Recovering Survival Probabilities from Equity Option Prices and Credit Default Swap Spreads

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Abstract

After the financial crisis onset in 2007, the interest in credit risk assessment has grown exponentially and there is an even more pressing need for efficient credit risk models. To this aim we introduce a credit risk structural model that is consistent with both the equity option and the credit default swap markets. It provides survival probabilities that can be used for market consistent pricing of over-the-counter structured products. Empirical evidence of the model accuracy and efficiency is given by using Goldman Sachs, J.P. Morgan Chase and Morgan Stanley market data prior to, during and after the Lehman Brothers default.

Keywords: Credit default swaps, Credit risk, Derivatives pricing, Structural models, Survival probabilities

1 Introduction

The financial crisis that began in 2007 has led and is leading to bankrupt many firms around the world. Given the importance of the theme, the interest in credit
risk is having an impressive growth, and not only within the business world. Of course international institutions and organizations have long recognized the need to include this type of risk in financial valuation (e.g. Basel Accords and International Accounting Standards). However, in spite of the amount of research on the topic, the debate is still open, especially because of the high demand for efficient and market consistent credit risk valuation models coming also from non-financial firms due to more severe regulation. The aim of this paper is to contribute to such debate by introducing a credit risk structural model which, unlike the models in the literature known to us, is consistent with both the option and the credit default swap markets.

1.1 Credit Default Swaps
Credit risk can be defined as the risk that a counterparty to a contract defaults and fails to meet its financial commitments. In order to offer protection against credit risk, J.P. Morgan introduced Credit Default Swaps (CDSs) in the 90s. These financial contracts have become very liquid in recent years and there is no longer the need to have a model for their valuation but rather CDS quotes are used to calibrate valuation models in order to take credit risk into account.

A CDS provides protection in the event of default of a specified reference entity that in this paper is assumed to be a firm. The protection buyer and the protection seller, enter into a contract with the following conditions. Considering a unit notional capital and a CDS maturing at time $T$, if the firm defaults at time $\tau$, with $t_0 < \tau < t_N = T$, the protection buyer receives from the protection seller an amount of money $LGD$ (loss given default, i.e. the capital loss, expressed in percentage terms, arising if the firm defaults). In return, at times $t_1, ..., t_N$ or until time $\tau$, the protection seller receives from the protection buyer a fixed cash flow $R_T$ (the so-called CDS spread, expressed in percentage terms).

The two streams are respectively known as protection leg and premium leg. Typically, $LGD = 1 - Rec$ where $Rec$ is the recovery rate, i.e. the amount recovered through bankruptcy procedures in event of a default, expressed as a percentage of face value. Assuming $LGD$ known at time $t$, with $0 < t < t_0$, the sum of discounted cash flows of the CDS can be written from the perspective of the protection buyer as:

$$\Sigma(t) = -\tilde{\delta}(t, \tau)(\tau - t_{x-1})R_T \mathbf{1}_{\{t_0 < \tau < t_N\}} - \sum_{i=1}^{N} \tilde{\delta}(t, t_i) \gamma_i R_T \mathbf{1}_{\{\tau \geq t_i\}} + \tilde{\delta}(t, \tau) \mathbf{1}_{\{t_0 < \tau < t_N\}}LGD$$

(1)
where \( \bar{v}(t, T) \) is the risk-free stochastic discount factor from \( T \) to \( t \), \( t_x \) is the first date \( t_i \) after \( \tau \), \( y_i \) is the fraction of a year between \( t_{i-1} \) and \( t_i \). Independently from the model chosen for the default time \( \tau \), the CDS can be priced by risk-neutral valuation\(^1\):

\[
CDS_T(t) = E_t^Q\left\{ \sum\{i\} \right\}
\]

where \( E_t^Q \) is the expectation at time \( t \) under the risk-neutral probability measure \( Q \). Usually a CDS maturing at time \( T \) is quoted through its spread \( R_T \), such that at inception the CDS value is null. Posing \( t = 0 \), assuming deterministic interest rates and known LGD, it follows that:

