Analytical Solution of Fractional Navier-Stokes

Equation Using Sumudu Transform

and Its Properties

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Abstract

In this paper, we introduce an analytical and approximate technique to obtain the solution of time-fractional Navier-Stokes equation in a tube, which is based on Fractional Sumudu Transform method and its differential and integral properties. By using an initial value, the solution of the equation has been obtained in a closed form then its numerical solution has been represented graphically. The results obtained by the proposed technique indicate that the approach is easy to implement and computationally very attractive. The outcomes further provide evidence of the usefulness of this method. The achieved outcomes are calculated using the symbolic calculus software Maple 16.

Keywords: Time Fractional Navier-Stokes equation, Fractional Sumudu Transform method, Fractional integrals and derivatives for Sumudu Transform

1. Introduction

Fractional differential operators have a long history, however in past hundred years extensive notice in fractional differential equations has been motivated due to their abundant applications in the areas of science, physics, engineering [1-2]. Numerous significant models are well described by fractional differential equations in electro-chemistry, electromagnetics, viscoelasticity, acoustics, material sciences and financial market [3]. Some essential outcomes associated to
explaining fractional differential equations found in Oldham and Spanier [4], Miller and Ross [5], Diethelm and Ford [6], Kilbas and Srivastava [7] and Samko et al [8]. Consequently, seeking solutions of fractional ordinary and partial differential equations still a significant task. Therefore, there have been attempts to develop the new methods for obtaining analytical and approximate solutions of fractional ordinary and partial differential equations arising in science and engineering. Recently, numerous methods have drawn special attention such as Adomian Decomposition method [9], Homotopy Analysis method [10], Homotopy Perturbation method [11], Variational Iteration method [12], Hamiltonian and He’s energy Balance approach [13], Optimal Homotopy Asymptotic method [14].

In a given work, we study the unsteady, one dimensional motion of a viscous fluid in a tube, the equations of motion which direct the flow field in a tube are the Navier-Stokes equations in cylindrical coordinates [15]. Next, we apply the proposed Fractional Sumudu transform method (FSTM) to solve the time-fractional Navier-Stokes equation. The Fractional Sumudu transform method (FSTM) is a combination of the Sumudu Transform method and its differential and integral properties. The objective of this paper is to spread the application of the Fractional Sumudu transform method (FSTM) to acquire a solution of the time-fractional Navier-Stokes equation. The benefits of this technique is its capability for attaining exact and approximate analytical solutions. It is worth mentioning that the proposed technique is capable of reducing the volume of the computational work as matched to the classical methods while still keeping the high accurateness of the numerical outcome, the size reduction amounts to a perfection of the performance of the approach.

The Navier-Stokes equation is the principal equation of computational fluid dynamics, relating pressure and external forces acting on a fluid to the reaction of the fluid flow. The Navier-Stokes and continuity equations are given as:

\[
\frac{\partial \psi}{\partial \tau} + (\psi \cdot \nabla)\psi = -\left(\frac{1}{\rho}\right)\nabla p + \nu \nabla^2 \psi \tag{1.1}
\]

with condition,

\[
\psi \cdot \nabla = 0 \tag{1.2}
\]

where \(\rho\) is the density, \(p\) is the pressure, \(\nu\) is the kinematics viscosity, \(\psi\) is the velocity and \(\tau\) is the time.

The central purpose of this work is to present approximate analytical solutions of time-fractional Navier-Stokes equation by using fractional Sumudu transform method (FSTM). The equations of motion which govern the flow field in a tube are the Navier-Stokes equations in cylindrical coordinates are given as

\[
D_\tau \psi(x, \tau) = -\frac{\partial p}{\rho \partial z} + \nu (D_x^2 \psi + \frac{1}{x} D_x \psi) \tag{1.3}
\]

