Abstract

In this paper, we extend the model presented by Duffie et al. [5], assuming that the agents pay the transaction costs, when trading assets in over the counter market. First of all, we determine a formula of equilibrium price, depending on transaction costs, and analyze the formula when vanishing search frictions. Moreover, we measure the impact on equilibrium price of the different transaction cost politics, depending to one who loads to pay them.

Mathematics Subject Classification: 62P20, 91B26

Keywords: search frictions, transaction costs, equilibrium price

1 Introduction

An Over-the-Counter (OTC) market is a decentralized market, without a central physical location. This characteristic of decentralized structure requires
buy and sell transactions to be directly negotiated by agents who need to contact one of the potential dealers in the market to learn about prices. Moreover, the characteristic of the market’s decentralization offers the possibility to bargain an acceptable price of trade which is different from bid-ask quotations which are normally posted by dealers (see, e.g., Bloomberg or Reuters).

The trading delays generated by search frictions, consequently, it take time to meet a suitable trading counter-party. In particular, since the information regarding trades is not “transparent”, OTC markets are often opaque; due to this opacity, Duffie [3] defines this kind of market as a “dark market”. The difficulties in searching counterparts and the bargaining problems affect on the liquidity level (and, consequently, influence asset prices) of the assets traded in these markets. Brunnermeier and Pedersen [2] show that the market liquidity is reduced if funding liquidity is bounded, because it is affected by funding of trades. The most observed difficulties are part of market friction’s problems; an investor can identify problems (for instance, the mis-information about a product or the taxes levied on the transactions) that might lead to alter her/his decision and, consequently, lead to not execute the trading. The interested reader can focus her/his attention to papers of Amihud and Mendelson [1], Garman [7], and Ho and Stoll [9], for theoretical considerations, or papers of Glosten and Milgrom [8], and Kyle [10], for an information of problems arising from asymmetric information.

In recent times, they have been carried out researches on the relationship between search frictions or liquidity and volatility of the market; in particular, these studies have been focused on the impacts of the search-and-bargaining in financial markets on asset price dynamics. Duffie et al. [4, 5] have developed search models, based on bargaining theory (see, e.g., Nash [12] or Rubinstein and Wolinsky [14]), explaining the relationship between volatility and trading volume, modeling search frictions in a single-asset over the counter market leading to illiquidity and assuming that the asset cannot be traded whenever one wants. Additionally, in Duffie et al. [4], the authors carry out some studies about the influence of the skills of the investors to find counter-parties on the bid-ask spread: “Bid-ask spreads are lower if investors can find each other more easily”. In their base model, the authors assume that risk-neutral agents can trade two assets: An interest-bearing deposit account and a fixed coupon perpetual bond; the interest rate is assumed constant. In Duffie et al. [5], the authors extended their model, examining the impact of the risk-aversion on equilibrium price; the fixed coupon perpetual bond is replaced by an asset with stochastic dividends while the interest rates remains constant. In their paper,
they compute steady-state prices assuming that all agents are risk-neutral and risk-averse and conclude that their model is a good first-order approximation to the risk-averse one. In the last section of their paper, the authors carry out an analysis on the negative influence of the aggregate liquidity shocks of the holding costs, on the prices. Other extensions have been studied in the last years. For instance, Vayanos and Wang [15] and Weill [16] extend the search friction problem, considering various asset markets; in particular, the authors show that if there exist price discrepancies between assets with identical pay-offs (so that the meeting between buyer and seller is more probable), these discrepancies generate search frictions. Gârleanu [6] considers a model where trades can occur with more than two agents and Lagos and Rocheteau [11] develop a search-theoretic model of financial intermediation in an over-the-counter market and analyze the distribution of asset holdings and liquidity measures are influenced by trading frictions.

As in Duffie et al. [4, 5], we consider an economy in terms of their baseline model and extend the risk-aversion model proposed by Duffie et al. [5], adding the transaction costs. In this paper, we assume that the transaction costs are a fixed percentage of equilibrium prices and, assume that the buyer and seller pay different transaction costs, and analyze the impact on the equilibrium price of the different policies related to transaction costs. With the Nash bargaining theory or the Rubinstein-type game (see, e.g., Nash [12] or Rubinstein and Wolinsky [14]), we get a general formula for the equilibrium price, when the buyer and seller pay different transaction costs. This formula is similar in spirit to the risk-aversion formula in Duffie et al. [5]. The impact of transaction costs on equilibrium price can be seen as decrease bargaining power of the seller. This reduction comes forward in any analyzed case: both when the transaction costs are shared between seller and buyer and they are paid by one of the counter-parties. In particular, we empirically show that the equilibrium price decreases with respect to interest rate $r$, and increases with respect to matching intensity $\lambda$ and bargaining power $q$. Moreover, we graphically measure the impact of transaction costs on the equilibrium price: As is naturally expected, the equilibrium price is higher when transaction costs aren’t paid. In particular, we study the behaviour of the equilibrium price when only one of the parties takes the payment of transaction costs upon oneself; the seller that pays transaction costs, tends to bargain a higher price $\overline{P}$ to sell her/his asset. Conversely, if transaction costs are paid by the buyer, it leads to bargaining a lower equilibrium price $\underline{P}$ to buy her/his asset. When transaction costs are paid in equal amounts, the equilibrium price $P$ is lower than $\overline{P}$ and is higher than $\underline{P}$. The price analysis is particularly interesting,
when vanishing search frictions. As it is naturally expected, this price is lower than the perfect liquidity price determined in Duffie et al. [5]. Moreover, Walrasian price decreases with respect to transaction costs.

