Mutually Orthogonal Graph Squares for
Disjoint Union of Paths

R. El-Shanawany¹, A. El-Mesady¹ and S. M. Shaaban²,³

¹ Faculty of Electronic Engineering
Department of Physics and Engineering Mathematics
Menoufia University, Menouf, Egypt
Corresponding author: R. El-Shanawany

² Department of Electrical Engineering, Northern Border University
Arar, 1321, Saudi Arabia

³ Department of Engineering Basic Sciences, Minoufiya University
Shebin El-Kom, 32511, Egypt

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Abstract

This paper gives some new results on mutually orthogonal graph squares (MOGS). These generalize mutually orthogonal Latin squares in an interesting way. As such, the topic is quite nice and should have broad appeal. MOGS have strong connections to core fields of finite algebra, cryptography, finite geometry, and design of experiments. We are concerned with the mutually orthogonal half starters method to construct the mutually orthogonal graph squares for disjoint union of paths.

Mathematics Subject Classification: 05C70, 05B30

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1 Introduction

In this paper, we investigate the mutually orthogonal graph squares (MOGS) of the complete bipartite graphs. MOGS have strong connections to core fields of finite algebra, cryptography, finite geometry, and design of experiments. To describe the many constructions for sets of MOGS, we must begin with covering the basic definitions for the graph squares (see also [1]).

Definition 1 Let $G$ be a subgraph of $K_{n,n}$ with $n$ edges. A square matrix $L$ of order $n$ is a $G$-square if every element in $\mathbb{Z}_n$ occurs exactly $n$ times and the graphs $G_i$ where $i \in \mathbb{Z}_n$ with $E(G_i) = \{(x,y) : L(x,y) = i\}$ are isomorphic to $G$. It is clear that the $G$-square represents the edge decomposition of $K_{n,n}$ by $G$.

Definition 2 The two graph squares have the property that, when superimposed, every ordered pair occurs exactly once. Thus the squares are orthogonal. A set of graph squares $L^1, \ldots, L^k$ is mutually orthogonal, or a set of MOGS, if $L^i$ and $L^j$ are orthogonal for every $1 \leq i < j \leq k$.

The $i$-translate of $G$ is defined as follows if $G$ is a subgraph of the complete bipartite graph $K_{n,n}$, and $i \in \mathbb{Z}_n$, then the graph $G + i$ with the edge set $E(G + i) = \{(p + i, q + i) : (p, q) \in E(G)\}$ is the $i$-translate of $G$. The length of the edge $e = (p, q) \in E(G)$ is defined by $d(e) = q - p$ where the sums and differences are calculated modulo $n$. For the graph $G$ if $|E(G)| = n$ and the lengths of all edges in $G$ are mutually distinct and equal to $\mathbb{Z}_n$, then the graph $G$ is said to be a half starter w.r.t. $\mathbb{Z}_n$. In [1], El-Shanawany has proved the following Theorem.

Theorem 3 The union of all translates of $G$ forms an edge decomposition of $K_{n,n}$, i.e.,

$$\bigcup_{z \in \mathbb{Z}_n} E(G + z) = E(K_{n,n})$$

iff $G$ is a half starter.
and the corresponding three mutually orthogonal P\(_4\) ∪ 2P\(_2\)-squares are defined as follows, where L\(^i\) is corresponding to v\(^i\)(G + j); i ∈ Z\(_3\), j ∈ Z\(_5\).

For more illustration, see Figures 2, 3, and 4, where the graph G\(_j\) \(\cong P_4 \cup 2P_2\) is corresponding to the entry j in the square L\(^i\); i ∈ Z\(_3\), j ∈ Z\(_5\).

Let us write N(n, G) for the maximal number of squares in the largest possible set of mutually orthogonal G-squares of order n. This paper gives some
new results on mutually orthogonal graph squares (MOGS). These generalize mutually orthogonal Latin squares (MOLS) in an interesting way. For a survey on MOLS, see [2]. The results in [3-6] motivated us to consider the MOGS for the disjoint union of paths.

2 Main Results

**Theorem 5** Let \( n \geq 5 \) be a prime. Then \( N(n, P_n \cup (n-3)P_2) \geq (n-2) \).

**Proof.** For \( k \in \mathbb{Z}_{n-2} \), let \( G_k \) be subgraphs of \( K_{n,n} \) with \( n \) edges. Define the \((n-2)\) half starter vectors as follows, for \( k = 0 \), we have \( v_0(G_0) = 0, v_i(G_0) = -2(i-1); 1 \leq i \leq n-1 \) and for \( k \in \mathbb{Z}_{n-2} \setminus \{0\} \), we have \( v_0(G_k) = 0, v_i(G_k) = (k+2)i - 2i + 2; 1 \leq i \leq n-1 \). Firstly, our task is to prove the orthogonality of \( G_0 \) and \( G_k \), \( k \in \mathbb{Z}_{n-2} \setminus \{0\} \), we have

\[
\{v_i(G_k) - v_i(G_0) : i \in \mathbb{Z}_n\} = \{(k+2)i - 2i + 2 - (-2i) = (k+2)i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n. \tag{1}
\]

