A Note on the Appell-Type Degenerate Twisted Tangent Numbers and Polynomials

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Abstract

In this paper, we construct the Appell-type degenerate twisted tangent numbers and polynomials. We also obtain some explicit formulas for Appell-type degenerate twisted tangent numbers and polynomials. Finally, we investigate the zeros of Appell-type degenerate twisted tangent polynomials by using computer.

Mathematics Subject Classification: 11B68, 11S40, 11S80

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1 Introduction

L. Carlitz constructed the degenerate Bernoulli polynomials (see [1]). P.T. Young studied the degenerate Bernoulli polynomials (see [7]). Feng Qi et al.[2] introduced the partially degenerate Bernoulli polynomials of the first kind in $p$-adic field. T. Kim introduced the Barnes’ type multiple degenerate Bernoulli and Euler polynomials (see [3]), Recently, Ryoo introduced the twisted tangent numbers and tangent polynomials (see [4, 5, 6]). In this paper, we introduce Appell-type degenerate twisted tangent numbers $T_{n,q}(\lambda)$ and tangent polynomials $T_{n,q}(x, \lambda)$. Throughout this paper we use the following notations. By $\mathbb{N}$ we denote the set of natural numbers, $\mathbb{C}$ denotes the complex number field,
and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $r$ be a positive integer, and let $\zeta$ be $r$th root of 1. We recall that the degenerate twisted tangent polynomials are defined by the generating function

$$
\sum_{n=0}^{\infty} T_{n,\zeta}(x,\lambda) \frac{t^n}{n!} = \frac{2}{\zeta(1 + \lambda t)^{2/\lambda} + 1}(1 + \lambda t)^{x/\lambda}.
$$

(1.1)

For $x = 0$, formula (1.1) reduces to the generating function of the degenerate twisted tangent numbers

$$
\sum_{n=0}^{\infty} T_{n,\zeta}(\lambda) \frac{t^n}{n!} = \frac{2}{\zeta(1 + \lambda t)^{2/\lambda} + 1}.
$$

(1.2)

## 2 Appell-type degenerate twisted tangent polynomials

In this section, we introduce Appell-type degenerate twisted tangent numbers and polynomials, and we obtain explicit formulas for them. Let $r$ be a positive integer, and let $\zeta$ be $r$th root of 1. Let us define the Appell-type degenerate twisted tangent numbers $T_{n,\zeta}(\lambda)$ and polynomials $T_{n,\zeta}(x,\lambda)$ as follows:

$$
\frac{2}{\zeta(1 + \lambda t)^{2/\lambda} + 1} = \sum_{n=0}^{\infty} T_{n,\zeta}(\lambda) \frac{t^n}{n!},
$$

(2.1)

$$
\left(\frac{2}{\zeta(1 + \lambda t)^{2/\lambda} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} T_{n,\zeta}(x,\lambda) \frac{t^n}{n!}.
$$

(2.2)

Note that $(1 + \lambda t)^{1/\lambda}$ tends to $e^t$ as $\lambda \to 0$. By (2.2), we see that

$$
\frac{d}{dx} T_{m,\zeta}(x,\lambda) = m T_{m-1,\zeta}(x,\lambda).
$$

(2.3)

By (2.3) we get

$$
\int_0^x \frac{d}{dt} \left( \frac{T_{n+1,\zeta}(t,\lambda)}{n+1} \right) dt = \int_0^x T_{n,\zeta}(t,\lambda) dt
$$

$$
= T_{n+1,\zeta}(x,\lambda) - T_{n+1,\zeta}(\lambda) \frac{n}{n+1}.
$$

(2.4)