The paper is organized as follows: In Section 2, we recall the basic search model of asset prices with risk-neutral agents, studied by Duffie et al. [4, 5]. Section 3, analyzes the problem of determining a formula for bargaining price when we assume that the agents pay transaction costs. Moreover, we give a formula to Walrasian price when the matching intensity $\lambda \to +\infty$. Section 4 is devoted to measuring the impact of transaction costs on the equilibrium price and to empirically show the behaviour of the equilibrium price with respect to one who pays transaction costs, in several cases. The mathematical proofs are contained in the appendix.

2 Model Search

In this Section we briefly summarize the basic model, proposed by Duffie et al. [5].

In a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where the filtration $\{\mathcal{F}_t : t \geq 0\}$ satisfies the usual conditions (see, e.g., Protter [13]), we assume that all agents are risk-averse and have an infinite lifetime with a constant time-preference rate $\beta > 0$ for consumption of a single not storable numeraire good; moreover, each agent can invest in a liquid risk-free asset with an interest rate of $r$, and trade a risky asset in an OTC market. To prevent Ponzi schemes, we assume that the liquid wealth process $X_t$ of each agent has a lower bound.

In the market, there are four agent types; they can have high or low liquidity (“$h$” and “$l$”, respectively), and in their turn, can own a large or small number of the shares (“$o$” or “$n$”, respectively). We denote the full set of agent types is $\mathcal{T} = \{ho, hn, lo, ln\}$. The agent’s intrinsic type is a Markov chain switching from low to high with intensity $\lambda_u$, and back with intensity $\lambda_d$. The switching processes are random and are assumed to be pairwise independent for any couple of investor so that the agents are incentivized to trade because low-type owners want to sell and high-type non-owners want to buy.

Let $\mu_\sigma(t)$ be the fraction at time $t$ of agents of type $\sigma \in \mathcal{T}$, it follows that the fractions of each type of agent add to 1 at any time $t$:

$$\sum_{\sigma \in \mathcal{T}} \mu_\sigma(t) = 1. \quad (1)$$
Since the agents can trade the risky asset, the trading requires searching for a counter-party. The search process is random and is driven by a Poisson process with intensity $\lambda$, reflecting search ability and/or efficiency in the OTC market. Only low-type owners want to sell their asset, while only high-type non-owners want to buy one. The type-agents $ln$ and $ho$ don’t trade.

In equilibrium, the rates of change of the fractions of the respective investors types are

$$
\begin{align*}
\dot{\mu}_{lo}(t) &= -2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_u\mu_{lo}(t) + \lambda_d\mu_{ho}(t), \\
\dot{\mu}_{ln}(t) &= +2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_u\mu_{ln}(t) + \lambda_d\mu_{hn}(t), \\
\dot{\mu}_{ho}(t) &= +2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_d\mu_{ho}(t) + \lambda_u\mu_{lo}(t), \\
\dot{\mu}_{hn}(t) &= -2\lambda\mu_{hn}(t)\mu_{lo}(t) - \lambda_d\mu_{hn}(t) + \lambda_u\mu_{ln}(t).
\end{align*}
$$

In [4], the authors show that there is a unique stable steady-state solution for $\{\mu(t) : t \geq 0\}$

In steady-state equilibrium, assuming that an agent owns either $\theta_{(n)}$ or $\theta_{(o)}$ shares of the asset, market clearing requires that

$$(\mu_{lo} + \mu_{ho})\theta_{(o)} + (\mu_{ln} + \mu_{hn})\theta_{(n)} = \Theta,$$

where $\Theta$ is the total supply of shares per investor; consequently, by (1), the fraction of large owners is

$$\mu_{lo} + \mu_{ho} = \frac{\Theta - \theta_{(n)}}{\theta_{(o)} - \theta_{(n)}}.$$

Consequently, it is possible compute Nash bargaining price of the risky asset $P$, by the determined steady-state fractions of investor types.

3 Equilibrium price with transactions costs

We assume that there are transaction costs in the financial market and all assets can be traded continuously. Let $(W_t, W^1_t, W^2_t, W^3_t, \ldots)$ be a sequence of independent Brownian motion with respect to a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

We assume that there are two traded securities:

- one risk-free asset defined by

$$
\begin{align*}
\begin{dcases}
 dB_t = r B_t \, dt \\
B_0 = b,
\end{dcases}
\end{align*}
$$

where $r$ is the constant interest rate;
• one risky stock paying a cash flow $D_t$ is described by the cumulative dividend process
\[
\begin{aligned}
dD_t &= m_D dt + \sigma_D dW_t, \\
D_0 &= d.
\end{aligned}
\]
The agent $i$ receives a cumulative endowment process $\eta^i_t$, with
\[
\begin{aligned}
d\eta^i_t &= m_\eta dt + \sigma_\eta (\rho^i dW^i_t + \sqrt{1 - (\rho^i)^2} dW^i_t), \\
\eta_0 &= \eta,
\end{aligned}
\]
where $\rho^i$ is the instantaneous correlation between the asset dividend and the endowment of agent $i$.

