Complete Convergence for Asymptotically Almost Negatively Associated Random Variables

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Abstract

In this paper, some sufficient conditions of complete convergence for asymptotically almost negatively associated random variables are obtained, which partially extend the corresponding ones for independent random variables and negatively associated random variables.

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1. Introduction

1.1. Complete convergence. A sequence of random variables \( \{U_n, n \geq 1\} \) is said to converge completely to a constant \( C \) if

\[
\sum_{n=1}^{\infty} \mathbb{P}( |U_n - C| > \varepsilon ) < \infty, \quad \text{for all } \varepsilon > 0.
\] (1.1)

The concept of complete convergence was introduced firstly by Hsu and Robbins [1]. In view of the Borel-Cantelli lemma, complete convergence implies that \( U_n \to C \) almost surely. The converse is true if \( \{U_n, n \geq 1\} \) are independent random variables. Hsu and Robbins [1] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables...
variables converges completely to the expected value if the variance of the summands is finite. Somewhat later, Erdős [2] proved the converse. We summarize their results as follow.

**Hsu-Robbins-Erdős Strong Law.** Let \( \{X, X_n, n \geq 1\} \) be a sequence of i.i.d. random variables with mean zero, and set \( S_n = \sum_{i=1}^{n} X_i \), \( n \geq 1 \), then \( \mathbb{E}X^2 < \infty \) is equivalent to the condition that

\[
\sum_{n=1}^{\infty} \mathbb{P}(\{|S_n| > \varepsilon n\}) < \infty, \quad \text{for all } \varepsilon > 0. \tag{1.2}
\]

Hsu-Robbins-Erdős strong law can be viewed as a result on the rate of convergence in the law of large numbers. The following theorem is a more general result which bridges the integrability of summands and the rate of convergence in the Marcinkiewicz-Zygmund strong law of large numbers.

**Theorem A.** Let \( 0 < r < 2 \), \( r \leq p \). Suppose that \( \{X, X_n, n \geq 1\} \) is a sequence of i.i.d. random variables with mean zero, and set \( S_n = \sum_{i=1}^{n} X_i \), \( n \geq 1 \), then \( \mathbb{E}|X|^p < \infty \) is equivalent to the condition that

\[
\sum_{n=1}^{\infty} n^{p/r-2} \mathbb{P}(\{|S_n| > \varepsilon n^{1/r}\}) < \infty, \quad \text{for all } \varepsilon > 0, \tag{1.3}
\]

and also equivalent to the condition that

\[
\sum_{n=1}^{\infty} n^{p/r-2} \mathbb{P}\left(\max_{1 \leq k \leq n} |S_k| > \varepsilon n^{1/r}\right) < \infty, \quad \text{for all } \varepsilon > 0. \tag{1.4}
\]

For \( r = p = 1 \), the equivalence between \( \mathbb{E}|X| < \infty \) and (1.3) is a famous result due to Spitzer [3]. For \( p = 2 \) and \( r = 1 \), the equivalence between \( \mathbb{E}X^2 < \infty \) and (1.3) is just Hsu-Robbins-Erdős strong law. For the general \( p, r \) satisfying the conditions of Theorem A, Katz [4], and later Baum and Katz [5] proved the equivalence between \( \mathbb{E}|X|^p < \infty \) and (1.3) and Chow [6] established the equivalence between \( \mathbb{E}|X|^p < \infty \) and (1.4).

For the i.i.d. case, related results are fruitful and detailed. It is natural to extend them to dependent cases, for examples, martingale difference, negatively associated sequence, mixing random variables and so on. In the present paper, we are interested in the asymptotically almost negatively associated random variables.

1.2. **Some notations.** First, let us recall some definitions as follow.