This model can be generalized by interchanging the first time derivative by a fractional derivative of order \(q\), \(0 < q \leq 1\). The time-fractional model for Navier-Stokes equation has the form of the operator equation as

\[
D_t^q \psi(x, \tau) = P + \nu \left(\psi_{xx} + \frac{2}{x} \psi_x\right), \quad 0 < q \leq 1 \tag{1.4}
\]
where \( P = -\frac{\partial \mu}{\partial z} \) and \( D_t^q \) represents the Caputo fractional derivative of order \( q \), where \( q \) is parameter telling the order of the time fractional derivatives. In the case of \( q = 1 \), the fractional equation reduces to the standard Navier-Stokes equation. The time-fractional Navier-Stokes equations have been studied by Momani and Odibat [15].

### 2. Basic Definitions of Fractional Calculus

In this section, we give some basic definitions and properties of Fractional Calculus and Sumudu transforms, which will be used in this paper.

**Definition:** A real function \( \psi(x), x > 0 \) is in the space \( K_\mu, \mu \in \mathbb{R} \) if there is a real number \( \lambda > \mu \) such that \( \psi(x) = x^\lambda \ g(x) \), where \( g(x) \in K \{0, \infty \} \) and it is in the space \( K_m^\mu \) if and only if \( \psi^{(m)} \in K_\mu \) for \( m \in \mathbb{N} \).

**Definition:** The Riemann-Liouville fractional integral operator of order \( q \) of a function \( (x) \in K_\mu, \mu \geq -1 \) is given as follows

\[
J^q \psi(x) = \frac{1}{\Gamma(q+1)} \int_0^x (x-t)^{q-1} \psi(t) \, dt, \quad q > 0, \quad x > 0
\]

The operator \( J^q \) has some properties, for \( q, \mu > -1 \) and \( C \) a real constant.

- \( J^0 \psi(x) = \psi(x) \)
- \( J^q C = \frac{C}{\Gamma(q+1)} x^q \)

**Definition:** The Caputo Fractional derivatives \( D^q \) of a function \( \psi(x) \) of any real number \( q \) such that \( m-1 < q < m, m \in \mathbb{N} \), for \( x > 0 \) and \( \psi \in C^m_{-1} \) as

\[
D^q \psi(x) = \frac{1}{\Gamma(m-q)} \int_0^x (x-t)^{m-q-1} \psi^{(m)}(t) \, dt, \quad q = m
\]

and has the following properties for \( m-1 < q < m, m \in \mathbb{N}, \mu \geq -1 \) and \( \psi \in C^m_{\mu} \)

- \( D^q J^q \psi(x) = \psi(x) \)
- \( J^q D^q \psi(x) = \psi(x) - \sum_{k=0}^{m-1} \psi^{(k)}(0+) \frac{x^k}{k!} \), for \( x > 0 \)

**Definition:** The Sumudu transform is explained over the set of functions:

\[
A = \{ \psi(\tau): \exists M, t_1, t_2 > 0, \ |\psi(\tau)| < Me^{t_1} \text{ if } \tau \in (-1)^l \times [0, \infty) \}
\]

given by

\[
F(u) = S[\psi(\tau)] = \int_0^\infty \frac{1}{u} e^{-\frac{t}{u}} \psi(t) \, dt
\]

The Sumudu transform is an integral transform, first proposed by Watugala in 1998, [16] to solve engineering problems [17]. The existence and uniqueness was given in [18]. For more information and properties of Sumudu transform and its derivatives, [19].