The consumption and trading of an investor must be consistent with the dynamics of his/her wealth. Specifically, the wealth process $X_t$ satisfies
\[
\begin{aligned}
dX_t &= (r X_t - c_t) dt + \theta_t dD_t + d\eta_t - P d\theta_t - d\xi(\theta_t), \\
X_0 &= x,
\end{aligned}
\]
where $c_t$ is the consumption, $\xi(\theta_t)$ is the process that represents transaction costs and $\theta_t$ denotes the associated asset-position process; that is
\[
\theta_t = \begin{cases}
\theta_{(o)} & \text{if the agent owns the stock,} \\
\theta_{(n)} & \text{otherwise.}
\end{cases}
\]

We make the following standing assumption:

**Assumptions 3.1.** The transaction costs are proportional to trading price $P \theta_t$:
\[
d\xi(\theta_t) = P \xi^\pm d\theta_t,
\]
where $\xi^+$ (or $\xi^-$) represents the share percentage of transaction costs for the seller (or for the buyer).

Moreover, we assume that $\xi^+$ is positive and is smaller than the bargaining power $q$, and $\xi^-$ is positive.

In this paper, as in Duffie et al. [5], the agents derive utility from the consumption of a numeraire good, and have a CARA utility function with constant coefficient of absolute risk aversion $\gamma > 0$. The goal is to maximize the following expectation
\[
\mathbb{E} \left[ \int_0^\infty e^{-\beta t} (1 - e^{-\gamma c_t}) dt | \mathcal{F}_0 \right],
\]
subject to the budget constraint (3), and $\beta$ is the constant time-preference rate. Then our value function is given by

$$V(x; \sigma) := -\inf_c \mathbb{E} \left[ -\int_0^\infty e^{-\beta t - \gamma c} \, dt \, \big| F_0 \right].$$

(4)

The solution of this dynamic programming problem is given by

$$V(x; \sigma) = -\inf_c \sup_h \mathbb{E} \left[ -\int_0^h e^{-\beta t - \gamma c} \, dt + e^{-\beta h} V(X_h; r_h; \sigma) \, \big| F_0 \right].$$

(5)

Under the following assumption in order to prohibit Ponzi schemes,

$$\lim_{T \to +\infty} e^{-\beta T} \mathbb{E} \left[ e^{-r\gamma X_T} \, \big| F_0 \right] = 0,$$

the function value (5) satisfies the following HJM equation

$$0 = \sup_c \left\{ -\beta V(x; \sigma) - e^{-\gamma c} + \mathcal{L} V(x; \sigma) \right. + \left. \sum_{\{\sigma': \sigma' \neq \sigma\}} \lambda(\sigma, \sigma') \left( V(x + (z(\sigma, \sigma') - \xi^\pm) P\theta; \sigma') - V(x; \sigma) \right) \right\},$$

(6)

where $\mathcal{L}$ is the infinitesimal operator of the process $X_t$ so defined

$$\mathcal{L} := \left[ r x - c + \theta m_D + m_\eta \right] \frac{\partial}{\partial x} + \frac{1}{2} \left[ \theta^2 \sigma_D^2 + \sigma_\eta^2 + 2 \rho \theta \sigma_D \sigma_\eta \right] \frac{\partial^2}{\partial x^2},$$

$\lambda(\sigma, \sigma')$ is the intensity of transition from $\sigma$ to $\sigma'$, $z(\sigma, \sigma')$ is -1, 1, or 0, depending on whether the transition is, respectively, a buy, a sell, or an intrinsic-type change and $\xi^\pm$ is $\xi^+$ or $\xi^-$, depending on whether transaction costs are paid by seller or the buyer, respectively.

**Theorem 3.2.** Under Assumption 3.1, in a steady-state equilibrium, we conjecture that the value function is given by

$$V(x; \sigma) = -e^{-\gamma r \left( x + \bar{a} \sigma + \bar{a} \right)},$$

(7)

where

$$\bar{a} = \frac{1}{r} \left( \frac{\ln(r)}{\gamma} + m_\eta - \frac{1}{2} r \gamma \sigma_\eta^2 - \frac{r - \beta}{r \gamma} \right).$$

