Abstract

This study combines the concept of connected domination in graphs introduced by Sampathkumar and Walikar [8] in 1979 and the concept of $k$-fair domination in graphs introduced by Caro, Hansberg and Henning [3] in 2011. Let $G = (V(G), E(G))$ be a simple graph. For an integer $k \geq 1$, a $k$-fair dominating set ($kfd$-set) is a dominating set $S \subseteq V(G)$ such that $|N(u) \cap S| = k$ for every $u \in V(G) \setminus S$. The $k$-fair domination number of $G$, denoted by $\gamma_{kfd}(G)$, is the minimum cardinality of a $kfd$-set. Let $G$ be a connected graph. A connected $k$-fair dominating set ($Ckfd$-set) is a $k$-fair dominating set $S \subseteq V(G)$ such that $\langle S \rangle$, the subgraph induced by $S$, is connected. The smallest cardinality of a
A set $S \subseteq V(G)$ is a dominating set in $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The minimum cardinality of a dominating set in $G$, denoted by $\gamma(G)$, is the domination number of $G$. Any dominating set in $G$ of cardinality $\gamma(G)$ is referred to as a $\gamma$-set in $G$. Sampathkumar and Walikar [8] introduced another variation called the connected domination in graphs. A dominating set $S \subseteq V(G)$ is called a connected dominating set in $G$ if $\langle S \rangle$, the subgraph induced by $S$, is connected. The connected domination number of $G$, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set in $G$.

Another domination parameter is fair domination, introduced by Caro, Hansberg and Henning [3] in 2011. For an integer $k \geq 1$, a $k$-fair dominating set ($kfd$-set) is a dominating set $S \subseteq V(G)$ such that $|N(u) \cap S| = k$ for every $u \in V(G) \setminus S$. The $k$-fair domination number of $G$, denoted by $\gamma_{kfd}(G)$, is the minimum cardinality of a $kfd$-set. Clearly, $k \leq \gamma_{kfd}(G) \leq |V(G)|$. Let $G$ be a connected graph. A connected $k$-fair dominating set (C$kfd$-set) is a $kfd$-set $S \subseteq V(G)$ such that $\langle S \rangle$ is connected. The connected $k$-fair domination number of $G$, denoted by $\gamma_{Ckfd}(G)$, is the minimum cardinality of a C$kfd$-set. A connected $k$-fair dominating set of cardinality $\gamma_{Ckfd}(G)$ is called a minimum connected $k$-fair dominating set or a $\gamma_{Ckfd}$-set.
Let $G$ be a connected graph. A fair dominating set (fd-set) in $G$ is a $kfd$-set for some integer $k \geq 1$. Thus, a dominating set $S \subseteq V(G)$ is an fd-set in $G$ if for every two distinct vertices $u$ and $v$ from $V(G) \setminus S$, $|N(u) \cap S| = |N(v) \cap S|$. The fair domination number of $G$, denoted by $\gamma_{fd}(G)$, is the minimum cardinality of an fd-set. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called a minimum fair dominating set or a $\gamma_{fd}$-set. Maravilla, Isla, and Canoy [5, 6, 7] characterized the fair dominating, $k$-fair dominating, and fair total dominating sets in the join, corona, lexicographic product, and Cartesian product of graphs and determined the bounds or exact values of the fair, $k$-fair, and fair total domination numbers, respectively, of these graphs.

2 Preliminary Results

Remark 2.1 For any connected graph $G$ of order $n \geq 2$ and a positive integer $k$,

$$1 \leq \gamma(G) \leq \gamma_{kfd}(G) \leq \gamma_{Ckfd}(G).$$

Lemma 2.2 [5] Let $G$ be a connected nontrivial graph and $k \in \mathbb{N}$. Then $\gamma_{kfd}(G) = 1$ if and only if $k = 1$ and $\gamma(G) = 1$.

Remark 2.3 [5] Let $G$ be a connected graph and let $k \in \mathbb{N}$. Then every $kfd$-set is an fd-set. In particular, $\gamma_{fd}(G) \leq \gamma_{kfd}(G)$.

Theorem 2.4 Let $G$ be a connected nontrivial graph and $k$ a positive integer. Then $\gamma_{Ckfd}(G) = 1$ if and only if $k = 1$ and $\gamma(G) = 1$.

Proof. Suppose that $\gamma_{Ckfd}(G) = 1$. Then $\gamma(G) = \gamma_{kfd}(G) = 1$ by Remark 2.1. Hence, by Lemma 2.2, $k = 1$. Conversely, Suppose that $k = 1$ and $\gamma(G) = 1$. Let $S = \{v\}$ be a $\gamma$-set in $G$. Then for all $w \in V(G) \setminus S$, $|N_G(w) \cap S| = |\{v\}| = 1$. Since $\langle S \rangle$ is connected, $S$ is a $\gamma_{C1fd}$-set in $G$. Hence, $\gamma_{Ckfd}(G) = 1$. □

The next result immediately follows from Theorem 2.4

Corollary 2.5 Let $G$ be $K_n$, $W_n$, $F_n$ or $K_{1,n}$. Then, $\gamma_{C1fd}(G) = 1$.