**Definition 1.1** The random variables \( \{X_n, n \geq 1\} \) is stochastically dominated by the random variable \( X \) if there exists a positive constant \( C \) such that,

\[
\mathbb{P}(|X_n| > x) \leq C \mathbb{P}(|X| > x) \tag{1.5}
\]

for all \( x > 0 \) and \( n \geq 1 \).
Definition 1.2 ([7]) A finite family of random variables \( \{X_k, 1 \leq k \leq n\} \) is said to be negatively associated (NA), if for any disjoint subsets \( A \) and \( B \) of \( \{1, 2, \ldots, n\} \) and any real coordinatewise nondecreasing functions \( f \) on \( \mathbb{R}^A \) and \( g \) on \( \mathbb{R}^B \),
\[
Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0,
\]
whenever the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

Definition 1.3 ([8]) A sequence \( \{X_n, n \geq 1\} \) of random variables is said to be asymptotically almost negatively associated (AANA) if there exists a nonnegative sequence \( q(n) \to 0 \) as \( n \to \infty \) such that
\[
Cov(f(X_n), g(X_{n+1}, X_{n+2}, \ldots, X_{n+k})) \leq q(n) [Var(f(X_n))Var(g(X_{n+1}, X_{n+2}, \ldots, X_{n+k}))]^{1/2}
\]
for all \( n, k \geq 1 \) and all coordinate-wise nondecreasing continuous functions \( f \) and \( g \) whenever the variances exist. \( \{q(n), n \geq 1\} \) is called the mixing coefficients of \( \{X_n, n \geq 1\} \).

Obviously, the family of AANA sequences contain NA sequence (with \( q(n) = 0, n \geq 1 \)). An example of AANA random variables which are not NA was constructed by Chandra and Ghosal ([8]).

Since the notion of AANA sequence was introduced by Chandra and Ghosal ([8]), the AANA properties have aroused wide interest because of numerous applications in reliability theory and multivariate statistical analysis. The probability inequality, moment inequality, probability limit theorems and application for AANA random variables were obtained. For more details on related results, one can refer to Chandra [8], Yuan[9], Wang[10, 11, 12], Wang[13], An[14], Ko[15], Tang[16], and so on.

Throughout the paper, \( C \) denote positive constants which may be different in variables places. \( I(A) \) be the indicator functions of the set \( A \). The \( a_n = O(b_n) \) denote that there exists a positive constant \( C \) such that \( a_n \leq Cb_n \).

2. Main results

In the section, we state our main results.

Theorem 2.1. Let \( \{X_n, n \geq 1\} \) be a sequence of AANA random variables with mixing coefficients \( \{q(n), n \geq 1\} \), and \( \sum_{n=1}^{\infty} q^2(n) < \infty \), which is stochastically dominated by \( X \). Let \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) be an array of constants such that for some \( 0 < \alpha < 2 \),
\[
\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n),
\]
Let \( b_n = n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}} \), if \( E(X_n) = 0 \) for \( 1 < \alpha < 2 \) and \( E|X|^{\alpha} \log(1+|X|) < \infty \), then for any \( \epsilon > 0 \),

\[
\sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \left| \sum_{i=1}^{n} a_{ni}X_{i} \right| > \epsilon b_{n} \right) < \infty. \tag{2.2}
\]

**Theorem 2.2.** Let \( \{X_{n}, n \geq 1\} \) be a sequence of AANA random variables with mixing coefficients \( \{q(n), n \geq 1\} \), and \( \sum_{n=1}^{\infty} q^{2}(n) < \infty \). Let 1 \( \leq p \leq 2 \), if
\[
\sum_{n=1}^{\infty} n^{-p} E|X_{n}|^{p} < \infty, \tag{2.3}
\]
and satisfy the Cesàro uniform integrability condition, i.e.,
\[
\lim_{x \to \infty} \sup_{n \geq 1} n^{-1} \sum_{k=1}^{n} E|X_{k}| I(|X_{k}| > x) = 0. \tag{2.4}
\]

Then for any \( \epsilon > 0 \),
\[
\sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_{k} - EX_{k}) \right| > \epsilon n \right) < \infty. \tag{2.5}
\]
In particular,
\[
n^{-1} \sum_{k=1}^{n} (X_{k} - EX_{k}) \to 0, \ a.s. \tag{2.6}
\]

3. **Proofs of Main results**

3.1. **Some lemmas.** To prove our results, we first give some lemmas as follows.