**Definition:** The Sumudu transform \( S[\psi(\tau)] \) of the Riemann-Liouville fractional integral is given as below [20-21],

\[
S[j^q \psi(\tau)] = u^q S[\psi(\tau)]
\]
**Definition:** The Sumudu transform $S[\psi(\tau)]$ of the Caputo fractional derivative is given as below [20-21].

$$S[D^q \psi(\tau)] = u^{-\alpha}S[\psi(\tau)] - \sum_{k=0}^{m-1} u^{-\alpha+k} \psi^{(k)}(0), \; m - 1 < q \leq m$$  \hspace{0.5cm} (2.6)

**Definition:** The Inverse Sumudu transform is given as [22]

$$S^{-1}[\sum_{k=0}^{m-1} u^{\alpha+k} \psi^{(k)}(0)] = \sum_{k=0}^{m-1} \frac{u^{\alpha+k} \psi^{(k)}(0)}{\Gamma(k+1)}$$  \hspace{0.5cm} (2.7)

### 3. Analysis of Proposed Fractional Sumudu Transform Method

In this section, we give the basic idea of Fractional Sumudu Transform method (FSTM) for the fractional partial differential equation.

To show the fundamental scheme of this technique, we let the given below nonlinear fractional partial differential equation

$$D^q_x \psi(x, \tau) + \mathcal{L}[\psi(x, \tau)] + \mathcal{N}[\psi(x, \tau)] = Q(x, \tau), \; \tau > 0, m - 1 < q \leq m$$  \hspace{0.5cm} (3.1)

with the initial condition

$$\psi(x, 0) = f_0(x)$$  \hspace{0.5cm} (3.2)

here $D^q_x = \partial^q / \partial \tau^q$ is the fractional Caputo derivative, $\mathcal{L}$ is the linear differential operator, $\mathcal{N}$ is the nonlinear differential operator and $Q(x, \tau)$ is the source term of the function $\psi(x, \tau)$.

Now, applying both sides the Sumudu transform (3.1), we have

$$S[D^q_x \psi(x, \tau)] + S[\mathcal{L}[\psi(x, \tau)] + \mathcal{N}[\psi(x, \tau)]] = S[Q(x, \tau)]$$  \hspace{0.5cm} (3.3)

Operating with the differential property of Sumudu transform,

$$u^{-\alpha}S[\psi(x, \tau)] - \sum_{k=0}^{m-1} u^{-\alpha+k} \psi^{(k)}(x, 0) + S[\mathcal{L}[\psi(x, \tau)] + \mathcal{N}[\psi(x, \tau)]] = S[Q(x, \tau)]$$  \hspace{0.5cm} (3.4)

On simplifying,

$$S[\psi(x, \tau)] = u^q \sum_{k=0}^{m-1} u^{-q+k} \psi^{(k)}(x, 0) + u^q S[[Q(x, \tau)] - [\mathcal{L}[\psi(x, \tau)] + \mathcal{N}[\psi(x, \tau)]]]$$  \hspace{0.5cm} (3.5)

Applying both sides the Sumudu inverse (3.5)

$$\psi(x, \tau) = \psi(x, 0) + S^{-1}[u^q S [[Q(x, \tau)] - [\mathcal{L}[\psi(x, \tau)] + \mathcal{N}[\psi(x, \tau)]]]]$$  \hspace{0.5cm} (3.6)

Operating with the integral property of Sumudu transform

$$\psi(x, \tau) = \psi(x, 0) + J^q_u \left[ [Q(x, \tau)] - [\mathcal{L}[\psi(x, \tau)] + \mathcal{N}[\psi(x, \tau)]] \right]$$  \hspace{0.5cm} (3.7)
Analytical solution of fractional Navier-Stokes equation

The Fractional Sumudu transform decomposition admits a solution in the form
\[
\psi(x, \tau) = \sum_{n=0}^{\infty} \psi_n(x, \tau)
\]  
(3.8)

Substituting (3.8) in (3.7), we get,
\[
\sum_{n=0}^{\infty} \psi_n(x, \tau) = \psi(x, 0) + \int_0^\tau \left[ Q(x, \tau) - \left[ \mathcal{L} \left[ \sum_{n=0}^{\infty} \psi_n(x, \tau) \right] + \mathcal{N} \left[ \sum_{n=0}^{\infty} \psi_n(x, \tau) \right] \right] \right] dt
\]  
(3.9)