\[
CDS_T(0) = -R_T \int_{t_0}^{T_N} v(0, u)(u-t_{x-1})d\mathbb{Q}(\tau > u) + \\
+ \sum_{i=1}^{N} v(0, t_i)y_i\mathbb{Q}(\tau > t_i) + LGD\int_{t_0}^{T_N} v(0, u)d\mathbb{Q}(\tau > u) = 0
\]

where \( v(t, T) \) is the risk-free discount factor from \( T \) to \( t \) and \( \mathbb{Q}(\tau > t) \) is the firm’s risk-neutral survival probability until \( t \). Using the credit spread market quotes for different maturities along with Eq. (3) allows recovering the survival probabilities necessary to calibrate the credit risk model.

1.2 Credit risk modeling

Over the years two main paradigms have become dominant to model credit risk: reduced form models (also known as intensity models) and structural models. Reduced form models describe the occurrence of a default by the first jump of an appropriate exogenous stochastic process. Jarrow and Turnbull (1995) \([17]\) is the first reduced form model introduced in the literature, soon followed by its extension Jarrow et al. (1997) \([16]\). Other well-known reduced-form models are Madan and Unal (1998) \([21]\), Lando (1998) \([18]\), Duffie and Singleton (1999)\([12]\). Reduced-form models are relatively easy to be calibrated to the CDS and bond markets data and are widely used for relative derivatives pricing (e.g. for CDS options pricing). However, they are not easy to implement in cases of derivatives valuation problems with more underlyings (e.g. First to Default Baskets and Credit Default Obligations) because calibrating the correlations among default times of different firms is not trivial.

\(^1\) For more details see, for instance, Alexander (2008), p. 142.
Structural models are based on the seminal work of Merton (1974) [23] and the default event occurrence depends on the evolution of the firm’s asset value. In particular in Merton (1974) [23] the firm has a single liability represented by a zero-coupon bond and the firm can default only at its maturity. More realistic models, such as Black and Cox (1976) [4], Longstaff and Schwartz (1995) [20], Briys and de Varenne (1997) [8], Cenci and Gheno (2005) [9], Hsu et al. (2010) [15] introduce more flexible debt structures and the possibility of default before the debt maturity as soon as firm’s asset value falls below a certain (deterministic or stochastic) threshold. Equity can therefore be nicely interpreted as a barrier option written on the firm’s asset value.

As reported in Bielecki and Rutkowski (2004) [3] structural models present some clear advantages over reduced-form models:

a) being the uncertainty based on the volatility of the firm’s asset value, the credit risk is measured in a standard way;

b) the default event is linked to the firm’s insolvency and the random default time is defined in an intuitive way;

c) dependent defaults and valuation problems with more underlyings are easy to handle through correlation of firms’ asset value processes corresponding to different firms.

In spite of recent studies by Leland (2004) [19] and Schaefer and Strebulaev (2008) [26], the majority of the relevant empirical literature seems to report a poor performance of the structural models respect to reduced-form models (see for instance, Duffee (1999) [11], Eom et al. (2004) [13], Driessen (2005) [10] and Bakshi et al. (2006) [2]). However in a very interesting article, Gündüz and Uhrig-Homburg (2013) [14] perform for the first time a rigorous empirical test of both model classes on the same dataset. Their study shows that reduced-form and structural models perform quite similarly and that neither approach consistently outclasses the other one. Hence they conclude that many of the differences documented in the literature so far are due to other reasons such as different input data, calibration methods, and sampling design.

Given the findings of Gündüz and Uhrig-Homburg (2013) [14] and the advantages reported above, the model we introduce and apply in this paper belongs to the class of the structural models.