Theorem 2.6 Let $G$ be $P_n$ or $C_n$, where $n \geq 3$. Then $\gamma_{C1fd}(G) = n - 2$.

Proof. Let $P_n = [v_1, v_2, ..., v_n]$ and $C_n = [v_1, v_2, ..., v_n, v_1]$. Clearly, $S = \{v_2, v_3, ..., v_{n-1}\}$ is a $\gamma_{C1fd}$-set of $P_n$ and $C_n$. It follows that $\gamma_{C1fd}(G) = |S| = n - 2$. □
Remark 2.7 Let $G$ be a connected graph and $k$ a positive integer. Then \( \gamma_{Ckfd}(G) = \gamma(G) \) if and only if $G$ has a $\gamma$-set which is a $Ckfd$-set.

Proposition 2.8 Let $G$ be a connected graph, $k$ a positive integer and $\gamma_{Ckfd}(G) = m < |V(G)|$. If $S$ is a $Ckfd$-set in $G$, then $S$ is not a $Ctfd$-set for every positive integer $t$ with $t > m$.

Proof. Let $S$ be a $\gamma_{Ckfd}$-set in $G$. Then $|S| = m < |V(G)|$. This implies that $S \neq V(G)$. Choose any $x \in V(G) \backslash S$. Then $|N_G(x) \cap S| \leq m$. Thus, $S$ is not a $Ctfd$-set for every positive integer $t$ with $t > m$. □

Theorem 2.9 [2] Let $G$ and $H$ be connected graphs. Then $C \subseteq V(G[H])$ is a dominating set in $G[H]$ if and only if $C = \bigcup (\{x\} \times T_x)$ and either

(a) $S$ is a total dominating set in $G$, or

(b) $S$ is a dominating set in $G$ and $T_x$ is a dominating set in $H$ for all $x \in S \backslash N_G(S)$.

Theorem 2.10 [5] Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively, and $k$ a positive integer with $1 \leq k \leq \max\{m, n\}$. Then $S \subseteq V(G + H)$ is a $kfd$-set of $G + H$ if and only if one of the following holds:

(a) $S = V(G + H)$.

(b) $S \subseteq V(G)$, $|S| = k$ and $S$ is a $kfd$-set of $G$.

(c) $S \subseteq V(H)$, $|S| = k$ and $S$ is a $kfd$-set of $H$.

(d) $S = S_G \cup S_H$, where $S_G$ is a $(k - |S_H|)fd$-set of $G$ and $S_H$ is a $(k - |S_G|)fd$-set of $H$.

(e) $S = V(G) \cup T$, where $|V(G)| = m < k$ and $T$ is a $(k - m)fd$-set of $H$.

(f) $S = D \cup V(H)$, where $|V(H)| = n < k$ and $D$ is a $(k - n)fd$-set of $G$.

Theorem 2.11 [5] Let $G$ and $H$ be nontrivial connected graphs. Then $C = \bigcup (\{x\} \times T_x) \subseteq V(G[H])$ is a $kfd$-set of $G[H]$ if and only if the following hold:

(i) $S$ is a dominating set of $G$.

(ii) For each $x \in S \cap N_G(S)$, $T_x = V(H)$ and $|V(H)| = r \leq k$ whenever $C \neq V(G[H])$ or $T_x$ is an $rfd$-set and $\sum_{x \in N_G(x) \cap S} |T_x| = k - r$. 

(iii) For each $x \in S \setminus N_G(S)$, $T_x = V(H)$ and $|V(H)| \leq k$ or $|T_x| = k$ and $T_x$ is a $kfd$-set of $H$.

(iv) For each $y \in V(G) \setminus S$, $\sum_{v \in N_G(y) \cap S} |T_v| = k$.

**Theorem 2.12** [5] Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively, and $k$ and $r$ be integers with $0 \leq r < k \leq \min\{m,n\}$. Then $C = \bigcup_{x \in V(G)} \{ \{x\} \times T_x \} \subseteq V(G \square H)$ is a $kfd$-set of $G \square H$ if and only if

(i) $V(H) \setminus T_x \subseteq N_H(T_x) \cup (\bigcup_{z \in N_G(x)} T_z)$ for each $x \in V(G)$, and

(ii) for each $x \in V(G)$, $T_x = V(H)$ or for each $a \in V(H) \setminus T_x$, either $|N_H(a) \cap T_x| = k$ and $|\{z : z \in N_G(x), a \in T_z\}| = 0$ or $|N_H(a) \cap T_x| = r < k$ and $a \in \bigcap_{i=1}^{k-r} T_{x_i}$, where $x_i \in N_G(x)$ for $i = 1, 2, \ldots, k - r$.