**Lemma 3.1.** ([9]) Let \( \{X_{n}, n \geq 1\} \) be a sequence of AANA random variables with mixing coefficients \( \{q(n), n \geq 1\} \). Let \( f_{1}, f_{2}, \cdots \) be all nondecreasing (or all nonincreasing) functions, then \( \{f_{n}(X_{n}), n \geq 1\} \) is still a sequence of AANA random variables with mixing coefficients \( \{q(n), n \geq 1\} \).

**Lemma 3.2.** ([9]) Let \( \{X_{n}, n \geq 1\} \) be an AANA sequence of zero mean random variables with mixing coefficients \( \{q(n), n \geq 1\} \). If \( \sum_{n=1}^{\infty} q^{2}(n) < \infty \), then there exists a positive constant \( C_{p} \) depending only on \( p \) such that
\[
E \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_{i} \right|^{p} \right) \leq C_{p} \sum_{i=1}^{n} E|X_{i}|^{p} \tag{3.1}
\]
for all \( n \geq 1 \) and \( 1 < p \leq 2 \).

**Lemma 3.3.** ([17]) Let \( X \) be a random variables, \( \{a_{ni}, 1 \leq i \leq n, n \geq 1\} \) be an array of positive constants such that (2.1) for some \( 0 < \alpha < 2 \). Let
Lemma 3.4. \((\text{[17]})\) Let \(X\) be a random variables, \(b_n = n^{1/\alpha}(\log n)^{1/\gamma}\) for some \(\gamma > 0\). Then
\[
\sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} \mathbb{P}(|a_{ni}| > b_n) \leq \begin{cases} 
C\mathbb{E}|X|^\alpha, & \text{if } \alpha > \gamma \\
C\mathbb{E}|X|^\alpha \log(1 + |X|), & \text{if } \alpha = \gamma \\
C\mathbb{E}|X|^{\gamma}, & \text{if } \alpha < \gamma 
\end{cases}
\] (3.2)

Lemma 3.5. \((\text{[18]})\) Let \(\{X_n, n \geq 1\}\) be a sequence of random variables which is stochastically dominated by a random variable \(X\). If \(\mathbb{E}|X|^p < \infty\) for some \(p > 0\), then for any \(t > 0\) and \(n \geq 1\), the following statements hold:
\[
\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}|X_k|^p \leq C\mathbb{E}|X|^p,
\] (3.4)
\[
\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}|X_k|^p I(|X_k| \leq t) \leq C \left[ \mathbb{E}|X|^p I(|X| \leq t) + t^p \mathbb{P}(|X| > t) \right]
\] (3.5)
and
\[
\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}|X_k|^p I(|X_k| > t) \leq C\mathbb{E}|X|^p I(|X| > t).
\] (3.6)

Lemma 3.6. Let \(1 \leq p \leq 2\), \(\{X_n, n \geq 1\}\) be a sequence of AANA random variables with mixing coefficients \(\{q(n), n \geq 1\}\), and \(\sum_{n=1}^{\infty} q^2(n) < \infty\). If \(\sum_{n=1}^{\infty} n^{-p}\mathbb{E}|X_n|^p < \infty\), then for any \(\varepsilon > 0\),
\[
\sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \left| \sum_{k=1}^{n} (X_k - \mathbb{E}X_k) \right| > n\varepsilon \right) < \infty.
\] (3.7)

Proof. By \(\sum_{n=1}^{\infty} n^{-p}\mathbb{E}|X_n|^p < \infty\) and Kronecker lemma, we can get that
\[
n^{-p} \sum_{k=1}^{n} \mathbb{E}|X_k|^p \rightarrow 0, \text{ as } n \rightarrow \infty.
\] (3.8)