1.3 Classical structural models

The Merton (1974) [23] model firm’s asset value dynamics can be described by the following stochastic differential equation:

\[ dV_t = \left( r - kV_t \right) V_t dt + \sigma V_t dz_t \]  

(4)
where \( r \) and \( k^V \) (constants) are the risk-free interest rate and the firm’s asset pay-out rate, \( \sigma \) (constant) is the firm’s asset volatility and \( z_t \) is a Wiener process under the risk-neutral probability measure \( Q \). Since assets can be financed through debt and/or equity, it follows that:

\[ V_t = D_t + S_t \]  

(5)

where \( D_t \) is the debt value and \( S_t \) is the equity value.

The Merton model in its simplest form assumes that the firm has a single liability represented by a zero-coupon bond with face value \( L \) and maturity \( T \). The firm can default only at time \( T \) if the event \( \{V_T < L\} \) occurs. Hence the default time can be expressed as \( \tau = T \mathbf{1}_{\{V_T < L\}} + \infty \mathbf{1}_{\{V_T \geq L\}} \) and the debt value \( D_t \ \forall t \leq T \) is given by:

\[ D_t = e^{-r(T-t)}E^Q_t\left[ \min(V_T,L) \right] = e^{-r(T-t)}E^Q_t\left[ L - (L-V_T)^+ \right] = e^{-r(T-t)}L - P_t \]  

(6)

where \( P_t \) is the value at time \( t \) of a European put option written on \( V_t \) with maturity \( T \) and strike price \( L \). Given Eq. (5), assuming no dividends and recalling the well-known put-call parity relation, yields:

\[ S_t = V_t - D_t = V_t - e^{-r(T-t)}L + P_t = C_t \]  

(7)

where \( C_t \) is the value in \( t \) of the corresponding call option written on the firm’s asset value.

Hence from Eq. (7) equityholders hold a call option on the firm’s asset value, while debtholders hold a risk-free zero-coupon bond and a short position on a put option on the same underlying. Following the traditional Black-Scholes-Merton (Black and Scholes (1973) [5] and Merton (1973) [22]) analysis, the debt value at time \( t \) can be expressed as:

\[ D_t = V_te^{-k^V(T-t)}N(-d_+(V_t,T-t)) + Le^{-r(T-t)}N(-d_-(V_t,T-t)) \]  

(8)

where \( N(\cdot) \) is the cumulative standard normal distribution function and

\[ d_\pm(V_t,T-t) = \frac{\ln(V_t/L) + \left(r - k^V \pm 0.5\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}. \]  

(9)
The survival probability of the firm is given by the probability associated with the event \( \{ V_T \geq L \} \):

\[
\mathbb{Q}(\tau \geq T) = \Phi\left( d_-(V_T, T-t) \right), \tag{10}
\]

The corresponding default probability can be easily obtained \((1 - \mathbb{Q}(\tau \geq T))\).

The main criticism to the Merton (1974) model is that the firm’s default takes place on the maturity \( T \) of the single zero-coupon bond issued by the firm. Black and Cox (1976) [4] (BC) address this issue assuming that default occurs as the firm’s asset value hits a lower barrier, allowing default to take place at any time \( t \leq T \). Such barrier represents the safety covenants on the debt and has the following time dependent exponential form:

\[
H_t = \begin{cases} 
K e^{-\gamma(T-t)} & \text{if } t < T \\
L & \text{if } t = T 
\end{cases} \tag{11}
\]

where \( K \) and \( \gamma \) are constant parameters and \( L \) is the nominal value of a zero-coupon bond maturing at time \( T \), with \( 0 < K < L \).

Given the process in Eq. (4) and the barrier in Eq. (11) the resulting default probability is (Bielecki and Rutkowski (2004) [3]):

\[
\mathbb{Q}(\tau < t) = 1 - \mathbb{Q}(\tau \geq t) = \Phi\left( \frac{\ln \left( \frac{L}{V_0} \right) - \phi t}{\sigma \sqrt{t}} \right) + \\
\left( \frac{H_0}{V_0} \right)^{2\phi\sigma^2} \Phi\left( \frac{\ln \left( \frac{H_0}{V_0} \right) + \phi t}{\sigma \sqrt{t}} \right) \tag{12}
\]

where \( \phi = r - k^V - \gamma - 0.5\sigma^2 \).