The technique shows a series solution for \( \psi(x, \tau) \) defined as
\[
\psi(x, \tau) = \sum_{n=0}^{\infty} f_n(x) \frac{\tau^n}{\Gamma(qn+1)}
\]  
(3.10)

Equating the terms on both sides of (3.10), we get the following relation
\[
\psi_{n+1}(x, \tau) = \int_0^\tau \left[ Q(x, \tau) - \left[ \mathcal{L} \left[ \psi_n(x, \tau) \right] + \mathcal{N} \left[ \psi_n(x, \tau) \right] \right] \right] dt = \sum_{n=1}^{\infty} f_n(x) \frac{\tau^n}{\Gamma(qn+1)}
\]  
(3.11)

and the functions \( \{f_n\}_{n=0}^{\infty} \) are given by
\[
\begin{align*}
    f_0 &= \psi_0 \\
    f_1 &= Q(x, \tau) - \left[ \mathcal{L} [f_0(x, \tau)] + \mathcal{N} [f_0(x, \tau)] \right] \\
    f_2 &= \mathcal{L} [f_1(x, \tau)] + \mathcal{N} [f_1(x, \tau)] \\
    f_3 &= \mathcal{L} [f_2(x, \tau)] + \mathcal{N} [f_2(x, \tau)] \\
    &\vdots \\
    f_{n+1} &= \mathcal{L} [f_n(x, \tau)] + \mathcal{N} [f_n(x, \tau)]
\end{align*}
\]  
(3.12)

So that the solution \( \psi(x, \tau) \) in series form is defined as
\[
\psi(x, \tau) = f_0 + f_1 \frac{\tau^q}{\Gamma(q+1)} + f_2 \frac{\tau^{2q}}{\Gamma(2q+1)} + f_3 \frac{\tau^{3q}}{\Gamma(3q+1)} + \ldots + f_n \frac{\tau^{nq}}{\Gamma(nq+1)}
\]  
(3.13)

4. Illustrative examples

In this section, we discuss the implementation of our proposed algorithm and investigate its accuracy on time - fractional Navier Stokes equation in a tube. The simplicity and accurateness of the proposed technique is showed through the following examples.

Example 1: Consider the following time - fractional Navier Stokes equation [15]:
\[
D_t^q \psi(x, \tau) = P + \psi_{xx} + \frac{1}{z} \psi_x, \quad 0 < q \leq 1
\]  
(4.1)

where \( P = -\frac{\partial p}{\partial z} \), subject to initial condition
\[
\psi(x, 0) = 1 - x^2 = f_0(x)
\]  
(4.2)

Now, applying the proposed Fractional Sumudu transform technique, using both sides the Sumudu transform (4.1),
\[
S[D_t^q \psi(x, \tau)] = S[P + \psi_{xx} + \frac{1}{z} \psi_x]
\]  
(4.3)
Operating differential property of Sumudu transform

\[ S[\psi(x, \tau)] = \psi(x, 0) + u^q S[P + \psi_{xx} + \frac{1}{x} \psi_x] \quad (4.4) \]

Using both sides the Inverse Sumudu (4.4)

\[ \psi(x, \tau) = \psi(x, 0) + S^{-1}[u^q S[P + \psi_{xx} + \frac{1}{x} \psi_x]] \quad (4.5) \]

Operating the integral property of Sumudu transform

\[ \psi(x, \tau) = \psi(x, 0) + J[\tau] q \left[ P + \psi_{xx} + \frac{1}{x} \psi_x \right] \quad (4.6) \]

The Sumudu transform decomposition admits a solution in the form

\[ \psi(x, \tau) = \sum_{n=0}^{\infty} \psi_n(x, \tau) \quad (4.7) \]

Substituting (4.7) in (4.6), we get,

\[ \sum_{n=0}^{\infty} \psi_n(x, \tau) = \psi(x, 0) + J[\tau] q \left[ P + \sum_{n=0}^{\infty} \psi_{nxx} + \frac{1}{x} \sum_{n=0}^{\infty} \psi_{nx} \right] \quad (4.8) \]