(8)
and the parameters $a_\sigma$ with $\sigma \in T$ satisfy the following system

\[
\begin{cases}
0 = r a_{lo} + \lambda_u e^{-r \gamma (a_{ho} - a_{lo})} - 1 + 2 \lambda \mu_{hn} e^{-r \gamma (P \bar{\theta}(1-\xi^-) + a_{ln} - a_{lo})} - 1 \\
0 = r a_{ln} + \lambda_u e^{-r \gamma (a_{hn} - a_{ln})} - 1 - (K(\theta) - \theta \bar{\delta}) \\
0 = r a_{ho} + \lambda_d e^{-r \gamma (a_{ho} - a_{ln})} - 1 - K(\theta) \\
0 = r a_{hn} + \lambda_d e^{-r \gamma (a_{hn} - a_{ln})} - 1 + 2 \lambda \mu_{lo} e^{-r \gamma (-P \bar{\theta}(1+\xi^-) + a_{ho} - a_{hn})} - 1 \\
- K(\theta(\bar{n}))
\end{cases}
\]

with

\[
\bar{\theta} = \theta(\bar{o}) - \theta(\bar{n}), \quad K(\theta) = \theta m_D - \frac{1}{2} r \gamma \left( \theta^2 \sigma_D^2 + 2 \rho \theta \sigma_D \sigma_n \right), \quad \bar{\delta} = r \gamma (\rho_l - \rho_h) \sigma_D \sigma_n.
\]

Then,

1. The optimal consumption rate is given by

\[
c^* = r(x + a_\sigma + \bar{a}) - \frac{\ln(r)}{\gamma}.
\]

2. The Nash bargaining equation is given by

\[
\frac{1-q}{1-\xi^+} \left[ 1 - e^{r \gamma (P \bar{\theta}(1-\xi^-) - (a_{ho} - a_{ln}))} \right] = \frac{q}{1+\xi^-} \left[ 1 - e^{-r \gamma (-P \bar{\theta}(1+\xi^-) + a_{ho} - a_{hn})} \right],
\]

where $q$ is the bargaining power of the seller.

The previous theorem extends the result in Duffie et al. [5]; in particular, the extension involves the system of equations characterizing the coefficients $a_\sigma$ given by (9), depending on the presence of transaction costs in the first and last equation. Moreover, the transaction costs also influence the bargaining equation (12). In the following proposition, we obtain a closed formula for the equilibrium price, applying the first order approximation to the exponential terms.

**Proposition 3.3.** Assume the same assumptions of Theorem 3.2. If the function value is given by (7)-(10) and the optimal consumption rate is given by (11), then the approximated equilibrium price $P$ is given by

\[
P \approx \frac{K(\theta(\bar{o})) - K(\theta(\bar{n}))}{\bar{\theta}} A(\xi^+, \xi^-) - \bar{\delta} B(\xi^+, \xi^-),
\]
where

\[ A(\xi^+, \xi^-) = \frac{(r + \lambda_u + \lambda_d)(1 - q)(1 + \xi^-) + q(1 - \xi^+) + 2\lambda\mu_{hn}q(1 - \xi^+) + 2\lambda\mu_{lo}(1 - q)(1 + \xi^-)}{D(\xi^+, \xi^-)}, \]

\[ B(\xi^+, \xi^-) = \frac{(1 - q)(1 + \xi^-)r + \lambda_d(1 - q)(1 + \xi^-) + q(1 - \xi^+) + 2\lambda\mu_{lo}(1 - q)(1 + \xi^-)}{D(\xi^+, \xi^-)}, \]

with

\[ D(\xi^+, \xi^-) = r(1 - \xi^+)(1 + \xi^-)[r + \lambda_u + \lambda_d + 2\lambda\mu_{hn}q + 2\lambda\mu_{lo}(1 - q)] \]

\[ + (\xi^+ + \xi^-)[2\lambda\mu_{hn}q(1 - \xi^+)\lambda_d - 2\lambda\mu_{lo}(1 - q)(1 + \xi^-)\lambda_u]. \]

### 3.1 Walrasian equilibrium price

A Walrasian equilibrium characterizes equality between demand and supply in each state and at every point in time and all agents can sell and buy instantly. A Walrasian allocation is efficient and all assets are held by agents of high type, if

\[ \mu_{lo} < \mu_{hn}. \]  

(17)

In steady-state equilibrium, the condition (17) holds if and only if all sellers can sell immediately and buyers are rationed; that is

\[ s := \mu_{lo} + \mu_{ho} < \frac{\lambda_u}{\lambda_u + \lambda_d}. \]  

(18)

**Proposition 3.4.** Under the condition (18), the Walrasian frictionless price \( P_\infty \) is given by

\[ P_\infty = \frac{K(\theta(o)) - K(\theta(n))}{\theta} \frac{1}{r(1 + \xi^-) + (\xi^+ + \xi^-)\lambda_d}. \]  

(19)

We can immediately note that:

\[ \text{since } (r + \lambda_d)\xi^- + \lambda_d\xi^+ > 0 \implies \frac{1}{r(1 + \xi^-) + (\xi^+ + \xi^-)\lambda_d} < \frac{1}{r}; \]

consequently, we price \( P_\infty < \bar{P}_\infty \), where \( \bar{P}_\infty \) is the perfect liquidity price, determined in [5].

Now, with the following corollary, we determine the Walrasian frictionless price when buyers and sellers pay same transaction costs.