Unfortunately, given the small number of parameters \( (\sigma, L, K \text{ and } \gamma) \), the BC model is not able to fit market data in a satisfactory way and hence produce reliable market consistent default probabilities.

2 The model

In this section we present a structural model that retains the advantages listed in Section 1.2 and that can also be calibrated to both the CDS and equity option market quotes. In order to accomplish this task, we assume the following firm’s asset value dynamics:
Recovering survival probabilities

\[ dV_t = \left( r_t - k_t \right) V_t \, dt + \sigma_t V_t \, dz_t \] (13)

where, unlike in Eq. (4), the risk-free interest rate, the pay-out rate and the volatility are deterministic functions of time.

Brigo and Morini (2006) [6] developed a structural model that generalizes the BC model with the following deterministic barrier type:

\[ H_t = H e^{\int_0^t \left( r_u - k_u - B \sigma_u^2 \right) du} \] (14)

where \( H \) and \( B \) are parameters chosen arbitrarily. Such model is able to calibrate efficiently and precisely CDS market quotes for a range of maturities through the time dependent function \( \sigma_u^2 \). Our aim is to also integrate equity option market prices in the valuation process in order to be able to price equity and hybrid credit/equity over-the-counter (OTC) structured products in a more accurate way.

For the stock price we assume the following dynamics:

\[ dS_t = \left( r_t - k_t^S \right) S_t \, dt + \sigma_t^S \left[ \alpha_t S_t + \left( 1 - \alpha_t \right) S_0 e^{\int_0^t \beta_u \, du} \right] dz_t \] (15)

where \( k_t^S \) is the firm’s equity pay-out rate, \( \alpha_t \) is the volatility skew parameter and \( \left( \sigma_t^S \right)^2 \) is a Nelson-Siegel (1987) type function:

\[ \left( \sigma_t^S \right)^2 = a + \left( b + c \right) e^{-\frac{t}{d}} - \frac{c t}{d} e^{-\frac{t}{d}} \] (16)

with \( a, b, c \) and \( d \) constant parameters. In support of this choice of \( dS_t \), Piterbarg (2005) [25] reports that the process in Eq. (15) is well suited to be calibrated to skewed volatility surfaces implied by equity option prices.

For the barrier we adopt the following functional form:

\[ H_t = H e^{\int_0^t \left( r_u - k_u - (0.5 + \beta_u) \sigma_u^2 \right) du} \] (17)

where \( H \) is a constant parameter and \( \beta_u \) is a deterministic function of time that allows perfect calibration to CDS market quotes \( R_j \) available for the maturities \( t_j \ (j = 1, \ldots, M) \). Survival probabilities \( \mathbb{Q} \left( \tau > t \right) \) can be deduced numerically by
splitting the time period taken into consideration in subintervals and applying the Chapman-Kolmogorov equation:

$$Q(\tau > t) = \sum_{V_t, \ldots, V_{t_i}} \frac{\partial Q(V_t \geq H_t, \tau > t \mid V_t, \tau > t_i)}{\partial V_t} \Delta V_t \times \frac{\partial Q(V_t \geq H_t, \tau > t_i \mid V_{t-1}, \tau > t_{i-1})}{\partial V_{t_i}} \Delta V_{t_i} \times \ldots \times \frac{\partial Q(V_t \geq H_t, \tau > t_1 \mid V_0)}{\partial V_{t_1}} \Delta V_{t_1}$$