The technique shows a series solution for \( \psi(x, \tau) \), we obtain

\[ \psi_{n+1}(x, \tau) = J[\tau] q \left[ P + \psi_{nxx} + \frac{1}{x} \psi_{nx} \right] = \sum_{n=1}^{\infty} f_n(x) \frac{\tau^n q}{\Gamma(qn+1)} \quad (4.9) \]

Equating the terms on both sides of (4.9), we get the following relation

\[ \psi_0(x, \tau) = \psi(x, 0) = f_0(x) = 1 - x^2 \]
\[ \psi_{n+1}(x, \tau) = f_n(x) = \sum_{n=1}^{\infty} f_n(x) \frac{\tau^n q}{\Gamma(qn+1)} \quad (4.10) \]

and the functions \( f_n \) are given by

\[ f_0 = \psi_0(x, \tau) = \psi(x, 0) = 1 - x^2 \]
\[ f_1 = P + f_0xx + \frac{1}{x} f_0x = P - 4 \]
\[ f_2 = f_1xx + \frac{1}{x} f_1x = 0 \]
\[ f_3 = f_2xx + \frac{1}{x} f_2x = 0 \]
\[ \vdots \]
\[ f_{n+1} = 0, \quad \forall \ n \geq 0 \quad (4.11) \]

So that the solution \( \psi(x, \tau) \) in series form is defined as

\[ \psi(x, \tau) = f_0 + f_1 \frac{\tau q}{\Gamma(q+1)} + f_2 \frac{\tau^2 q}{\Gamma(2q+1)} + f_3 \frac{\tau^3 q}{\Gamma(3q+1)} + \cdots + f_n \frac{\tau^n q}{\Gamma(nq+1)} \quad (4.12) \]

Now the solution \( \psi(x, \tau) \) in closed form is obtained as

\[ \psi(x, \tau) = \sum_{n=0}^{\infty} \psi_n(x, \tau) = 1 - x^2 + \left( P - 4 \right) \frac{\tau q}{\Gamma(q+1)} \quad (4.13) \]
which represents the exact solution for (4.1) and setting \( q = 1 \) in (4.13), we reproduce the solution of the classical Navier-Stokes equation as follows

\[
\psi(x, \tau) = 1 - x^2 + (P - 4)\tau
\]

(4.14)

The solution of \( \psi(x, \tau) \) w.r.t \( x \) and \( \tau \) when \( q = 0.5, 0.7, 0.8, 0.9, 1 \) and \( \tau = 1 \) is given in Figure 4.1

![Figure 4.1](image)

Fig 4.1: The behaviour of the solution for different values of \( q \) at \( P = 1 \) and \( \tau = 1 \)

The following surfaces show the solution of \( \psi(x, \tau) \) for (4.1) with respect to \( x \) and \( \tau \) when \( P = 1 \) and \( q = 0.5, 1 \) is given in Figure 4.2 (a) and Figure 4.2 (b).

![Figure 4.2(a)](image)  ![Figure 4.2(b)](image)

Fig 4.2(a): Plot of the solution when \( q = 1 \)  Fig 4.2(b): Plot of the solution when \( q = 0.5 \)

The above result is in complete agreement with Momani and Odibat [15].

**Example 2:** Consider the following time-fractional Navier-Stokes equation [15]:

\[
D^q_t \psi(x, \tau) = \psi_{xx} + \frac{1}{x} \psi_x , \quad 0 < q \leq 1
\]

subject to initial condition

\[
\psi(x, 0) = x = f_0(x)
\]

(4.15)

(4.16)