Hence for any \(\varepsilon > 0\), we have that for \(n\) large enough
\[
n^{-1} \left| \sum_{k=1}^{n} \mathbb{E}X_k I(|X_k| > n) \right| \leq n^{-1} \sum_{k=1}^{n} \mathbb{E}|X_k| I(|X_k| > n)
\leq n^{-1} \sum_{k=1}^{n} n^{-1-p} \mathbb{E}|X_k|^p I(|X_k| > n) \leq n^{-p} \sum_{k=1}^{n} \mathbb{E}|X_k|^p \leq \frac{\varepsilon}{2}.
\] (3.9)
By Lemma 3.2, \( \{X_k I(|X_k| \leq n) - \mathbb{E}X_k I(|X_k| \leq n), n \geq 1, 1 \leq k \leq n \} \) is a sequence of AANA random variables with zero mean. From (3.9), Lemma 3.3 and Markov’s inequality, we can get that

\[
\sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \left| \sum_{k=1}^{n} (X_k - \mathbb{E}X_k) \right| > \varepsilon n \right) \leq \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{n} P (|X_k| > n)
\]

\[
+ \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \left| \sum_{k=1}^{n} (X_k I(X_k) \leq n) - \mathbb{E}X_k I(X_k \leq n) \right| > \frac{\varepsilon n}{2} \right)
\]

\[
\leq \sum_{n=1}^{\infty} n^{-1-p} \sum_{k=1}^{n} \mathbb{E}|X_k|^p + C \sum_{n=1}^{\infty} n^{-3} \sum_{k=1}^{n} \mathbb{E}|X_k|^2 \mathbb{I}(|X_k| \leq n)
\]

(3.10)

\[
\leq \sum_{n=1}^{\infty} n^{-p-1} \sum_{k=1}^{n} \mathbb{E}|X_k|^p + C \sum_{n=1}^{\infty} n^{-p-1} \sum_{k=1}^{n} \mathbb{E}|X_k|^p
\]

\[
\leq C \sum_{k=1}^{\infty} \mathbb{E}|X_k|^p \sum_{n=k}^{\infty} n^{-p-1} \leq C \sum_{k=1}^{\infty} k^{-p} \mathbb{E}|X_k|^p < \infty.
\]

\[\square\]

3.2. Proof of Theorem 2.1. With loss of generality, we may assume that \( \sum_{i=1}^{n} |a_{ni}|^\alpha \leq n \), and \( a_{ni} \geq 0 \) for all \( 1 \leq i \leq n \) and \( n \geq 1 \). For \( n \geq 1 \), and \( 1 \leq i \leq n \), let

\[ X_{ni} = -b_n I(a_{ni}X_i < -b_n) + a_{ni}X_i I(|a_{ni}X_i| \leq b_n) + b_n I(a_{ni}X_i > b_n), \]

and

\[ Z_{ni} = (a_{ni}X_i + b_n)I(a_{ni}X_i < -b_n) + (a_{ni}X_i - b_n)I(a_{ni}X_i > b_n). \]

It is easy to see that for \( \forall n \geq 1, 1 \leq i \leq n \), we have \( X_{ni} + Z_{ni} = a_{ni}X_i \).

Then

\[
\sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \left| \sum_{i=1}^{n} a_{ni}X_i \right| > \epsilon b_n \right)
\]

\[
\leq \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \bigcup_{i=1}^{n} \{ |a_{ni}X_i| > b_n \} \right) + \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \left| \sum_{i=1}^{n} X_i \right| > \epsilon b_n, \bigcap_{i=1}^{n} \{ |a_{ni}X_i| \leq b_n \} \right)
\]

\[
\leq \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} \mathbb{P} (|a_{ni}X_i| > b_n) + \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \left| \sum_{i=1}^{n} X_{ni} \right| > \epsilon b_n \right) \triangleq I_1 + I_2.
\]

(3.11)

Therefore, to prove (2.2), it is enough to show that \( I_1 < \infty \) and \( I_2 < \infty \). By the Lemma 3.3 and the condition \( \mathbb{E}|X|^\alpha \log(1 + |X|) < \infty \), we have

\[
I_1 \leq C \mathbb{E}|X|^\alpha \log(1 + |X|) < \infty.
\]