(18)

where $t_i$ is the first date before $t$ ($t_i < t \leq t_M$, $i = 0, \ldots, M - 1$) and the joint distribution of the first passage time and $V_t$ is:

$$Q(V_t \geq x, \tau > t) = N \left[ \ln(H_t/x) + \ln(V_0/H) + \beta_{i+1} \int_{t_i}^{t} \sigma_u^2 du \right] \left[ \frac{\int_{t_i}^{t} \sigma_u^2 du}{\sqrt{\int_{t_i}^{t} \sigma_u^2 du}} \right]$$

$$- (H/V_0)^{2\beta_{i+1}} N \left[ \ln(H_t/x) + \ln(H/V_0) + \beta_{i+1} \int_{t_i}^{t} \sigma_u^2 du \right] \left[ \frac{\int_{t_i}^{t} \sigma_u^2 du}{\sqrt{\int_{t_i}^{t} \sigma_u^2 du}} \right]$$

(19)

with $x \geq H_t$ and $V_0 \geq H^2$.

Eq. (18) jointly with Eq. (3) allows the survival probabilities calibration to the CDS market quotes.

The firm’s asset volatility $\sigma_t$ and the initial firm value $V_0$ needed to compute $Q(V_t \geq x, \tau > t)$ in Eq. (19) can be determined as follows.

As pointed out in Section 1.3, within structural models the equity can be interpreted as an option (plain vanilla or barrier) written on the firm’s asset value, hence $S_t = f(V_t)$. Applying Itô’s lemma to Eq. (13) yields:

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2 For further details on the derivation of Eq. (19) see Brigo et al. (2011).
Recovering survival probabilities

\[ dS_t = \left( r_t - k_t^S \right) S_t dt + \sigma_t V_t \frac{\partial S_t}{\partial V_t} dz_t. \]  

(20)

By comparing Eqs. (15) and (20) it follows that:

\[ \sigma_t^S \left[ \alpha_t S_t + (1 - \alpha_t) S_0 e^{\int_0^t r_u du} \right] = \sigma_t V_t \frac{\partial S_t}{\partial V_t}. \]  

(21)

Eq. (21) links the equity options market to the firm’s asset value process. Moreover since \( S_t = f(V_t) \) and \( S_0 \) is known, the following relation holds:

\[ S_0 = e^{-\int_0^{t_M} r_u du} E_0^Q \left[ g(V_{t_M}) \right] = e^{-\int_0^{t_M} r_u du} E_0^Q \left[ V_{t_M} - H_{t_M} \right] \]  

(22)

By solving Eqs. (21) and (22) the asset volatility \( \sigma_t \) and the initial firm value \( V_0 \) can be inferred. Thence the survival probabilities can be computed numerically.

3 Application and results

3.1 Data

The US housing market crisis started in 2007 and had a significant impact on the global banking system. In fact many large financial institutions held a huge amount of low-grade mortgages and the burst of the housing bubble questioned their solvency. The peak of the crisis can be identified when the Lehman Brothers default occurred in September 2008, which caused a great turmoil in the global financial markets. Given these facts we apply and test the model introduced in Section 2 to three bulge bracket US banks, namely Goldman Sachs (GS), J.P. Morgan Chase (JPM) and Morgan Stanley (MS), considering a period spanning from 2007 to 2009. In particular we apply the model to three dates. We choose 10 October 2008 because, given the credit spread market quotes, it can be considered the most critical date around the Lehman Brothers default occurred on 14 September 2008. The other two dates being 10 October 2007 and 10 October 2009.

The market data needed to calibrate the model at each date for each bank consist of CDS spreads for a given range of maturities (1, 3, 5, 7, 10 years), equity option prices for different strike prices and maturities (ideally from 1 month to 2 years), the equity price and the interest rate term structure. Fig. 1 depicts the CDS spreads for the three banks considered and shows their different sensitivity to the Lehman Brothers default as measured by credit spreads. MS is consistently the riskiest of
the group in terms of credit risk while JPM is the least risky. In particular the MS spread pattern is very impressive: the 1-year CDS spread ranges from 42 basis points (bp) in 2007 to 131 bp in 2009, passing through 2427 bp in 2008 (while GS and JPM 1-year CDS spread patterns respectively are 35-779-90 and 13-114-37 bp).