**Theorem 2.13** Let $a$ and $b$ be positive integers such that $a \leq b$ and $k = 1$ or $k = 2$. Then there exists a connected graph $G$ such that $\gamma_{kfd}(G) = a$ and $\gamma_{Ckfd}(G) = b$.

**Proof.** Consider the following cases:

Case 1. $k = 1$.

Subcase 1. $a = b$

Let $G = G_1$ be the graph shown in Figure 1.

It is clear that the set $A = \{x_i : i = 1, 2, \ldots, a\}$ is both a $\gamma_{1fd}$-set and a $\gamma_{C1fd}$-set in $G_1$. It follows that $\gamma_{1fd}(G_1) = \gamma_{C1fd}(G_1) = |A| = a = b$.

Subcase 2. $a < b$

Let $G = G_2$ be the graph shown in Figure 2.

It is clear that the set $A = \{x_i : i = 1, 2, \ldots, a\}$ is a $\gamma_{1fd}$-set and set $B = A \cup \{z_i : i = 1, 2, \ldots, a\}$ is a $\gamma_{C1fd}$-set in $G_2$. It follows that $\gamma_{1fd}(G_2) = |A| = a$ and $\gamma_{C1fd}(G_2) = |B| = a + a = 2a$. Let $b = 2a$. Then $a < b$ and $\gamma_{C1fd}(G_2) = b$. 

Figure 1: A graph $G$ with $\gamma_{1fd}(G) = \gamma_{C1fd}(G) = a$
Subcase 1.

Case 2.

Let \( \gamma \) be the graph shown in Figure 3.

It is clear that the set \( A = \{ x_i : i = 1, 2, \ldots a \} \) is both a \( \gamma_{2fd} \)-set and a \( \gamma_{C2fd} \)-set in \( G \). It follows that \( \gamma_{2fd}(G_3) = \gamma_{C2fd}(G_3) = |A| = a = b \).

Subcase 2. \( a < b \)

Let \( G = G_4 \) be the graph shown in Figure 4.

Clearly, the set \( A = \{ x_i : i = 1, 2, \ldots c \} \cup \{ y_j : j = 1, 2, \ldots, c \} \cup \{ z_l : l = 1, 2, \ldots, c \} \) is a \( \gamma_{2fd} \)-set and set \( B = A \cup \{ w_i : i = 1, 2, \ldots, c \} \) is a \( \gamma_{C2fd} \)-set in \( G_4 \). It follows that \( \gamma_{2fd}(G_4) = |A| = 3c \) and \( \gamma_{C2fd}(G_4) = |B| = 3c + c = 4c \). Let \( a = 3c, b = 4c \). Then \( a < b \), \( \gamma_{2fd}(G_4) = a \), and \( \gamma_{C2fd}(G_4) = b \). \( \square \)

**Corollary 2.14** Let \( m \) be a positive integer and \( k = 1 \) or \( k = 2 \). Then there exists a connected graph \( G \) such that \( \gamma_{Ckfd}(G) - \gamma_{kfd}(G) = m \), that is, the difference \( \gamma_{Ckfd}(G) - \gamma_{kfd}(G) \) can be made arbitrarily large.

Figure 2: A graph \( G \) with \( \gamma_{1fd}(G) = a < \gamma_{C1fd}(G) = 2a = b \)

Figure 3: A graph \( G \) with \( \gamma_{2fd}(G) = \gamma_{C2fd}(G) = a \)

Figure 4: A graph \( G \) with \( \gamma_{2fd}(G) = 3c = a < \gamma_{C2fd}(G) = 4c = b \)
3 Connected $k$-Fair Domination in the Join of Graphs

The join $G+H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

**Theorem 3.1** Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively, and $k$ a positive integer with $1 \leq k \leq \max\{m,n\}$. Then $S \subseteq V(G+H)$ is a C$k$fd-set in $G+H$ if and only if one of the following holds:

(a) $S = V(G+H)$.

(b) $S \subseteq V(G)$, $|S| = k$ and $S$ is a C$k$fd-set in $G$.

(c) $S \subseteq V(H)$, $|S| = k$ and $S$ is a C$k$fd-set in $H$.

(d) $S = S_G \cup S_H$, where $S_G$ is a $(k-|S_H|)$fd-set in $G$ and $S_H$ is a $(k-|S_G|)$fd-set in $H$.

(e) $S = V(G) \cup T$, where $|V(G)| = m < k$ and $T$ is a $(k-m)$fd-set in $H$.