**Corollary 3.5.** In the same hypothesis of Proposition 3.4, and if the transaction costs are fairly allocated (that is, \( \xi^+ = \xi^- = \xi \)), we have the Walrasian price \( P_\infty \) is given by

\[ P_\infty = \frac{K(\theta(o)) - K(\theta(n))}{\theta} \frac{1}{r + \xi(r + 2\lambda_d)}. \]
Moreover, the price $P_\infty$ is decreasing in $\xi$ and the following limits hold:

$$\lim_{\xi \to 0} P_\infty = \frac{K(\theta_0) - K(\theta_n)}{\bar{\theta}r}, \quad \text{and} \quad \lim_{\xi \to +\infty} P_\infty = 0.$$ 

The rise of transactions cost $\xi^-$ implies the reduction of equilibrium price. When vanishing search frictions, the price is given by (19), and we can observe that the price falls to 0 when the transaction costs increase; in particular, the assumption that $\xi^+$ is smaller than $q$ is not needed.

4 Transaction cost politics and numerical results

This section is devoted to the politics of transaction costs, and explains how the latter influence the equilibrium prices in OTC market.

Notwithstanding that the equilibrium price is given by (13), when the bargaining power $q$ is replaced by $q_1 = q_1(\xi^+, \xi^-) := \frac{q(1-\xi^-)}{(1-q)(1+\xi^+)+q(1-\xi^-)}$, the terms $A(\xi^+, \xi^-)$, $B(\xi^+, \xi^-)$ and $D(\xi^+, \xi^-)$, defined in (14), (15) and (16), respectively, become

$$A(\xi^+, \xi^-) = \frac{r+\lambda_u+\lambda_d+2\lambda_{\mu_h}q_1+2\lambda_{\mu_o}(1-q_1)}{D_1(\xi^+, \xi^-)},$$
$$B(\xi^+, \xi^-) = \frac{(1-q_1)r+\lambda_d+2\lambda_{\mu_o}(1-q_1)}{D_1(\xi^+, \xi^-)},$$

where

$$D_1(\xi^+, \xi^-) = \frac{D(\xi^+, \xi^-)}{(1-q)(1+\xi^+)+q(1-\xi^-)}.$$

Now, we analyze three particular cases: First of all, we study the behaviour of the equilibrium price when the buyer and the seller pay the same transaction costs (that is $\xi^+ = \xi^- = \xi$). In this case, the parameters $A(\xi^+, \xi^-)$ and $B(\xi^+, \xi^-)$ become

$$A(\xi, \xi) = \frac{r+\lambda_u+\lambda_d+2\lambda_{\mu_h}q_1+2\lambda_{\mu_o}(1-q_1)}{D_1(\xi, \xi)},$$
$$B(\xi, \xi) = \frac{(1-q_1)r+\lambda_d+2\lambda_{\mu_o}(1-q_1)}{D_1(\xi, \xi)},$$

where $q_1 = q_1(\xi, \xi) = \frac{q(1-\xi)}{(1-q)(1+\xi)+q(1-\xi)}$. We can observe that the equilibrium price decreases when the parameter $\xi$ increases but it has to be smaller than 1, since Assumption 3.1 holds. Moreover, since $(1-q)(1+\xi)+q(1-\xi) > 0$, it follows that

$$q_1 > q \iff 1-\xi > (1-q)(1+\xi)+q(1-\xi) \implies \xi < 0,$$
which contradicts Assumption 3.1. Consequently, the introduction of transaction costs leads to a distortion of bargaining power for involved agents type in the buying and selling; the presence of transaction costs raise the bargaining power of the buyer, and reduce the bargaining power of the seller.

Now, we analytically study the behaviour of the equilibrium price, when transaction costs are paid only by the buyer or seller. Assuming that only the seller pays transactions costs (that is, $\xi^- = 0$ and $\xi^+ = 2\xi > 0$), we have that

$$A(2\xi, 0) = \frac{r + \lambda_u + 2\lambda_h q_1 + 2\lambda_l (1 - q_1)}{D_1(2\xi, 0)}$$

$$B(2\xi, 0) = \frac{(1 - q_1) r + \lambda_d + 2\lambda_l (1 - q_1)}{D_1(2\xi, 0)}$$

where $q_1 = q_1(2\xi, 0) = \frac{q(1 - 2\xi)}{(1 - q) + q(1 - 2\xi)}$. The equilibrium price, given by (13), decreases, as $2\xi$ tends to $1^-$. When the transaction costs are paid only by the buyer (that is, $\xi^+ = 0$ and $\xi^- = 2\xi > 0$), we have that

$$A(0, 2\xi) = \frac{r + \lambda_u + \lambda_d + 2\lambda_h q_1' + 2\lambda_l (1 - q_1)}{D_1(0, 2\xi)}$$

$$B(0, 2\xi) = \frac{(1 - q_1) r + \lambda_d + 2\lambda_l (1 - q_1)}{D_1(0, 2\xi)}$$

where $q_1 = q_1(0, 2\xi) = \frac{q}{(1 - q)(1 + 2\xi) + q}$. In this case, the equilibrium price arrives to 0, when $\xi$ tends to $+\infty$.