(3.12)
To prove $I_2 < \infty$, we first show that
\[ b_n^{-1} \left| \sum_{i=1}^{n} \mathbb{E} X_{ni} \right| \to 0, \quad n \to \infty. \] (3.13)

For $1 < \alpha < 2$, note that $\mathbb{E} X_i = 0$ and $X_{ni} + Z_{ni} = a_{ni}X_i$, then $\mathbb{E} X_{ni} = -\mathbb{E} Z_{ni}$. By Markov’s inequality and Lemma 3.5, we have
\[ b_n^{-1} \left| \sum_{i=1}^{n} \mathbb{E} X_{ni} \right| \leq b_n^{-1} \sum_{i=1}^{n} \mathbb{E} |Z_{ni}| \]
\[ \leq \sum_{i=1}^{n} \mathbb{P}(|a_{ni}X_i| > b_n) + b_n^{-1} \sum_{i=1}^{n} \mathbb{E}|a_{ni}X_i|I(|a_{ni}X_i| > b_n) \]
\[ \leq Cb_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} \mathbb{E}|X|^{\alpha} + Cb_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} \mathbb{E}|X|^{\alpha} \]
\[ \leq C\mathbb{E}|X|^{\alpha}(\log n)^{-1} \to 0, \quad n \to \infty. \] (3.14)

For $0 < \alpha \leq 1$, by Lemma 3.5 and Markov’s inequality, then
\[ b_n^{-1} \left| \sum_{i=1}^{n} \mathbb{E} X_{ni} \right| \leq b_n^{-1} \sum_{i=1}^{n} \mathbb{E}|X_{ni}| \]
\[ \leq \sum_{i=1}^{n} \mathbb{P}(|a_{ni}X_i| > b_n) + b_n^{-1} \sum_{i=1}^{n} \mathbb{E}|a_{ni}X_i|I(|a_{ni}X_i| \leq b_n) \]
\[ \leq \sum_{i=1}^{n} \mathbb{P}(|a_{ni}X| > b_n) + b_n^{-1} \sum_{i=1}^{n} \mathbb{E}|a_{ni}X|I(|a_{ni}X| \leq b_n) \]
\[ \leq C \sum_{i=1}^{n} \mathbb{P}(|a_{ni}X| > b_n) + Cb_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} \mathbb{E}|X|^{\alpha} \]
\[ \leq C\mathbb{E}|X|^{\alpha}(\log n)^{-1} \to 0, \quad n \to \infty. \] (3.15)

Hence, to prove $I_2 < \infty$, it is enough to prove that
\[ I_3 = \sum_{n=1}^{\infty} \mathbb{P} \left( \left| \sum_{i=1}^{n} (X_{ni} - \mathbb{E} X_{ni}) \right| > \epsilon b_n \right) < \infty, \quad \text{for all } \epsilon > 0. \] (3.16)

From Lemma 3.1, it is easy to see that for any fixed $n \geq 1$, $\{X_{ni} - \mathbb{E} X_{ni}, 1 \leq i \leq n\}$ are AANA random variables with mean zero. By Markov’s inequality
and Lemma 3.2, we get that