The volatility surfaces implied by equity option prices quoted on the Chicago Board Options Exchange (CBOE) needed to calibrate the model are shown in Figs. 2, 3 and 4. In 2008 GS and JPM can be considered equivalent in terms of equity market risk as measured by the implied volatility (about 80% for both considering 6 months and 1-year at-the-money (ATM) options) while MS is again the riskiest (about 250% and 190% respectively for 6 months and 1-year ATM options). On the other hand in 2007 and 2009 the situation is rather homogeneous: implied volatilities are respectively within the ranges $[20\%, 45\%]$ and $[33\%, 51\%]$ across all maturities and moneyness available for the options written on the three banks$^3$.

As far as stock prices are concerned JPM exhibits almost constant levels (+11% in 2008, −10% in 2009) while GS and MS show significant changes (respectively −62% in 2008 and +113% in 2009, and −86% in 2007 and +232% in 2009). New York Stock Exchange (NYSE) prices are shown in Fig. 5.

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$^3$ JPM volatilities for maturities longer than 1 year are available on Bloomberg only for the first date we take into consideration (10 October 2007).
Fig. 1. CDS spreads for GS, JPM, MS in 2007 (top), 2008 (middle), 2009 (bottom). Spreads are expressed in bp. Source: Bloomberg.
Fig. 2. 10 October 2007 volatility surface for call and put options written on GS (top), JPM (middle) and MS (bottom). Moneyness is expressed in terms of $S/K$ and maturity is in years. Source: Bloomberg.
Fig. 3. 10 October 2008 volatility surface for call and put options written on GS (top), JPM (middle) and MS (bottom). Moneyness is expressed in terms of $S/K$ and maturity is in years. Source: Bloomberg.
Fig. 4. 10 October 2009 volatility surface for call and put options written on GS (top), JPM (middle) and MS (bottom). Moneyness is expressed in terms of $S/K$ and maturity is in years. Source: Bloomberg.
3.2 Model implementation

The application of the model can be summarized in three steps:

**Step 1.** Stock price process calibration
In order to estimate $\sigma_t^S$ and $\alpha_t$ for the stock price process in Eq. (15), the following optimization problem is solved:

$$
\min_{\alpha_t, a, b, c, d} \left( \frac{\sigma_{MDL}^S - \sigma_{MKT}^S}{\sigma_{MKT}^S} \right)^2
$$

where $\alpha_t$ is the volatility skew parameter, $a, b, c, d$ are the parameters in Eq. (16), $\sigma_{MDL}^S$ is the equity volatility surface implied by the model and $\sigma_{MKT}^S$ is the volatility surface implied by equity option market prices (see Figs. 2, 3 and 4).

**Step 2.** Firm’s asset value process and barrier calibration
Given $S_0$, $\sigma_t^S$, $\alpha_t$ and the CDS spread market quotes $R_j$ ($j=1, 3, 5, 7, 10$. See Fig. 1), Eqs. (3), (20) and (21) allow to recover $V_0$, $\sigma_t$ and $\beta_j$ ($j=1, 3, 5, 7, 10$):
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\[
\begin{align*}
\left[ \begin{array}{l}
CDS_{t_j}(0) = 0 & j = 1, 3, 5, 7, 10 \\
\nu(0,t_M)E_0^Q\left[V_{t_M} - H_{t_M}\mid \tau > t_M\right] = S_0 \\
\sigma_t S_t \left[ \alpha_t S_t + (1 - \alpha_t)S_0 e^{\int_0^t \rho_u du} \right] = \sigma_t \frac{\partial S_t}{\partial \nu_t}
\end{array} \right]
\tag{24}
\end{align*}
\]

Given $\beta_j$ ($j = 1, 3, 5, 7, 10$) the barrier can be obtained from Eq. (17).