(f) $S = D \cup V(H)$, where $|V(H)| = n < k$ and $D$ is a $(k-n)$fd-set in $G$.

**Proof.** Suppose that $S$ is C$k$fd-set in $G+H$. Suppose further that $S \neq V(G+H)$. Consider the following cases:

Case 1. $S \subseteq V(G)$ or $S \subseteq V(H)$

Assume first that $S \subseteq V(G)$ and let $x \in V(H)$. Then $|N_{G+H}(x) \cap S| = |S| = k$. If $S = V(G)$, then $S$ is a C$k$fd-set in $G$ since $G$ is connected. If $S \neq V(G)$, then $|N_G(v) \cap S| = |N_{G+H}(v) \cap S| = k$ for every $v \in V(G) \setminus S$. Thus, $S$ is a $k$fd-set in $G$. Moreover, since $\langle S \rangle$ is connected in $G+H$ by assumption, $S$ is a C$k$fd-set in $G$. Similarly, if $S \subseteq V(H)$, then $|S| = k$ and $S$ is a C$k$fd-set in $H$.

Case 2. $S_G = S \cap V(G) \neq \emptyset$ and $S_H = S \cap V(H) \neq \emptyset$

If $S_G = V(G)$, then $S_H \neq V(H)$ and $m < k$. It follows from Theorem 2.10 that $S_H$ is a $(k-m)$fd-set in $H$. Similarly, if $S_H = V(H)$, then $S_G \neq V(G)$, $n < k$, and $S_G$ is a $(k-n)$fd-set in $G$. If $S_G \neq V(G)$ and $S_H \neq V(H)$, then $S = S_G \cup S_H$, where $S_G$ is a $(k-|S_H|)$fd-set in $G$ and $S_H$ is a $(k-|S_G|)$fd-set in $H$ by Theorem 2.10.

For the converse, $S$ is clearly a C$k$fd-set in $G+H$ if Statement (a), (b), or (c) holds. Suppose Statement (e) holds, that is, $S = V(G) \cup T$, where $|V(G)| = m < k$ and $T$ is a $(k-m)$fd-set in $H$. Let $x \in V(G+H) \setminus S$, that is, $x \in V(H) \setminus T$. Then $|N_{G+H}(x) \cap S| = |V(G)| + |N_H(x) \cap T| = m+(k-m) = k$, hence $S$ is a $k$fd-set in $G+H$. Moreover, $\langle S \rangle = G+\langle T \rangle$, hence $\langle S \rangle$ is connected in $G+H$. 

Therefore, \( S \) is a \( Ckfd \)-set in \( G + H \). Similarly, \( S \) is a \( Ckfd \)-set in \( G + H \) if Statement (f) holds. Finally, suppose Statement (d) holds, that is, \( S = S_G \cup S_H \), where \( S_G \) is a \( (k - |S_H|)fd \)-set in \( G \) and \( S_H \) is a \( (k - |S_G|)fd \)-set in \( H \). Let \( u \in V(G + H) \setminus S \). Suppose further that \( u \in V(G) \setminus S_G \). Then, 
\[
|N_{G+H}(u) \cap S| = |S_H| + |N_G(u) \cap S_G| = |S_H| + (k - |S_H|) = k.
\]
Next, suppose further that \( u \in V(H) \setminus S_H \). Then, 
\[
|N_{G+H}(u) \cap S| = |S_G| + |N_H(u) \cap S_H| = |S_G| + (k - |S_G|) = k.
\]
Hence, \( S \) is a \( kfd \)-set in \( G + H \). Moreover, \( \langle S \rangle = \langle S_G \rangle + \langle S_H \rangle \), hence \( \langle S \rangle \) is connected in \( G + H \). Therefore, \( S \) is a \( Ckfd \)-set in \( G + H \).

**Corollary 3.2** Let \( G \) and \( H \) be nontrivial connected graphs of orders \( m \) and \( n \), respectively, and \( k \) a positive integer with \( 1 \leq k \leq \max\{m, n\} \). If \( G \) or \( H \) has a \( Ckfd \)-set \( S \) with \( |S| = k \), then \( \gamma_{Ckfd}(G + H) = k \). Moreover, if one of Statement (d), (e), or (f) of Theorem 3.1 holds, respectively, then 

\[
\gamma_{Ckfd}(G + H) = \gamma_{(k - |S_H|)}fd(G) + \gamma_{(k - |S_G|)}fd(H) = m + \gamma_{(k - m)}fd(H),
\]

or 

\[
\gamma_{Ckfd}(G + H) = n + \gamma_{(k - n)}fd(G),
\]

respectively. If none of Statements (b) to (f) of Theorem 3.1 holds, then \( \gamma_{Ckfd}(G + H) = m + n \).