Even in the last two cases analyzed, we can show that the incorporation of transaction costs leads to a distortion of bargaining power of involved agents type in the buying and selling; the presence of transaction costs raise the bargaining power of the buyer, and reduce the bargaining power of the seller. Through simulations, we measure the impact of transaction costs on the equilibrium price. We adopt the same input parameters used by Duffie et al. [5], that we recall hereunder in the following tables:

<table>
<thead>
<tr>
<th>$\mu_{ho}$</th>
<th>$\mu_{lo}$</th>
<th>$\mu_{hn}$</th>
<th>$\mu_{ln}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7972</td>
<td>0.0028</td>
<td>0.1118</td>
<td>0.0882</td>
</tr>
</tbody>
</table>

Table 1: Steady-state masses

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\lambda_u$</th>
<th>$\lambda_d$</th>
<th>$r$</th>
<th>$\beta$</th>
<th>$q$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>625</td>
<td>5</td>
<td>0.5</td>
<td>0.05</td>
<td>0.05</td>
<td>0.5</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Table 2: Base-case parameters for Duffie, Gârleanu and Pedersen model
<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\rho_h$</th>
<th>$\rho_l$</th>
<th>$m_\eta$</th>
<th>$\sigma_\eta$</th>
<th>$m_D$</th>
<th>$\sigma_D$</th>
<th>$\theta_{(o)}$</th>
<th>$\theta_{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>-0.5</td>
<td>0.5</td>
<td>10000</td>
<td>10000</td>
<td>1</td>
<td>0.5</td>
<td>20000</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: Risk-aversion case parameters for Duffie, Gârleanu and Pedersen model

The choice of the rates of change of the fractions of the respective investors types as in Table 1 implies that 79.72% of all investors are high-type investors owning an asset, 11.18% are of high-type but do not own an asset, and are thus potential buyers. Only 0.28% of investors are willing to sell their asset in this steady state equilibrium. The class of buyers is more numerous than the class of sellers, so all $lo$-type investors are able to sell their asset to a market maker, when they meet it. On the other hand, not all of the $hn$-type investors can buy an asset when they meet a market maker. Consequently, buyers are rationed.

Given that the switching intensities $\lambda_u$ and $\lambda_d$ as in Table 2, we have that an investor is a high-type investor 90.91% of the time. Moreover, the investor stays a high-type for $\frac{1}{\lambda_d} = 2$ years on average. A low-type investor is of low-type 9.09% of the time and stays low for $\frac{1}{\lambda_2} = 0.2$ years on average. Setting the search intensity $\lambda = 625$, we have that each investor locates other investors every other day on average; in particular, it is reasonable to expect every investor to interact with five investors every day, since $\frac{2\lambda}{250} = 5$.

Analyzing the behaviour of the equilibrium price, we can observe that the equilibrium price and Walrasian price decay to 0, when the transaction costs grow (see Figure 1). In particular, Walrasian price is higher than equilibrium price.
Figure 1: Comparison between paths of equilibrium price with respect to trans-
action costs.

In Figure 2, we graph the trends of equilibrium prices with respect to search
intensity $\lambda$, when transaction costs are assumed $\xi^\pm = 0.1\%$. We can observe
that equilibrium price is a growing function with respect to search intensity $\lambda$.
Moreover, the equilibrium price is highest if there aren’t transaction costs with
respect to our case. If the transaction costs are paid only by the buyer or seller,
the price is lower or higher than the price in our case; A seller who takes over
the transaction costs is more inclined to sell his asset at a higher price. Con-
versely, if transaction costs are paid only by the buyer, they are lower.

Figure 2: Comparison among the equilibrium prices with respect to search
intensity $\lambda$.

In the next figures, we analyze the behaviour of the equilibrium prices with
respect to bargaining power $q$ (see Figure 3), with respect to the number of
the lo-type investors (see Figure 4), and with respect to the interest rate $r$
(see Figure 5). We can observe that the prices increase with respect to $q$ and
decrease with respect to $\mu_{lo}$ and $r$, as it is natural to expect. Also in these cases, the impact of transaction costs on the equilibrium price is the same. The equilibrium price is higher when transaction costs are not paid. If only one of the parties pays transaction costs, the previous considerations still hold; it is higher (or lower), if transaction costs are paid by seller (or holder).

Figure 3: Comparison among the equilibrium prices with respect to bargaining power $q$.

Figure 4: Comparison among the equilibrium prices with respect to $\mu_{lo}$.

Figure 5: Comparison among the equilibrium prices with respect to $r$. 

5 Conclusions

From our research, it is clear that the presence of transaction costs lowers both the equilibrium price and the Walrasian price. An increase in transaction costs leads the agents to bargain a lower equilibrium price. In particular, we have noticed that if the transaction costs are paid only by the seller (or by the buyer), the latter will tend to negotiate a higher (or lower) equilibrium price to sell (or to buy) the asset with respect to assume that the transaction costs are paid only by the buyer (or by the seller) or else fairly allocated. When vanishing transaction costs, we find the equilibrium price formulas and the Walrasian price in Duffie et al. [5]. The presence of transaction costs raise the bargaining power of the buyer, and reduce the bargaining power of the seller.

The empirical analyses show that:

- Increasing the matching intensity $\lambda$ results in an increase in the equilibrium price,

- an increase in the $r$ interest rate or the number of $lo$-agents, leads to a reduction in the equilibrium price.

6 Appendix: Mathematical proofs

In this appendix, we report the proofs of the mathematical results.

Proof of Theorem 3.2.