**Step 3. Survival probabilities computation**

Survival probabilities for times $t_j$ ($j = 1, 3, 5, 7, 10$), are obtained numerically from Eqs. (18) and (19).

### 3.3 Results

Given the volatility surface implied by equity option market prices depicted in Figs. 2, 3 and 4 the optimization problem in Eq. (23) is solved and the resulting values for the volatility skew parameter $\alpha_t$ and the stock volatility $\sigma_t^S$ are reported respectively in Table 1 and 2. The results are rather accurate as shown by the root mean square errors (RMSEs) summarized in Table 3.

<table>
<thead>
<tr>
<th>t</th>
<th>Bank</th>
<th>1M</th>
<th>2M</th>
<th>3M</th>
<th>6M</th>
<th>12M</th>
<th>18M</th>
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<td>-0.7300</td>
<td>-0.8560</td>
<td>-0.9555</td>
<td>-0.5566</td>
<td>-0.1309</td>
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<td></td>
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<td>0.4154</td>
<td>0.6292</td>
<td>0.6958</td>
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<tr>
<td>2009</td>
<td>GS</td>
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**Table 1.** Volatility skew parameters $\alpha_t$ for GS, JPM and MS from 2007 to 2009. Maturities are in months.
Table 2. Stock volatility $\sigma^S_t$ for GS, JPM and MS from 2007 to 2009. Maturities are in months. Source: Bloomberg.

<table>
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<tr>
<th>Bank</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>GS</td>
<td>0.0163</td>
<td>0.0010</td>
<td>0.0116</td>
</tr>
<tr>
<td>JPM</td>
<td>0.0249</td>
<td>0.0009</td>
<td>0.0144</td>
</tr>
<tr>
<td>MS</td>
<td>0.0065</td>
<td>0.0013</td>
<td>0.0042</td>
</tr>
</tbody>
</table>

Table 3. RMSE for volatility skew and stock volatility estimation.

By solving system (24) (for each date, bank and maturity), we obtain $V_0$, $\sigma_t$ and $\beta_j$ ($j = 1, 3, 5, 7, 10$). The model asset volatility term structures are reported in Table 4 and the barrier levels obtained by calibration are shown in Fig. 6 as proportion of the initial firm’s asset values.

Table 4. Firm’s asset volatilities for GS, JPM and MS from 2007 to 2009. Maturities are in years.

During the calibration process the parameter $H$ in Eq. (17) is assumed to be equal to $H_1$. 
Fig. 6. GS, JPM and MS barrier levels as proportion of the initial firm’s asset values. 10 October 2007 (top), 10 October 2008 (middle) and 10 October 2009 (bottom).
Survival probabilities computed numerically from Eqs. (18) and (19) using the method introduced in Aluigi et al. (2013) are reported in Table 5. In 2007 the three banks considered have almost identical default probability term structures with values in the range [0%, 7%]. After the Lehman Brothers default the change is abrupt. In 2008 the GS and MS default probabilities’ ranges for maturities longer than 3 years are respectively [26%, 53%] and [54%, 72%], while JPM default probability term structure is considerably lower ([7%, 24%]). In 2009 the situation tends to normalize with all values in the range [2%, 22%], where the upper extreme is the MS 10-year default probability.

Table 6 summarizes the performance of the model proposed in this paper. CDS pricing errors obtained from Eq. (3) jointly with the survival probabilities recovered (Table 5) are negligible. Computational times are very fast and allow the model to be used by firms for market consistent valuations of their OTC financial instruments.