By (2.4), we have the following theorem.
Theorem 2.1 For \( n \in \mathbb{Z}_+ \), we have
\[
\int_0^x T_{n,\zeta}(t, \lambda) dt = \frac{T_{n+1,\zeta}(x, \lambda) - T_{n+1,\zeta}(\lambda)}{n+1}.
\]
From (2.2), we note that
\[
\sum_{n=0}^{\infty} \lim_{\lambda \to 0} T_{n,\zeta}(x, \lambda) \frac{t^n}{n!} = \lim_{\lambda \to 0} \left( \frac{2}{\zeta(1 + \lambda t)^{2/\lambda} + 1} \right) e^{xt} = \left( \frac{2}{\zeta e^{2t} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_{n,\zeta}(x) \frac{t^n}{n!}.
\]
Thus, we get
\[
\lim_{\lambda \to 0} T_{n,\zeta}(x, \lambda) = T_{n,\zeta}(x), \quad (n \geq 0),
\]
where, \( T_{n,\zeta}(x) \) are the usual twisted tangent polynomials(see [5]). From (2.2), we have
\[
\sum_{n=0}^{\infty} T_{n,\zeta}(x, \lambda) \frac{t^n}{n!} = \left( \sum_{m=0}^{\infty} T_{m,\zeta}(\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} x^l \frac{t^l}{l!} \right) = \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} T_{l,\zeta}(\lambda) x^{n-l} \frac{t^n}{n!}.
\]
Therefore, by (2.2) and (2.5), we obtain the following theorem.

Theorem 2.2 For \( n \geq 0 \), we have
\[
T_{n,\zeta}(x, \lambda) = \sum_{l=0}^{n} \binom{n}{l} T_{l,\zeta}(\lambda) x^{n-l}.
\]
From (2.1), we can derive the following recurrence relation:
\[
2 = (\zeta(1 + \lambda t)^{2/\lambda} + 1) \sum_{n=0}^{\infty} T_{n,\zeta}(\lambda) \frac{t^n}{n!}
= \zeta(1 + \lambda t)^{2/\lambda} \sum_{n=0}^{\infty} T_{n,\zeta}(\lambda) \frac{t^n}{n!} + \sum_{n=0}^{\infty} T_{n,\zeta}(\lambda) \frac{t^n}{n!}
= \left( \sum_{l=0}^{\infty} \zeta(2|\lambda|) \frac{t^l}{l!} \sum_{m=0}^{\infty} T_{n,\zeta}(\lambda) \frac{t^m}{m!} \right) + \sum_{n=0}^{\infty} T_{n,\zeta}(\lambda) \frac{t^n}{n!}
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} \zeta(2|\lambda|) T_{n-l,\zeta}(\lambda) + T_{n,\zeta}(\lambda) \right) \frac{t^n}{n!}.
\]
By comparing of the coefficients \( \frac{n^n}{n!} \) on the both sides of (2.6), we have the following theorem.

**Theorem 2.3** For \( n \in \mathbb{Z}_+ \), we have

\[
\zeta \sum_{l=0}^{n} \binom{n}{l} (2|\lambda|)_{l} T_{n-l,\zeta}(\lambda) + T_{n,\zeta}(\lambda) = \begin{cases} 
2, & \text{if } n = 0, \\
0, & \text{if } n \neq 0.
\end{cases}
\]

By (2.2), we get

\[
\sum_{n=0}^{\infty} T_{n,\zeta}(1-x,\lambda) \frac{t^n}{n!} = \frac{2}{\zeta(1+\lambda t)^{2/\lambda} + 1} e^{(1-x)t}
\]

\[
= \frac{2}{(1+\lambda t)^{2/\lambda} + 1} e^{x} e^{-xt}
\]

\[
= \left( \sum_{n=0}^{\infty} T_{n,\zeta}(1,\lambda) \frac{t^n}{n!} \right) \left( \sum_{l=0}^{\infty} (-x)^l \frac{t^l}{l!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} T_{n-l,\zeta}(1,\lambda) (-x)^l \right) \frac{t^n}{n!}.
\]

(2.7)

By comparing of the coefficients \( \frac{n^n}{n!} \) on the both sides of (2.7), we have the following theorem.