We conclude from (1) that \( G_0 \) and \( G_k \), \( k \in \mathbb{Z}_{n-2} \setminus \{0\} \) are orthogonal. Secondly, our task is to prove the orthogonality of \( G_r \) and \( G_s \) where \( r \neq s \) and \( 1 \leq r, s \leq n-3 \), we have

\[
\{v_i(G_r) - v_i(G_s) : i \in \mathbb{Z}_n\} = \{(r+2)i - 2i + 2 - (s+2)i + 2i - 2 = (r-s)i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n. \tag{2}
\]
We conclude from (2) that $G_r$ and $G_s$ are orthogonal. Hence, the half starters $G_k$, $k \in \mathbb{Z}_{n-2}$ are mutually orthogonal. It remains to prove the isomorphism of the half starters $G_k$, $k \in \mathbb{Z}_{n-2}$. From the vectors of these half starters, for each half starter, there are $(n-2)$ pendant vertices and one vertex with degree two in each independent set of $K_{n,n}$.

In the following two Theorems 6 and 7 we retrieve the previous result by new techniques.

**Theorem 6** Let $n \geq 5$ be a prime. Then $N(n, P_4 \cup (n-3)P_2) \geq (n - 2)$.

**Proof.** Let $G_k$ be subgraphs of $K_{n,n}$ with $n$ edges. Define the $(n-2)$ half starter vectors as follows, $v_i(G_j) = i^{n-1} + (j+1)i; i \in \mathbb{Z}_n, j \in \mathbb{Z}_{n-2}$. Our task is to prove the orthogonality of the $n-2$ half starter vectors, let $k, l \in \mathbb{Z}_{n-2}$ and $k \neq l$, then

$$\{v_i(G_k) - v_i(G_l) : i \in \mathbb{Z}_n\} = \{(k - l)i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n.$$

It remains to prove the isomorphism of the $n-2$ half starters. From the vectors of these half starters, for each half starter, there are $n-2$ pendant vertices and one vertex with degree two in each independent set of $K_{n,n}$. ■

**Theorem 7** Let $n \geq 5$ be a prime. Then $N(n, P_4 \cup (n-3)P_2) \geq n - 2$.

**Proof.** We have $n - 2$ half starter vectors defined by

$$v(G_i) = (0, i + 2, 2i + 3, 3i + 4, 4i + 5, \ldots, (n - 1)i)$$

where $i \in \mathbb{Z}_{n-2}$. Our task is to prove the orthogonality of the $n-2$ half starter vectors, let $k, l \in \mathbb{Z}_{n-2}$, and $k \neq l$, then

$$\{v_i(G_k) - v_i(G_l) : i \in \mathbb{Z}_n\} = \{(k - l)i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n.$$

It remains to prove the isomorphism of the $n-2$ half starters. From the vectors of these half starters, for each half starter, there are $n-2$ pendant vertices and one vertex with degree two in each independent set of $K_{n,n}$. ■

Now, we introduce the following Corollary to construct 4 mutually orthogonal $2P_4 \cup 5P_2$-squares and then a Conjecture.

**Corollary 8** $N(11, 2P_4 \cup 5P_2) \geq 4$.

**Proof.** We have 4 half starter vectors defined by

$$v(G_{\frac{i}{2} - \alpha}) = (0, i, 2i + 2, 3i + 3, 4i + 4, 5i + 5, 6i + 6, 7i + 7, 8i + 8, 9i + 9, 10i + 9)$$
where \( i = 0, 2, 6, 8 \), \( \alpha = 0 \) if \( i = 0, 2 \) and \( \alpha = 1 \) if \( i = 6, 8 \). Our task is to prove the orthogonality of the four half starter vectors, let \( k, l \in \{0, 2, 6, 8\} \), and \( k \neq l \), then

\[
\{v_i(G_k) - v_i(G_l) : i \in \mathbb{Z}_{11}\} = \{(k - l)i : i \in \mathbb{Z}_{11}\} = \mathbb{Z}_{11}.
\]

It remains to prove the isomorphism of the four half starters. From the vectors of the four half starters, for each half starter, there are 7 pendant vertices and two vertices with degree two in each independent set of \( K_{11,11} \).

**Conjecture 9** \( N(mx + ny, mP_{x+1} \cup nP_{y+1}) \geq 4 \), where \( mx + ny \) is a prime number.

Now, we introduce the following Corollary to construct 3 mutually orthogonal \( P_4 \cup P_5 \cup 4P_2 \)-squares and then a Conjecture.