**Proof.** Suppose \( G \) or \( H \) has a \( Ckfd \)-set \( S \) with \( |S| = k \). Then, by Theorem 3.1, \( S \) is a \( Ckfd \)-set in \( G + H \). Thus, 
\[
\gamma_{Ckfd}(G + H) \leq |S| = k.
\]
Clearly, 
\[
\gamma_{Ckfd}(G + H) \geq k.
\]
Therefore, \( \gamma_{Ckfd}(G + H) = k \). Next, suppose Statement (d), (e), or (f) of Theorem 3.1 holds, respectively. Suppose further that \( S_G \) is a \( \gamma_{(k - |S_H|)}fd \)-set in \( G \) and \( S_H \) is a \( \gamma_{(k - |S_G|)}fd \)-set in \( H \) for Statement (d), \( T \) is a \( \gamma_{(k - m)}fd \)-set in \( H \) for Statement (e), and \( D \) is a \( \gamma_{(k - n)}fd \)-set in \( G \) for Statement (f). Then by Theorem 3.1 each of the corresponding set \( S \) is a \( Ckfd \)-set in \( G + H \). Moreover, each set \( S \) is clearly a \( \gamma_{Ckfd} \)-set in \( G + H \). The respective desired conclusion now follows. If none of Statements (b) to (f) holds, then \( V(G + H) \) is a \( \gamma_{Ckfd} \)-set in \( G + H \), hence \( \gamma_{Ckfd}(G + H) = m + n \). 

\[\Box\]

4 Connected \( k \)-Fair Domination in the Corona of Graphs

The **corona** of two graphs \( G \) and \( H \), denoted by \( G \circ H \), is the graph obtained by taking one copy of \( G \) of order \( n \) and \( n \) copies of \( H \), and then joining the \( i \)-th vertex of \( G \) to every vertex in the \( i \)-th copy of \( H \). For every \( v \in V(G) \), we denote by \( H^v \) the copy of \( H \) whose vertices are joined or attached to the vertex \( v \). For each \( v \in V(G) \), the subgraph \( \langle v \rangle + H^v \) of \( G \circ H \) will be denoted by \( v + H^v \).

**Theorem 4.1** Let \( G \) be a nontrivial connected graph, let \( H \) be any graph, and let \( k \) be a positive integer with \( k \leq |V(H)| \). Then \( C \subseteq V(G \circ H) \) is a \( Ckfd \)-set in \( G \circ H \) if and only if \( C = V(G) \cup B \) where \( B = \emptyset \) or
$B = \bigcup_{v \in V(G)} S_v$, where $S_v = V(H^v)$ or $S_v$ is a $(k - 1)fd$-set in $H^v$ for some $k > 1$ for each $v \in V(G)$.

Proof. Let $C$ be a $Ckfd$-set in $G \circ H$ where $k \leq |V(H)|$. Then $\langle C \rangle$ is connected. Further, let $C = A \cup B$, where $A \subseteq V(G)$ and $B \subseteq \bigcup_{v \in V(G)} V(H^v)$.

Consider the following cases:

Case 1. $B = \emptyset$

Since $C$ is a dominating set in $G \circ H$, $A = V(G)$; that is, $C = V(G)$. Hence, $C$ is a $C1fd$-set in $G \circ H$.

Case 2. $B \neq \emptyset$

Since $\langle C \rangle$ is connected and $C$ is a dominating set, $A = V(G)$. Thus, $C = V(G) \cup B$. Let $v \in V(G)$ and let $S_v = V(H^v) \cap C$.

If $S_v = V(H^v)$, then we are done. Suppose that $S_v \neq V(H^v)$ and let $x \in V(H^v) \setminus S_v$. Hence,

$$|N_{G \circ H}(x) \cap C| = |N_{H^v}(x) \cap S_v| + 1 = k > 1.$$  

It follows that $|N_{H^v}(x) \cap S_v| = k - 1$. Thus, $S_v$ is a $(k - 1)fd$-set in $H^v$. In either case, $C$ is a $Ckfd$-set in $G \circ H$. \hfill \Box

Corollary 4.2 Let $G$ be a nontrivial connected graph of order $m$ and $H$ be any graph of order $n$, and let $k$ be a positive integer with $1 \leq k \leq n$. Then

$$\gamma_{Ckfd}(G \circ H) = \begin{cases} m, & \text{if } k = 1 \\ m(1 + \gamma_{(k-1)fd}(H)), & \text{if } k \geq 2. \end{cases}$$  