Now, applying the proposed Fractional Sumudu transform technique on (4.15) and (4.16), we obtained the following relation

\[
\psi(x, \tau) = \sum_{n=0}^{\infty} \psi_n(x, \tau) = \psi(x, 0) + \int_0^\tau \left[ \sum_{n=0}^{\infty} \psi_{nxx} + \frac{1}{x} \sum_{n=0}^{\infty} \psi_{nx} \right] d\tau
\]

\[
= \sum_{n=0}^{\infty} f_n(x) \frac{\tau^{\frac{nq}{2}}}{\Gamma\left(qn+1\right)}
\]

(4.17)
Equating the terms on both sides of (4.17), we get the following relation

$$\psi_{n+1}(x, \tau) = f_n(x, \tau) = \frac{r^n}{\Gamma(qn+1)}$$

(4.18)

and the functions \((f_n)_{n=0}^{\infty}\) are given by

$$f_0 = \psi(0, \tau) = \psi(0, 0) = x$$

$$f_1 = f_0 \tau + \frac{1}{x} f_0 \tau = \frac{1}{x}$$

$$f_2 = f_1 \tau + \frac{1}{x} f_1 \tau = \frac{1}{x^2}$$

$$f_3 = f_2 \tau + \frac{1}{x} f_2 \tau = \frac{1}{x^3}$$

$$f_4 = f_3 \tau + \frac{1}{x} f_3 \tau = \frac{1}{x^4}$$

$$f_5 = f_4 \tau + \frac{1}{x} f_4 \tau = \frac{1}{x^5}$$

$$f_n = f_{n-1} \tau + \frac{1}{x} f_{n-1} \tau = \frac{1}{x^n}$$

(4.19)

So that the solution \(\psi(x, \tau)\) in series form is defined as

$$\psi(x, \tau) = f_0 + f_1 \frac{\tau}{\Gamma(q+1)} + f_2 \frac{\tau^2}{\Gamma(2q+1)} + f_3 \frac{\tau^3}{\Gamma(3q+1)} + f_4 \frac{\tau^4}{\Gamma(4q+1)} + f_5 \frac{\tau^5}{\Gamma(5q+1)} + \cdots + f_n \frac{\tau^n}{\Gamma(nq+1)}$$

(4.20)

From (4.19), the solution \(\psi(x, \tau)\) becomes

$$\psi(x, \tau) = x + \frac{1}{x} \frac{\tau^q}{\Gamma(q+1)} + \frac{1}{x^3} \frac{\tau^{2q}}{\Gamma(2q+1)} + \frac{1}{x^5} \frac{\tau^{3q}}{\Gamma(3q+1)} + \frac{1}{x^7} \frac{\tau^{4q}}{\Gamma(4q+1)} + \frac{1}{x^9} \frac{\tau^{5q}}{\Gamma(5q+1)} + \cdots + \frac{1}{x^{2n-1}} \frac{\tau^{nq}}{\Gamma(nq+1)}$$

(4.21)

Now the solution \(\psi(x, \tau)\) in closed form is obtained as

$$\psi(x, \tau) = \sum_{n=0}^{\infty} \psi_n(x, \tau) = x + \sum_{n=1}^{\infty} \frac{1}{x^{2n-1}} \frac{\tau^{nq}}{\Gamma(nq+1)}$$

(4.22)

which represents the exact solution for (4.15) and setting \(q = 1\) in (4.22), we reproduce the solution of the classical Navier-Stokes equation as follows

$$\psi(x, \tau) = \sum_{n=0}^{\infty} \frac{1}{x^{2n-1}} \frac{\tau^{n}}{n!}$$

(4.23)

The above result is in complete agreement with Momani and Odibat [15].

5. Conclusion

In this paper, a concept of the Sumudu transform and its derivatives is successfully applied for time - fractional Navier Stokes equation in a tube by
using Fractional Sumudu transform method (FSTM). Two examples from the literature [15] are presented to determine the efficiency and simplicity of the proposed method. The achieved outcomes are calculated using the symbolic calculus software Maple 16. This scheme (FSTM) was clearly very efficient and powerful technique in finding the analytical solutions as well as numerical solutions of fractional partial differential equations.

References


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