**Corollary 10** \( N(11, P_4 \cup P_5 \cup 4P_2) \geq 3 \).

**Proof.** We have 3 half starter vectors defined by

\[ v(G_{i-\alpha}) = (0, i+2, 2i+4, 3i+6, 4i+1, 5i+4, 6i+7, 7i+10, 8i+2, 9i+5, 10i+8) \]

where \( i = 0, 1, 4 \), \( \alpha = 0 \) if \( i = 0, 1 \) and \( \alpha = 2 \) if \( i = 4 \). Our task is to prove the orthogonality of the three half starter vectors, let \( k, l \in \{0, 1, 2\} \), and \( k \neq l \), then

\[
\{v_i(G_k) - v_i(G_l) : i \in \mathbb{Z}_{11}\} = \{(k - l)i : i \in \mathbb{Z}_{11}\} = \mathbb{Z}_{11}.
\]

It remains to prove the isomorphism of the 3 half starters. From the vectors of the 3 half starters, for each half starter, there are 7 pendant vertices and two vertices with degree two in one independent set and 5 pendant vertices and 3 vertices with degree two in the other independent set of \( K_{11,11} \).

**Conjecture 11** \( N(mx + ny + lz, mP_{x+1} \cup nP_{y+1} \cup lP_{z+1}) \geq 3 \), where \( mx + ny + lz \) is a prime number.

In the following Theorem, we have MOGS for disjoint union of copies of \( P_4 \) and copies of \( P_2 \).

**Theorem 12** Let \( n \geq 7 \) be a prime. Then \( N(n, 2P_4 \cup (n - 6)P_2) \geq n - 4 \).

**Proof.** We have \( n - 4 \) half starter vectors defined by

\[ v(G_i) = (0, i+3, 2i+6, 3i+9, \ldots, (n-3)i+3(n-3), 1+(n-2)(i+3), (n-1)(i+4)) \]
where \( i \in \mathbb{Z}_{n-4} \). Our task is to prove the orthogonality of the \( n-4 \) half starter vectors, let \( k, l \in \mathbb{Z}_{n-4} \), and \( k \neq l \), then
\[
\{v_i(G_k) - v_i(G_l) : i \in \mathbb{Z}_n \} = \{(k - l)i : i \in \mathbb{Z}_n \} = \mathbb{Z}_n.
\]

It remains to prove the isomorphism of the \( n-4 \) half starters. From the vectors of these half starters, for each half starter, there are \( n-4 \) pendant vertices and two vertices with degree two in each independent set of \( K_{n,n} \).

In the following Corollary, we prove that \( N(15, H_1) \geq 3 \) where \( H_1 \) is isomorphic to \( G \) and
\[
E(G) = \{(14, j) : j \in \{0, 3, 10, 13\}\} \cup \{(11, j) : j \in \{3, 4, 9, 13\}\} \cup \{(i, 0) : i \in \{0, 5, 9\}\} \cup \{(6, 9), (9, 3), (6, 3), (5, 10)\}.
\]

**Corollary 13** \( N(15, H_1) \geq 3 \).

**Proof.** We have 3 half starter vectors defined by
\[
v(G_i) = (0, i + 14, 2i + 11, 3i + 6, 4i + 14, 5i + 5, 6i + 9, 7i + 11, 8i + 11, 9i + 9, 10i + 5, 11i + 14, 12i + 6, 13i + 11, 14i + 14)
\]

where \( i \in \mathbb{Z}_3 \). Our task is to prove the orthogonality of the 3 half starter vectors, let \( k, l \in \mathbb{Z}_3 \), and \( k \neq l \), then
\[
\{v_i(G_k) - v_i(G_l) : i \in \mathbb{Z}_{15} \} = \{(k - l)i : i \in \mathbb{Z}_{15} \} = \mathbb{Z}_{15}.
\]

It remains to prove the isomorphism of the 3 half starters. From the vectors of these half starters, for each half starter, there are two vertices with degree four, three vertices with degree two, and one pendant vertex in each independent set of \( K_{15,15} \).

### 3 Conclusion

In conclusion, we summarize the results of the paper in Table 1.

<table>
<thead>
<tr>
<th>Conclusion</th>
<th>Expression</th>
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<tbody>
<tr>
<td>1-N(n, P_4 \cup (n-3)P_2) \geq (n-2)</td>
<td></td>
</tr>
<tr>
<td>2-N(11, 2P_4 \cup 5P_2) \geq 4</td>
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</tr>
<tr>
<td>3-N(11, P_4 \cup P_5 \cup 4P_2) \geq 3</td>
<td></td>
</tr>
<tr>
<td>4-N(n, 2P_4 \cup (n-6)P_2) \geq n-4</td>
<td></td>
</tr>
<tr>
<td>5-N(15, H_1) \geq 3</td>
<td></td>
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</tbody>
</table>

Table 1: Summary of the results.
References


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