Proof. Clearly, $\gamma_{Ckfd}(G \circ H) = m$ if $k = 1$. So suppose that $k \geq 2$. It follows from Theorem 4.1 that if $C$ is a $Ckfd$-set in $G \circ H$, then $C = V(G) \cup B$ where $B = \bigcup_{v \in V(G)} S_v$ and $S_v$ is a $\gamma_{(k-1)fd}$-set in $H^v$ for each $v \in V(G)$. Thus,

$$\gamma_{Ckfd}(G \circ H) = |C| = m + \sum_{v \in V(G)} |S_v| = m + m(\gamma_{(k-1)fd}(H)) = m(1 + \gamma_{(k-1)fd}(H)).$$  

This proves the assertion. \hfill \Box

5 Connected $k$-Fair Domination in the Lexicographic Product of Graphs

The lexicographic product of two graphs $G$ and $H$, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.
Theorem 5.1 Let $G$ and $H$ be nontrivial connected graphs. If $C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ is a C$k$fd-set in $G[H]$, then the following hold:

(i) $S$ is a connected dominating set in $G$.

(ii) For each $x \in S \cap N_G(S)$, $T_x = V(H)$ and $|V(H)| = r \leq k$ whenever $C \neq V(G[H])$ or $T_x$ is an rfd-set and $\sum_{z \in N_G(x) \cap S} |T_z| = k-r$.

(iii) For each $y \in V(G) \setminus S$, $\sum_{v \in N_G(y) \cap S} |T_v| = k$.

Proof. Suppose that $C$ is a C$k$fd-set in $G[H]$. Then $S$ is a dominating set in $G$ by Theorem 2.11. We claim that $S$ is connected. Let $x, y \in S$ such that $x \neq y$ and $xy \notin E(G)$. Suppose further that $(x, a), (y, b) \in C$. Then $a \in T_x$ and $b \in T_y$ and $(x, a)(y, b) \notin E(G[H])$. Since $C$ is connected, there is a geodesic $[(x, a), (x_1, a_1), \ldots, (x_m, a_m), (y, b)]$ from $(x, a)$ to $(y, b)$ in $(C)$, where $x_i \in S$ for each $i$. Hence, $[x, x_1, \ldots, x_m, y]$ is a path from $x$ to $y$.

Therefore, $S$ is connected, thus (i) holds. By Theorem 2.11, (ii) and (iii) hold. \qed

The converse of Theorem 5.1 is not true. To see this, consider the graph shown in Figure 5.

![Figure 5: The graph $P_3[P_3]$](image)

Let $S = \{x\}$, a connected dominating set in $G$. Now, $S \cap N_G(S) = \emptyset$ and for each $y \in V(G) \setminus S$, $N_G(y) \cap S = \{x\}$ and $|T_x| = 2$. $C$ is a 2fd-set but not a C$2$fd-set in $G[H]$ since $C$ is not connected.

Theorem 5.2 Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively, and let $k = nt$ for some integer $t \leq m$. Then $C = S \times V(H)$ is a C$k$fd-set in $G[H]$ if and only if $S$ is a C$t$fd-set in $G$.

Proof. Let $C = S \times V(H)$ be a C$k$fd-set in $G[H]$. Then by Theorem 5.1, $S$ is a connected dominating set in $G$. Let $u \in V(G) \setminus S$, $a \in V(H)$. Then $(u, a) \in V(G[H]) \setminus C$. Since $C$ is a kfd-set,

$$|N_{G[H]}(u, a) \cap C| = |N_G(u) \cap S| \cdot |V(H)| = |N_G(u) \cap S| \cdot n = k,$$
hence $S$ is a $tfd$-set in $G$, where $t = k/n$. Therefore, $S$ is a $Ctfd$-set in $G$.

Conversely, suppose that $C = S \times V(H)$ and $S$ is a $Ctfd$-set in $G$. Since $S$ is connected, $C$ is clearly connected. By Theorem 2.9, $C$ is a dominating set in $G[H]$. We claim that $C$ is a $kfd$-set. Let $(u, a) \in V(G[H]) \setminus C$. Since $S$ is a $tfd$-set in $G$ where $t = k/n$,

$$\left| N_{G[H]}(u, a) \cap C \right| = \left| N_G(u) \cap S \right| \cdot |V(H)| = tn = k,$$

thus $C$ is a $kfd$-set. Therefore, $C$ is a $Ckfd$-set in $G[H]$. \hfill \Box

The next result immediately follows from Theorem 5.1.

**Corollary 5.3** Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively, and let $k = nt$ for some integer $t \leq m$. Then $\gamma_{Ckfd}(G[H]) \leq \gamma_{Ctfd}(G) \cdot n$.

