
   z_n &= \alpha_n x_n + (1 - \alpha_n) Ku_n, \\
   s_n &= \alpha_n y_n + (1 - \alpha_n) Gu_n, \\
   C_{n+1} \cap Q_{n+1} &= \{ (x, y) \in C_n \times Q_n : \| z_n - x \|^2 + \| s_n - y \|^2 \\
   &\quad \leq \| x_n - x \|^2 + \| y_n - y \|^2 \}, \\
   x_{n+1} &= P_{C_{n+1}} x_1, \\
   y_{n+1} &= P_{Q_{n+1}} y_1,
\end{align*}
\tag{3.23}
\]

where \( K = (1 - \xi)I + \xi T((1 - \eta)I + \eta T), G = (1 - \xi)I + \xi S((1 - \eta)I + \eta S), \) and the following conditions are satisfied:

1. \( \alpha_n \in (0, 1), \liminf_{n \to \infty} \alpha_n > 0, L \geq 1; \)

2. \( \gamma_n \in (0, \max(\frac{1}{\| A \|^2}, \frac{1}{\| B \|^2})), \liminf_{n \to \infty} \gamma_n > 0; \)

3. \( 0 < a < \xi < \eta < b < \frac{1}{1 + \sqrt{1 + L^2}}. \)

If \( \Omega = \{(p, q) \in F(T) \times F(S) \text{ such that } Ap = Bq \} \neq \emptyset \), \( T \) and \( S \) are demiclosed at zero, then the sequences \( \{(x_n, y_n)\} \) converges strongly to a point \((x^*, y^*) \in \Omega.\)
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References


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