Similarly to what is contained in Duffie et al. [5], we have:

1. If the function $V$, defined in (4), satisfies the following HJB equation (6), the first order necessary conditions allows us to get the optimal controls, that is

$$\gamma e^{-\gamma c} - \frac{\partial}{\partial x} V = 0, \quad \Rightarrow \quad c^* = -\frac{1}{\gamma} \ln \left( \frac{1}{\gamma} \frac{\partial}{\partial x} V \right). \quad (20)$$

Putting (20) in (6), we obtain the HJB equation as follows:

$$0 = -\beta V(x; \sigma) - e^{-\gamma c^*} + [r x - c^* + \theta m_D + m_n] \frac{\partial}{\partial x} V(x; \sigma)
+ \frac{1}{2} \left[ \theta^2 \sigma_D^2 + \sigma_n^2 + 2 \rho \theta \sigma_D \sigma_n \right] \frac{\partial^2}{\partial x^2} V(x; \sigma)
+ \sum_{\{\sigma', \sigma' \neq \sigma\}} \lambda(\sigma, \sigma') \left( V(x + (z(\sigma, \sigma') - \xi^\pm)P \theta; \sigma') - V(x; \sigma) \right). \quad (21)$$


Following Duffie et al. [5], let us “try” a solution to this differential equation of the form (7). Substituting the following partial derivatives in (21)

\[ \frac{\partial}{\partial x} V(x; \sigma) = r \gamma e^{-\gamma(x+a_\sigma+\bar{a})}, \]
\[ \frac{\partial^2}{\partial x^2} V(x; \sigma) = -(r \gamma)^2 e^{-\gamma(x+a_\sigma+\bar{a})}, \]

it follows that the consumption rate is given by (11) and, setting \( \bar{a} \) as in (8), HJM equation is satisfied if \( a_\sigma \) solves the following equation

\[
0 = \gamma r a_\sigma + \sum_{(\sigma', \sigma') \neq (\sigma, \sigma)} \lambda(\sigma, \sigma') \frac{(V(x, (z(\sigma), \xi) + a_\sigma, \bar{a}) - V(x, r, \sigma)) - V(x, r, \sigma)}{V(x, r, \sigma)} - \gamma r \theta m_D + \frac{1}{2} (\gamma r)^2 (\theta^2 \sigma_D^2 + 2 \rho \theta \sigma_D \sigma_D) ;
\]

consequently, we obtain the system (9).

2. Solving the following equation

\[
q \left( e^{-r \gamma (-P \bar{\theta} (1 + \xi^-) + a_{ho} - a_{hn})} - \frac{1}{r \gamma} \right) \left[ \frac{\partial}{\partial P} \left( e^{-r \gamma (-P \bar{\theta} (1 - \xi^+) + a_{lo} - a_{ln})} - \frac{1}{r \gamma} \right) \right] = (1 - q) \left( e^{-r \gamma (-P \bar{\theta} (1 - \xi^+) + a_{lo} - a_{ln})} - \frac{1}{r \gamma} \right) \left[ \frac{\partial}{\partial P} \left( e^{-r \gamma (-P \bar{\theta} (1 + \xi^-) + a_{ho} - a_{hn})} - \frac{1}{r \gamma} \right) \right],
\]

we obtain (12)

Proof of Proposition 3.3.

Applying the first order approximation to Nash bargaining equation (12), we obtain that the equilibrium price \( P \) is approximated by

\[
P \approx \frac{1 - q}{\bar{\theta} (1 - \xi^+)} a_l + \frac{q}{\bar{\theta} (1 + \xi^-)} a_h ,
\]

where \( a_l = a_{lo} - a_{ln} \) and \( a_h = a_{ho} - a_{hn} \).

By the nonlinear system (9), we obtain the following linearized system of equations:

\[
\begin{aligned}
0 &= r a_{lo} - \lambda_u(a_{ho} - a_{lo}) - 2 \lambda \mu_h (P \bar{\theta} (1 - \xi^+) + a_{ln} - a_{lo}) - (K(\theta_{(o)}) - \theta_{(o)}), \\
0 &= r a_{ln} - \lambda_u(a_{ho} - a_{ln}) - (K(\theta_{(n)}) - \theta_{(n)}), \\
0 &= r a_{ho} - \lambda_d(a_{lo} - a_{ho}) - K(\theta_{(o)}), \\
0 &= r a_{hn} - \lambda_d(a_{ln} - a_{ho}) - 2 \lambda \mu_o (-P \bar{\theta} (1 + \xi^-) + a_{ho} - a_{hn}) - K(\theta_{(n)}).
\end{aligned}
\]
Considering (22), subtracting the second and the fourth equation from first and the third equation in (23), respectively, and rearranging the terms in \( a_l \) and \( a_h \), we obtain the following linear system in matrix form:

\[
\begin{pmatrix}
A & -B & 0 \\
-C & D & 0 \\
\frac{1-q}{1-\xi^+} & \frac{q}{1+\xi^-} & -1
\end{pmatrix}
\begin{pmatrix}
a_l \\
a_h \\
P\bar{\theta}
\end{pmatrix}
= \begin{pmatrix}
K(\theta(o)) - K(\theta(n)) - \bar{\delta} \bar{\theta} \\
K(\theta(o)) - K(\theta(n)) \\
0
\end{pmatrix},
\]

where

\[
A = r + \lambda_u + 2 \lambda \mu_{hn} q, \quad B = \lambda_u + 2 \lambda \mu_{hn} q \frac{1-\xi^+}{1+\xi^-},
\]

\[
C = \lambda_d + 2 \lambda \mu_{lo} (1-q) \frac{1+\xi^-}{1-\xi^+}, \quad D = r + \lambda_d + 2 \lambda \mu_{lo} (1-q).
\]