**Proof.** Let $S$ be a $\gamma_{Ctfd}$-set in $G$, where $t = k/n$. Then $C = S \times V(H)$ is a $Ckfd$-set in $G[H]$ by Theorem 5.2. Hence, $\gamma_{Ckfd}(G[H]) \leq |C| = |S| \cdot |V(H)| = \gamma_{Ctfd}(G) \cdot n$. \hfill \Box

**Remark 5.4** The strict inequality in Corollary 5.3 can be attained. However, the given upper bound is sharp.

To see this, consider the graphs shown in Figure 6. The shaded vertices in each graph form a $\gamma_{Ckfd}$-set. Thus,

$$\gamma_{C3fd}(P_5[P_3]) = 9 = 3(3) = \gamma_{C1fd}(P_5) \cdot 3 = \gamma_{C1fd}(P_5) \cdot |V(P_3)|$$

and

$$\gamma_{C3fd}(C_4[P_3]) = 4 < 6 = \gamma_{C1fd}(C_4) \cdot 3 = \gamma_{C1fd}(C_4) \cdot |V(P_3)|$$

![Figure 6: The graphs $P_5[P_3]$ and $C_4[P_3]$.](image)
6 Connected $k$-Fair Domination in the Cartesian Product of Graphs

The *Cartesian product* of two graphs $G$ and $H$, denoted by $G \square H$, is the graph with vertex-set $V(G \square H) = V(G) \times V(H)$ and edge-set $E(G \square H)$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G \square H)$ if and only if either $u_1u_2 \in E(G)$ and $v_1 = v_2$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

Let $C \subseteq V(G \square H)$. The $G$-projection $C_G$ and $H$-projection $C_H$ of $C$ are, respectively, the sets $C_G = \{x \in V(G) : (x, a) \in C \text{ for some } a \in V(H)\}$ and $C_H = \{b \in V(H) : (y, b) \in C \text{ for some } y \in V(G)\}$. Further, if $x \in C_G$, then $T_x = \{a \in C_H : (x, a) \in C\}$, and if $a \in C_H$, then $D_a = \{x \in C_G : (x, a) \in C\}$.

**Lemma 6.1** Let $G$ and $H$ be nontrivial connected graphs. Then $C_1 = S_1 \times V(H)$ and $C_2 = V(G) \times S_2$ are connected sets in $G \square H$ if and only if $S_1$ and $S_2$ are connected sets in $G$ and $H$, respectively.

**Proof.** Suppose $S_1$ and $S_2$ are connected sets in $G$ and $H$, respectively. Let $C_1 = \{x \in V(G) : (x, a) \in C \text{ for some } a \in V(H)\}$ and $C_2 = \{b \in V(H) : (y, b) \in C \text{ for some } y \in V(G)\}$. Consider the following cases:

Case 1. $x = y$

Then $a \neq b$. Since $H$ is a connected graph, there exists an $a$-$b$ geodesic $[a, a_1, a_2, ..., a_n, b]$ in $H$. Thus, $[(x, a), (x, a_1), (x, a_2), ..., (x, a_n), (x, b)]$ is a geodesic from $(x, a)$ to $(y, b)$ in $\langle C \rangle$.

Case 2. $x \neq y$

Since $H$ is connected, there exists an $a$-$b$ geodesic $[a, a_1, a_2, ..., a_m, b]$ in $H$. Moreover, since $S_1$ is connected, there exists an $x$-$y$ geodesic $[x, x_1, x_2, ..., x_n, y]$ in $\langle S_1 \rangle$. If $a = b$, then $[(x, a), (x_1, a), ..., (x_n, a), (y, a)]$ is a geodesic from $(x, a)$ to $(y, b)$ in $\langle C \rangle$. If $a \neq b$, then $[(x, a), (x_1, a), ..., (x_n, a), (y, a), (y, a_1), ..., (y, a_m), (y, b)]$ is a path from $(x, a)$ to $(y, b)$ in $\langle C \rangle$.

Thus, in either case, $C_1$ is a connected set in $G \square H$.

Similarly, it can be shown that $C_2 = V(G) \times S_2$ is a connected set in $G \square H$.

For the converse, suppose $C = S \times V(H)$ is a connected set in $G \square H$. Let $x, y \in S$ where $x \neq y$ and $xy \notin E(G)$. Then $(x, a), (y, b) \in C$ for some $a, b \in V(H)$ and $(x, a) \neq (y, b)$, $(x, a)(y, b) \notin E(G \square H)$. Consider the following cases:

Case 1. $a = b$

Then there is a geodesic $[(x, a), (x_1, a), ..., (x_n, a), (y, b)]$ from $(x, a)$ to $(y, b)$ in $\langle C \rangle$. It follows that $[x, x_1, x_2, ..., x_n, y]$ is an $x$-$y$ geodesic in $\langle S \rangle$.

