A Note on the Homogeneous Neumann Boundary Condition

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Abstract
We have shown that the Laplacian possesses an eigenvalue equal a zero, i.e., we have proven that there is a nonzero function $u$ (having the homogeneous Neumann boundary condition) such that $\Delta u = 0$. Finally, we have tested the effect of this zero eigenvalue on the solutions of the heat equation, the wave equation and the Poisson equation.

Keywords: Homogeneous Neumann, zero eigenvalue, Laplacian

1 Introduction
Let $\Omega$ be the domain of $\mathbb{R}^n$ and let $\eta = (\eta_1, \ldots, \eta_n)$ be the outward unit normal vector to $\partial \Omega$. Then, the Laplace eigenproblem with Neumann condition is given by [1]

$$\begin{cases}
\Delta u = \lambda u, & \text{in } \Omega \\
\partial_\eta u = 0, & \text{on } \partial \Omega.
\end{cases} \quad (1)$$
Here, $\partial_\eta u$ is the normal derivative
$$\partial_\eta u = \eta_1 u_{x_1} + \cdots + \eta_n u_{x_n}.$$ 

Therefore, we need to find a nontrivial function $u$ to show that the Laplacian has a zero eigenvalue that satisfies Eq. (1) with $\lambda = 0$. Now, out the one dimensional case [2]
\[\Delta u = 0, \quad u'(0) = u'(1) = 0,\] (2)
whose solutions are the constant functions, we assume in a general way that the constant functions can be our own functions with $\lambda = 0$. It is easy to see that for $u = k$ ($k$ constant) we have
\[
\Delta u = u_{x_1 x_1} + \cdots + u_{x_n x_n}, \quad \partial_\eta u = \eta_1 u_{x_1} + \cdots + \eta_n u_{x_n} = 0,
\]
showing that the Laplacian has a zero eigenvalue. We can see that here count all the constants as a single function, since if $u$ satisfies (1), then $ku$ is true for any $k \in \mathbb{R}$ [3]. To verify if there is any eigenfunction, we can see the identity
\[\int_\Omega |\Delta u|^2 = 0,\] (3)
in the current case with $\lambda = 0$. This quantity (integration) is nonnegative, meaning that must be zero everywhere, i.e. we have the gradient of $u$ vanishing everywhere and so $u$ must be constant.

Now, let $u_0, u_1, \ldots$ be the Laplace eigenfunctions with the corresponding eigenvalues $\lambda_0, \lambda_1, \ldots$. Then, for $k \in \mathbb{N}$ we have
\[
\begin{cases}
\Delta u_k = \lambda_k u_k, & \text{in } \Omega \\
\partial_\eta u_k = 0, & \text{on } \partial \Omega.
\end{cases}
\]
Suppose that the eigenvalues are arranged as follows
$$\lambda_0 = 0 > \lambda_1 \geq \lambda_2 \geq \cdots$$
We can see that this means that $u_0$ is a constant function, then we can assume that $u_0 = 1$. So, any function $h : \Omega \rightarrow \mathbb{R}$ with finite energy can be decomposed as [4]
\[h = \sum_{k=0}^{\infty} h_k^* u_k,\] (4)
with $\{h^*\}$ the coordinates of $h$ in the basis $\{u_k\}$ and where $h_k^* u_k$ is the $k$-mode of $h$. Here, $\lambda_k \to -\infty$ as $k \to \infty$. 
2 The Poisson problem

Let us now treat the Poisson problem [5]

\[
\begin{aligned}
\Delta w &= h, \quad \text{in } \Omega \\
\partial_\eta w &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

(5)

Expanding the unknown solution \( w \) (in terms of the eigenfunctions) we have

\[
w = \sum_{k=0}^\infty w_k^* u_k,
\]

then

\[
\Delta w = \sum_{k=0}^\infty w_k^* \Delta u_k = \sum_{k=0}^\infty w_k^* \lambda_k u_k = \sum_{k=0}^\infty h_k^* u_k.
\]