<table>
<thead>
<tr>
<th>t</th>
<th>Bank</th>
<th>1Y</th>
<th>3Y</th>
<th>5Y</th>
<th>7Y</th>
<th>10Y</th>
<th>CPU Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>2007</td>
<td>GS</td>
<td>99.42%</td>
<td>98.41%</td>
<td>96.68%</td>
<td>95.14%</td>
<td>92.71%</td>
<td>2.7144</td>
</tr>
<tr>
<td></td>
<td>JPM</td>
<td>99.78%</td>
<td>98.94%</td>
<td>97.60%</td>
<td>96.45%</td>
<td>94.40%</td>
<td>2.6052</td>
</tr>
<tr>
<td></td>
<td>MS</td>
<td>99.31%</td>
<td>98.15%</td>
<td>96.63%</td>
<td>95.10%</td>
<td>92.50%</td>
<td>2.8392</td>
</tr>
<tr>
<td>2008</td>
<td>GS</td>
<td>87.52%</td>
<td>74.52%</td>
<td>64.99%</td>
<td>58.63%</td>
<td>46.96%</td>
<td>2.3868</td>
</tr>
<tr>
<td></td>
<td>JPM</td>
<td>98.13%</td>
<td>92.71%</td>
<td>87.28%</td>
<td>82.72%</td>
<td>76.29%</td>
<td>2.1528</td>
</tr>
<tr>
<td></td>
<td>MS</td>
<td>64.67%</td>
<td>46.74%</td>
<td>40.01%</td>
<td>32.54%</td>
<td>27.19%</td>
<td>2.8392</td>
</tr>
<tr>
<td>2009</td>
<td>GS</td>
<td>98.52%</td>
<td>94.93%</td>
<td>90.88%</td>
<td>87.41%</td>
<td>82.58%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>JPM</td>
<td>99.39%</td>
<td>97.36%</td>
<td>94.31%</td>
<td>92.01%</td>
<td>88.43%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>MS</td>
<td>97.85%</td>
<td>93.30%</td>
<td>88.35%</td>
<td>83.89%</td>
<td>77.82%</td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Survival probabilities for GS, JPM and MS from 2007 to 2009. Maturities are in years.

<table>
<thead>
<tr>
<th>t</th>
<th>Bank</th>
<th>1Y</th>
<th>3Y</th>
<th>5Y</th>
<th>7Y</th>
<th>10Y</th>
<th>CPU Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>2007</td>
<td>GS</td>
<td>-6.7E-16</td>
<td>-2.776E-14</td>
<td>-3.331E-14</td>
<td>-3.02E-14</td>
<td>-2.887E-14</td>
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<tr>
<td></td>
<td>JPM</td>
<td>-7.2E-16</td>
<td>2.087E-14</td>
<td>1.954E-14</td>
<td>1.044E-14</td>
<td>3.55E-15</td>
<td>2.6052</td>
</tr>
<tr>
<td></td>
<td>MS</td>
<td>1.61E-15</td>
<td>-4.174E-14</td>
<td>-5.151E-14</td>
<td>-5.729E-14</td>
<td>-4.974E-14</td>
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<tr>
<td>2008</td>
<td>GS</td>
<td>8.9E-16</td>
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<td>0</td>
<td>-3.55E-15</td>
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<tr>
<td></td>
<td>JPM</td>
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<td>6.7E-16</td>
<td>1.554E-14</td>
<td>1.243E-14</td>
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<td>MS</td>
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<td>-3.464E-14</td>
<td>-2.842E-14</td>
<td>-3.908E-14</td>
<td>2.4960</td>
</tr>
</tbody>
</table>

Table 6. CDS pricing errors for GS, JPM and MS from 2007 to 2009. CPU time is in seconds.

5 The numerical results in this paper have been obtained using Matlab R2010a running on a PC Intel Core i5-2520M 2.50 GHz.
4 Conclusions

In this paper we introduce a credit risk structural model that is consistent with both the equity option and the credit default swap markets. It provides survival probabilities that can be used for market consistent pricing of OTC structured products. Empirical evidence of the model accuracy and efficiency is given by using GS, JPM and MS market data prior to, during and after the Lehman Brothers default.

Given that the CDS options market continues to grow, a promising direction for further research would be to introduce a stochastic recovery rate process in order to retrieve additional information from CDS options prices.

References


[19] Hayne E. Leland, Predictions of default probabilities in structural models of


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