**Theorem 2.4** For \( n \in \mathbb{Z}_+ \), we have

\[
T_{n,\zeta}(1-x,\lambda) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} T_{n-l,\zeta}(1,\lambda)x^l.
\]

From (2.2), we have

\[
\sum_{n=0}^{\infty} T_{n,\zeta}(x+y,\lambda) \frac{t^n}{n!} = \frac{2}{\zeta(1+\lambda t)^{2/\lambda} + 1} e^{(x+y)t}
\]

\[
= \frac{2}{\zeta(1+\lambda t)^{2/\lambda} + 1} e^{xt} e^{yt}
\]

\[
= \left( \sum_{n=0}^{\infty} T_{m,\zeta}(x,\lambda) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} y^n \frac{t^n}{n!} \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} T_{l,\zeta}(x,\lambda) y^{n-l} \right) \frac{t^n}{n!}.
\]

(2.8)

Therefore, by (2.8), we have the following theorem.
**Theorem 2.5** For $n \in \mathbb{Z}_+$, we have

$$T_{n,\zeta}(x + y, \lambda) = \sum_{l=0}^{n} \binom{n}{l} T_{l,\zeta}(x, \lambda)y^{n-l}.$$ 

Then, it is easy to deduce that $T_{n,\zeta}(x, \lambda)$ are polynomials of degree $n$. Here is the list of the first Appell-type degenerate twisted tangent’s polynomials.

- $T_{0,\zeta}(x, \lambda) = \frac{2}{1 + \zeta};$
- $T_{1,\zeta}(x, \lambda) = -\frac{4\zeta}{(1 + \zeta)^2} + \frac{2x}{(1 + \zeta)^2} + \frac{2\zeta x}{(1 + \zeta)^2};$
- $T_{2,\zeta}(x, \lambda) = -\frac{8\zeta}{(1 + \zeta)^3} + \frac{4\zeta^2}{(1 + \zeta)^3} + \frac{8\zeta^2}{(1 + \zeta)^3} - \frac{4\lambda\zeta^2}{(1 + \zeta)^3} - \frac{8\zeta x}{(1 + \zeta)^2} + \frac{2x^2}{1 + \zeta};$
- $T_{3,\zeta}(x, \lambda) = -\frac{16\zeta}{(1 + \zeta)^4} + \frac{24\lambda\zeta}{(1 + \zeta)^4} - \frac{8\zeta^2}{(1 + \zeta)^4} + \frac{8\lambda^2\zeta^2}{(1 + \zeta)^4} - \frac{64\zeta^2}{(1 + \zeta)^4} + \frac{16\lambda^2\zeta^2}{(1 + \zeta)^4} - \frac{16\zeta^3}{(1 + \zeta)^4} - \frac{24\lambda\zeta^3}{(1 + \zeta)^4} - \frac{8\lambda^2\zeta^3}{(1 + \zeta)^4} - \frac{24\lambda^2\zeta^3}{(1 + \zeta)^4} - \frac{24\zeta^3}{(1 + \zeta)^4} + \frac{12\lambda\zeta x}{(1 + \zeta)^3} + \frac{12\zeta x}{(1 + \zeta)^3} + \frac{2\zeta^2 x}{(1 + \zeta)^3} + \frac{2x^3}{1 + \zeta};$

## 3 Distribution of Zeros of the Appell-type degenerate twisted tangent polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the Appell-type degenerate twisted tangent polynomials $T_{n,\zeta}(x, \lambda)$. We investigate the beautiful zeros of the $T_{n,\zeta}(x, \lambda)$ by using a computer. Let $\zeta = e^{\frac{2\pi i}{N}}$ in $\mathbb{C}$ and $\lambda = \frac{1}{10}$. We plot the zeros of the Appell-type degenerate twisted tangent polynomials $T_{n,\zeta}(x, \lambda)$ for $n = 30$ and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 30$ and $N = 1$. In Figure 1(top-right), we choose $n = 30$ and $N = 3$. In Figure 1(bottom-left), we choose $n = 30$ and $N = 5$. In Figure 1(bottom-right), we choose $n = 30$ and $N = 7$. Stacks of zeros of $T_{n,\zeta}(x, \lambda)$ for $1 \leq n \leq 40$ from a 3-D structure are presented(Figure 2). In Figure 2(left), we choose $1 \leq n \leq 30$, $\zeta = e^{\frac{2\pi i}{N}}$ and $\lambda = \frac{1}{10}$. In Figure 2(right), we choose $1 \leq n \leq 30$, $\zeta = e^{\frac{2\pi i}{N}}$ and $\lambda = \frac{1}{10}$. Our numerical results for approximate solutions of real zeros of $T_{n,\zeta}(x, \lambda)$ are displayed(Tables 1, 2).
Figure 1: Zeros of $T_{n,\zeta}(x, \lambda)$