\[
I_3 = \sum_{n=1}^{\infty} n^{-1} \mathbb{E} \left( \left| \sum_{i=1}^{n} (X_{ni} - \mathbb{E}X_{ni}) \right| > \epsilon b_n \right) \leq \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \mathbb{E} \left( \sum_{i=1}^{n} (X_{ni} - \mathbb{E}X_{ni}) \right)^2 \\
\leq \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i=1}^{n} \mathbb{E} (X_{ni} - \mathbb{E}X_{ni})^2 \leq \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i=1}^{n} \mathbb{E} (X_{ni}^2) \\
\leq C \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i=1}^{n} \mathbb{E} (a_{ni}X_i)^2 I(|a_{ni}X_i| \leq b_n) + b_n^2 \mathbb{P}(|a_{ni}X_i| > b_n) \\
\leq C \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i=1}^{n} \mathbb{E} (a_{ni}X)^2 I(|a_{ni}X| \leq b_n) + b_n^2 \mathbb{P}(|a_{ni}X| > b_n) \\
\leq C \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i=1}^{n} \mathbb{E} (a_{ni}X)^2 I(|a_{ni}X| \leq b_n) + C \sum_{n=1}^{\infty} n^{-1} \sum_{i=1}^{n} \mathbb{P}(|a_{ni}X| > b_n) \\
\triangleq I_4 + I_5. \tag{3.17}
\]

Similar to \( I_1 < \infty \), we have \( I_5 < \infty \). To prove \( I_4 < \infty \), we divide \( \{a_{ni}, 1 \leq i \leq n\} \) into three subsets \( \{a_{ni} : |a_{ni}| \leq 1/(\log n)^s\}, \{a_{ni} : 1/(\log n)^s < |a_{ni}| \leq 1\}, \{a_{ni} : |a_{ni}| > 1\} \), where \( s = 1/(2 - \alpha) \). Then

\[
I_4 = \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i=1}^{n} \mathbb{E} (a_{ni}X)^2 I(|a_{ni}X| \leq b_n) \\
= \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i : |a_{ni}| \leq 1/(\log n)^s} \mathbb{E} (a_{ni}X)^2 I(|a_{ni}X| \leq b_n) \\
+ \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i : 1/(\log n)^s < |a_{ni}| \leq 1} \mathbb{E} (a_{ni}X)^2 I(|a_{ni}X| \leq b_n) \\
+ \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i : |a_{ni}| > 1} \mathbb{E} (a_{ni}X)^2 I(|a_{ni}X| \leq b_n) \\
\triangleq I_{41} + I_{42} + I_{43}. \tag{3.18}
\]

By Lemma 3.4, we get that \( I_{43} < \infty \). Note that

\[
\sum_{i : |a_{ni}| \leq 1/(\log n)^s} |a_{ni}|^\alpha \leq n(\log n)^{-s\alpha},
\]
and \(E|X|^\alpha < \infty\), then

\[
I_{41} = \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i: |a_{ni}| \leq 1/(\log n)^s} \mathbb{E}(a_{ni}X)^2 I(|a_{ni}X| \leq b_n)
\]

\[
\leq C \sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i: |a_{ni}| \leq 1/(\log n)^s} \mathbb{E}(a_{ni}X)^\alpha I(|a_{ni}X| \leq b_n) \tag{3.19}
\]

\[
\leq C \sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i: |a_{ni}| \leq 1/(\log n)^s} |a_{ni}|^\alpha \leq C \sum_{n=1}^{\infty} n^{-1} (\log n)^{-1-s\alpha} < \infty.
\]

Note that

\[
\sum_{i: 1/(\log n)^s < |a_{ni}| \leq 1} |a_{ni}|^2 \leq n,
\]

and \(E|X|^\alpha < \infty\), then

\[
I_{42} = \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i: 1/(\log n)^s < |a_{ni}| \leq 1} \mathbb{E}(a_{ni}X)^2 I(|a_{ni}X| \leq b_n)
\]

\[
\leq \sum_{n=1}^{\infty} n^{-1} b_n^{-2} \sum_{i: 1/(\log n)^s < |a_{ni}| \leq 1} a_{ni}^2 \mathbb{E}X^2 I(|X| \leq b_n (\log n)^s)
\]

\[
\leq C \sum_{n=1}^{\infty} b_n^{-2} \mathbb{E}X^2 I(|X| \leq n^{1/\alpha} (\log n)^{s+1/\alpha})
\]

\[
\leq C \sum_{n=1}^{\infty} b_n^{-2} \sum_{i=1}^{n} \mathbb{E}X^2 I\left((i - 1)^{1/\alpha} (\log(i - 1))^{s+1/\alpha} < |X| \leq i^{1/\alpha} (\log i)^{s+1/\alpha}\right)
\]