Now, with the individual coordinates we get

\[
\lambda_k w_k^* = h_k^* \quad \text{then} \quad 0 \cdot w_0^* = h_0^* \quad \text{and} \quad w_k^* = \frac{h_k^*}{\lambda_k}, \quad k = 1, 2, \ldots,
\]

where \( \lambda_0 = 0 \) and \( \lambda_k \neq 0 \) for \( k > 0 \). Therefore, as a result of division by \( \lambda_k \), the high frequency modes of \( u \) are much smaller than the corresponding modes of \( h \). We can see that the existence of the zero eigenvalue for the Poisson problem, the right hand side of \( h \) must have a vanishing zero-mode \( (h_0^* = 0) \). Then, if this is true, the function

\[
w = w_0^* u_0 + \sum_{k=0}^\infty \frac{h_k^*}{\lambda_k} u_k = w_0^* \cdot 1 + \sum_{k=0}^\infty \frac{h_k^*}{\lambda_k} u_k,
\]

with \( w_0^* \in \mathbb{R} \) satisfies the Poisson equation. Now, if \( w \) is a solution, then we have that for any constant \( k \), \( w + k \) is also a solution. Therefore, by multiplying Eq. (4) by \( u_0 \) and integrating on \( \Omega \) we get

\[
h_k^* \int_{\Omega} |u_0|^2 = \int_{\Omega} hv_0,
\]

then

\[
h_k^* = \frac{1}{vol(\Omega)} \int_{\Omega} h,
\]

i.e., \( h_k^* \) is just the average of \( h \) over \( \Omega \).

3 The Heat problem

Let us now treat the heat problem [6]
\begin{equation}
\begin{cases}
w_t = \Delta w \\
w|_{t=0} = h,
\end{cases}
\end{equation}

with the homogeneous Neumann B.C. Now, we have

\[ w(x, t) = \sum_{k=0}^{\infty} w_k^*(t) u_k(t), \]

then

\[ w_t = \sum_{k=0}^{\infty} w_k'^* u_k, \quad \Delta w = \sum_{k=0}^{\infty} \lambda_k w_k^*(t) u_k(t) \]

Here, \( w_k'^* \) is the time derivative of \( w_k^* \). Now, substituting into the heat equation we get

\[ w_k'^* = \lambda_k w_k^*, \]

which is solved by \( w_k^* = A_k e^{\lambda_k t} \). Therefore, from the initial condition we have \( A_k = h_k^* \) and

\[ w(x, t) = \sum_{k=0}^{\infty} h_k^* e^{\lambda_k t} u_k(x) = h_0^* + \sum_{k=1}^{\infty} h_k^* e^{\lambda_k t} u_k(x), \]

where \( \lambda_0 = 0 \) and \( u_0 = 1 \).

Here, there is a mode (i.e. the zero mode) that does nor evolve in time and as all other modes are decaying \( (t \to \infty) \), then the solution approaches that mode. In other words,

\[ w(x, t) \to h_0^*. \]

Finally, the zero mode is simply the average of the initial \( h \).

4 The Wave problem

Let us consider the wave equation [7]

\begin{equation}
\begin{cases}
w_{tt} = \Delta w \\
w|_{t=0} = h, \\
w_t|_{t=0} = g
\end{cases}
\end{equation}

Here, we consider the homogeneous Neumann boundary condition. Then, we have
Homogeneous Neumann boundary condition

\[ w_k'' = \lambda_k w_k^* , \]

whose solution is given by

\[ w_0^*(t) = A_0 + B_0 t, \quad w_k^*(t) = A_k \cos(\omega_k t) + B_k \sin(\omega_k t), \]

for \( k = 1, 2, \ldots \). Here, \( \omega_k = \sqrt{-\lambda_k} \). Therefore, we get

\[ w(x, t) = A_0 + B_0 t + \sum_{k=0}^{\infty} (A_k \cos(\omega_k t) + B_k \sin(\omega_k t)) u_k(t), \]

here we have taken into account that \( u_0 = 1 \). Now, from the initial conditions we have \( A_k = h_k^* \) for all \( k \) and also \( B_0 = g_0^* \) and \( B_k = g_k^* / \omega_k \), this for nonzero \( k \). Then,

\[ w(x, t) = h_0^* + g_0^* t + \sum_{k=1}^{\infty} (h_k^* \cos(\omega_k t) + g_k^* / \omega_k \sin(\omega_k t)) u_k(x) \]

The effect of the zero eigenvalue is that there is a mode (i.e. zero mode) that does not oscillate in time. Depending on the average of \( g \), the zero mode may be evolving linearly in time, and all other modes are oscillating around this mode [4].

5 Conclusion

In this work we have considered a bounded domain and we have assumed that the homogeneous Neumann boundary condition is imposed on all problems, i.e., Poisson, Heat and Wave equations. We have shown that the Laplacian possesses an eigenvalue equal a zero, i.e., we have proven that there is a nonzero function \( u \) (having the homogeneous Neumann boundary condition) such that \( \Delta u = 0 \). Finally, we have tested the effect of this zero eigenvalue on the solutions of the heat equation, the wave equation and the Poisson equation.

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