Table 1. Numbers of real and complex zeros of $T_{n,\zeta}(x, \lambda)$

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>real zeros</th>
<th>complex zeros</th>
<th>real zeros</th>
<th>complex zeros</th>
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<tbody>
<tr>
<td></td>
<td>$\zeta = e^{\frac{2\pi i}{10}}, \lambda = \frac{1}{10}$</td>
<td>$\zeta = e^{\frac{2\pi i}{3}}, \lambda = \frac{1}{10}$</td>
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<td>5</td>
<td>4</td>
<td>0</td>
<td>9</td>
</tr>
</tbody>
</table>
Appell-type degenerate twisted tangent polynomials

Figure 2: Stacks of zeros of $T_{n,\xi}(x,\lambda)$ for $1 \leq n \leq 30$

Plot of real zeros of $T_{n,\xi}(x,\lambda)$ for $1 \leq n \leq 30$ structure are presented (Figure 3). We observe a remarkably regular structure of the complex roots of the

Figure 3: Real zeros of $T_{n,\xi}(x,\lambda)$ for $\xi = e^{2\pi i}, \lambda = \frac{1}{10}$ and $1 \leq n \leq 30$

Appell-type degenerate tangent polynomials $T_{n,\xi}(x,\lambda)$. We hope to verify a remarkably regular structure of the complex roots of the Appell-type degenerate tangent polynomials $T_{n,\xi}(x,\lambda)$ (Table 1). Next, we calculated an approximate solution satisfying $T_{n,\xi}(x,\lambda), x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $T_{n,\xi}(x,\lambda) = 0, \xi = e^{2\pi i}, \lambda = 1/10, x \in \mathbb{R}$

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
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<td>1.00000</td>
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<tr>
<td>2</td>
<td>0.051317, 1.9487</td>
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<tr>
<td>3</td>
<td>0.69275, 1.1041, 2.5886</td>
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<tr>
<td>4</td>
<td>−1.2451, 0.15070, 2.1935, 2.9010</td>
</tr>
<tr>
<td>5</td>
<td>−1.5498, −0.83199, 1.2049</td>
</tr>
</tbody>
</table>
Finally, we consider the more general problems. How many zeros does $T_{n,\zeta}(x,\lambda)$ have? Find the numbers of complex zeros $C_{T_{n,\zeta}(x,\lambda)}$ of $T_{n,\zeta}(x,\lambda)$, $\text{Im}(x) \neq 0$. Since $n$ is the degree of the polynomial $T_{n,\zeta}(x,\lambda)$, the number of real zeros $R_{T_{n,\zeta}(x,\lambda)}$ lying on the real plane $\text{Im}(x) = 0$ is then $R_{T_{n,\zeta}(x,\lambda)} = n - C_{T_{n,\zeta}(x,\lambda)}$, where $C_{T_{n,\zeta}(x,\lambda)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{T_{n,\zeta}(x,\lambda)}$ and $C_{T_{n,\zeta}(x,\lambda)}$. Using computers, many more values of $n$ have been checked. It still remains unknown if the open problem fails or holds for any value $n$. We are able to decide if $T_{n,\zeta}(x,\lambda) = 0$ has $n$ distinct solutions (see Tables 1, 2).

References


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