\[
= C \sum_{i=1}^{\infty} \mathbb{E}X^2 I\left((i - 1)^{1/\alpha} (\log(i - 1))^{s+1/\alpha} < |X| \leq i^{1/\alpha} (\log i)^{s+1/\alpha}\right)
\]

\[
\cdot \sum_{n=i}^{\infty} n^{-2/\alpha} (\log n)^{-2/\alpha}
\]

\[
\leq C \sum_{i=1}^{\infty} \mathbb{E}X^2 I\left((i - 1)^{1/\alpha} (\log(i - 1))^{s+1/\alpha} < |X| \leq i^{1/\alpha} (\log i)^{s+1/\alpha}\right) (\log i)^{-2/\alpha}
\]

\[
\cdot \sum_{n=i}^{\infty} n^{-2/\alpha}
\]

\[
\leq C \sum_{i=1}^{\infty} \mathbb{E}X^2 I\left((i - 1)^{1/\alpha} (\log(i - 1))^{s+1/\alpha} < |X| \leq i^{1/\alpha} (\log i)^{s+1/\alpha}\right) (\log i)^{-2/\alpha} n^{1-2/\alpha}
\]

\[
\leq C E|X|^\alpha < \infty.
\]

(3.20)
3.3. **Proof of Theorem 2.2.** For given \( \varepsilon > 0 \), by the Cesàro uniform integrality condition, there is a constant \( x = x(\varepsilon) > 0 \), such that for all \( n \geq 1 \),

\[
\frac{1}{n} \left| \sum_{k=1}^{n} \mathbb{E} X_k I(|X_k| > x) \right| \leq \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} |X_k| I(|X_k| > x) \leq \frac{\varepsilon}{4}. \tag{3.21}
\]

For \( x = x(\varepsilon) \), we have

\[
\sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i - \mathbb{E} X_i) \right| > cn \right) \\
\leq \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i I(|X_i| \leq x) - \mathbb{E} X_i I(|X_i| \leq x)) \right| > \frac{n\varepsilon}{4} \right) \tag{3.22} \\
+ \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i I(|X_i| > x) - \mathbb{E} X_i I(|X_i| > x)) \right| > \frac{3n\varepsilon}{4} \right) \\
\triangleq J_1 + J_2.
\]

To prove (2.5), it is enough to prove \( J_1 < \infty \) and \( J_2 < \infty \). By Markov’s inequality and Lemma 3.2,

\[
J_1 = \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i I(|X_i| \leq x) - \mathbb{E} X_i I(|X_i| \leq x)) \right| > \frac{n\varepsilon}{4} \right) \\
\leq C \sum_{n=1}^{\infty} n^{-3} \mathbb{E} \left[ \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i I(|X_i| \leq x) - \mathbb{E} X_i I(|X_i| \leq x)) \right|^2 \right] \\
\leq C \sum_{n=1}^{\infty} n^{-3} \sum_{i=1}^{n} \mathbb{E} X_i^2 I(|X_i| \leq x) \leq C \sum_{n=1}^{\infty} n^{-2} < \infty. \tag{3.23}
\]
By (3.21), Markov’s inequality and Lemma 3.6, then

\[
J_2 = \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (X_i I(|X_i| > x) - \mathbb{E}X_i I(|X_i| > x)) \right| > \frac{3n\epsilon}{4} \right)
\]

\[
\leq \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i I(|X_i| > x) \right| > \frac{n\epsilon}{2} \right)
\]

\[
\leq \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \sum_{i=1}^{n} |X_i| I(|X_i| > x) > \frac{n\epsilon}{2} \right)
\]

\[
\leq \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \sum_{i=1}^{n} (|X_i| I(|X_i| > x) - \mathbb{E}|X_i| I(|X_i| > x)) \right| > \frac{n\epsilon}{4} \right) < \infty.
\]

(3.24)

REFERENCES


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