The linear system above admits a unique solution; the third component of the solution is given by

\[
P\bar{\theta} = \frac{\det \begin{vmatrix}
A & -B & K(\theta(o)) - K(\theta(n)) - \bar{\delta} \bar{\theta} \\
-C & D & K(\theta(o)) - K(\theta(n)) \\
\frac{1-q}{1-\xi^+} & \frac{q}{1+\xi^-} & 0
\end{vmatrix}}{\det \begin{vmatrix}
A & -B & 0 \\
-C & D & 0 \\
\frac{1-q}{1-\xi^+} & \frac{q}{1+\xi^-} & -1
\end{vmatrix}}
= \left[ K(\theta(o)) - K(\theta(n)) \right] \frac{(B+D) \frac{1-q}{1-\xi^+} + (C+A) \frac{q}{1+\xi^-}}{AD-BC} - \bar{\delta} \bar{\theta} \frac{C \frac{q}{1+\xi^-} + D \frac{1-q}{1-\xi^+}}{AD-BC}.
\]

Substituting (25) in above solution, we obtain the statement. \(\square\)

**Proof of Proposition 3.4.**

To prove the result, we need to show that if \( s := \mu_{ho} + \mu_{lo} < \frac{\lambda_u}{\lambda_u + \lambda_d} = y \), then the following limits hold:

\[
\lim_{\lambda \to +\infty} \lambda \mu_{lo} = \frac{\lambda_u s}{2(y-s)}, \quad \text{and} \quad \lim_{\lambda \to +\infty} \lambda \mu_{hn} = +\infty.
\]

From the relation (1), it follows that \( \mu_{hn} = y - s + \mu_{lo} \); consequently, it is enough to prove only the first limit.

From the equations (1) and (2), we obtain that \( \mu_{lo} \) solves the following quadratic equation

\[
2 \lambda \mu_{lo}^2 - [2\lambda(y-s) + \lambda_d \lambda_u] \mu_{lo} - \lambda_d s = 0.
\]
Since (26) admits a unique positive solution, using the linear approximation, we have that
\[
\lambda \mu_{t_0} = \frac{-[2\lambda(y-s)+\lambda_d+\lambda_u]+|2\lambda(y-s)+\lambda_d+\lambda_u|}{4}\sqrt{1+\frac{8\lambda_\mu}{|[2\lambda(y-s)+\lambda_d+\lambda_u]|^2}}
\]
\[
\approx \frac{-[2\lambda(y-s)+\lambda_d+\lambda_u]+|2\lambda(y-s)+\lambda_d+\lambda_u|+\frac{1}{2} \frac{8\lambda_\mu}{|2\lambda(y-s)+\lambda_d+\lambda_u|}}{4} \xrightarrow{\lambda \to +\infty} \frac{\lambda_d s}{2(y-s)}.
\]

To end the proof, it is enough to observe that, when \(\lambda \to +\infty\), the terms
\[
\frac{(r+\lambda_u+\lambda_d)\left(\frac{1}{1-\xi^+} + \frac{q}{1+\xi^-}\right)+2\lambda \mu_{t_0} - \frac{1}{1-\xi^-}}{r(r+\lambda_u+\lambda_d+2\lambda \mu_{t_0})^2[1-q] + (\xi^+ + \xi^-)2\lambda \mu_{t_0}^2(1-\xi^-\lambda_u)}
\]
and
\[
\frac{1-q}{1-\xi^+} r+\lambda_d\left(\frac{1}{1-\xi^+} + \frac{q}{1+\xi^-}\right)+2\lambda \mu_{t_0} - \frac{1}{1-\xi^-}
\]
\[
\xrightarrow{\xi \to +\xi^-} \frac{q}{1+\xi^-} - \lambda_d
\]
tend to \(\frac{q}{1+\xi^-} - \lambda_d\) and 0, respectively.

**Proof of Corollary 3.5.**
The first part of the proof is straightforward consequence of Proposition 3.3. Since
\[
\frac{\partial}{\partial \xi} P_\infty = -\frac{K(\theta_{(0)}) - K(\theta_{(n)})}{\theta} \frac{r + 2\lambda_d}{(r + \xi(r + 2\lambda_d))^2} < 0,
\]
the price is decreasing with respect to \(\xi\). The calculation of limits is easy.

**Acknowledgements.** We would like to express our very great appreciation to Giovanna Nappo, Associate Professor of Sapienza University of Rome, for her valuable and constructive suggestions and for her comments that greatly improved the manuscript. Her willingness to give her time so generously has been very much appreciated.

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Federico Flore and Marisa Cenci


Received: July 25, 2018; Published: September 18, 2018