Case 2. $a \neq b$

Since $H$ is connected, there is an $a$-$b$ geodesic $[a, a_1, a_2, ..., a_m, b]$ in $H$. Since $C$ is connected, there is a path $[(x, a), (x_1, a), ..., (x_n, a), (y, a)$
such that \((y, a_1, \ldots, y, a_m, y, b)\) from \((x, a)\) to \((y, b)\) in \((C)\). It follows that \([x, x_1, x_2, \ldots, x_n, y]\) is a path from \(x\) to \(y\) in \(S\).

Thus, in either case, \(S\) is a connected set in \(G\).

Similarly, it can be shown that if \(C = V(G) \times S\) is a connected set in \(G \square H\), then \(S\) is a connected set in \(H\). \(\Box\)

**Theorem 6.2** Let \(G\) and \(H\) be nontrivial connected graphs. Then \(C_1 = S_1 \times V(H)\) and \(C_2 = V(G) \times S_2\) are \(Ckfd\)-sets in \(G \square H\) if and only if \(S_1\) and \(S_2\) are \(Ckfd\)-sets in \(G\) and \(H\), respectively.

**Proof.** Suppose \(C_1 = S_1 \times V(H)\) is a \(Ckfd\)-set in \(G \square H\).

By Lemma 6.1, \(S_1\) is a connected set in \(G\) since \(C_1\) is connected. We show that \(S_1\) is a \(kfd\)-set in \(G\). Let \(x \in V(G) \setminus S_1\). Then \((x, a) \in V(G \square H) \setminus C_1\) for any \(a \in V(H)\). Since \(C_1\) is a dominating set in \(G \square H\), there exists a \((y, a) \in C_1\) such that \((x, a)(y, a) \in E(G \square H)\). Thus, \(y \in S_1\) and \(xy \in E(G)\). Hence, \(S_1\) is a dominating set in \(G\). Moreover, since \(C_1 = S_1 \times V(H)\) is a \(kfd\)-set in \(G \square H\), then \(|N_G(x) \cap S_1| = |N_{G \square H}(x, a) \cap C_1| = k\). Thus, \(S_1\) is a \(kfd\)-set in \(G\). Therefore, \(S_1\) is a \(Ckfd\)-set in \(G\).

Similarly, we can show that if \(C_2 = V(G) \times S_2\) is a \(Ckfd\)-set in \(G \square H\), then \(S_2\) is a \(Ckfd\)-set in \(H\).

For the converse, let \(S_1\) be a \(Ckfd\)-set in \(G\). Then by Lemma 6.1, \(C_1 = S_1 \times V(H)\) is a connected set in \(G \square H\). We claim that \(C_1 = S_1 \times V(H)\) is a \(kfd\)-set in \(G \square H\). Let \((x, a) \in V(G \square H) \setminus C_1\). Then \(x \in V(G) \setminus S_1\). Since \(S_1\) is a dominating set in \(G\), there exists \(z \in S_1\) such that \(xz \in E(G)\). This implies that \((x, a)(z, a) \in E(G \square H)\). Moreover, \((z, a) \in C_1\), thus \(C_1\) is a dominating set in \(G \square H\). Now,

\[
|N_{G \square H}((x, a)) \cap C_1| = |N_G(x) \cap S_1| = k.
\]

Thus, \(C_1\) is a \(kfd\)-set in \(G \square H\). Therefore, \(C_1\) is a \(Ckfd\)-set in \(G \square H\).

Similarly, if \(S_2\) is a \(Ckfd\)-set in \(H\), then \(C_2 = V(G) \times S_2\) is a \(Ckfd\)-set in \(G \square H\). \(\Box\)

**Corollary 6.3** Let \(G\) and \(H\) be connected graphs. Then

\[
\gamma_{Ckfd}(G \square H) \leq \min\{\gamma_{Ckfd}(H) \cdot |V(G)|, \gamma_{Ckfd}(G) \cdot |V(H)|\}.
\]

**Remark 6.4** The strict inequality in Corollary 6.3 can be attained. However, the given upper bound is sharp.
To see this, consider the graphs shown in Figure 7. The shaded vertices in each graph form a $\gamma_{C_{kd}}$-set. Thus, $\gamma_{C_{1kd}}(P_3 \square P_3) = 5 = \min\{1(5), 3(3)\} = \min\{\gamma_{C_{1kd}}(P_3) \cdot |V(P_3)|, \gamma_{C_{1kd}}(P_3) \cdot |V(P_3)|\}$, and $\gamma_{C_{2kd}}(P_4 \square P_4) = 12 < \min\{\gamma_{C_{2kd}}(P_4) \cdot |V(P_4)|, \gamma_{C_{2kd}}(P_4) \cdot |V(P_4)|\} = \min\{4(4), 4(4)\} = 16.$

![Figure 7: The graphs $P_3 \square P_3$ and $P_4 \square P_4$](image)

**References**


Connected k-fair domination in the join, corona, lexicographic and ...

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