Non Isolated Periodic Orbits of a Fixed Period for Quadratic Dynamical Systems

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Abstract
Given a dynamical system in $\mathbb{R}^3$ and $\gamma_\tau$ a periodic orbit of period $\tau > 0$, non isolated among periodic orbits of period $\tau$ (for any neighborhood $V \subset \mathbb{R}^3$ of $\gamma_\tau$, there exist another periodic orbit $\gamma'_\tau \subset V$ of period $\tau$), a way of obtaining non isolated periodic orbits of a fixed period $\tau$ for polynomial dynamical systems is emphasized. As an application, we build up a counter example to answer a referee’s question.

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1 Introduction

For any quadratic polynomial system

$$\dot{P} = T(P) + Q(P)$$

(1)

where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping and $Q : \mathbb{R}^n \to \mathbb{R}^n$ a homogeneous mapping of degree 2 i.e $Q(\alpha P) = \alpha^2 Q(P)$ for all $\alpha \in \mathbb{R}$ and $P \in \mathbb{R}^n$, we can
associate a structure of non-associative algebra (non-associative means not necessarily associative) given by the multiplication table:

$$\beta(P_1, P_2) = \frac{1}{2} \left[ Q(P_1 + P_2) - Q(P_1) - Q(P_2) \right].$$

Since Markus article [4], many results on polynomial dynamical systems have been obtained from the study of the corresponding algebra (see for instance [1] and [3]).

In [3], Kinyon and Sagle considered derivations and automorphisms of quadratic systems which are derivations and automorphisms of the corresponding non-associative algebra that are commuting with $T$ and stated the following theorem:

**Theorem 1.1.** Given a quadratic polynomial system in $\mathbb{R}^n$, we let $\mathcal{P}_\tau$ be the set of periodic orbits which are of period $\tau > 0$. Let $\gamma_\tau \in \mathcal{P}_\tau$ and $P$ be an element of $\gamma_\tau$, if $\mathcal{P}_\tau$ consists of isolated orbits and if there exists a derivation $D$ of the quadratic system for which $DP \neq 0$ then there is a nonzero $a \in \mathbb{R}$ such that $\gamma_\tau(t) = (\exp tG)P$, for any $t \in \mathbb{R}$ where $G = a^{-1}D$.

They also gave a remark on page 99 raised by the referee who asked whether it is possible to give a counterexample when $\mathcal{P}_\tau$ does not consist of isolated orbits.

The aim of this work is to give a family of counterexamples built from quadratic differential systems in $\mathbb{R}^2$ all having 0 as a center. As it will be seen from the following paragraph, two questions need to be answered. The first is how we can build up examples with non isolated periodic orbits of fixed period $\tau > 0$ and the second is trying to obtain nontrivial derivations leading to the desired conclusion.

To answer the first point, we will transform a given quadratic system in $\mathbb{R}^2$ having the origin as a center into a homogeneous quadratic one in $\mathbb{R}^3$ using the homogenization process (see [3]). This homogenization leads to cones, each cone consisting in periodic orbits and we will come up with non isolated periodic orbits of fixed period $\tau > 0$ in $\mathbb{R}^3$. For the second point, it turns out that in dimension 2, most of the examples having the origin as a center, if not all, correspond to non-associative algebras with trivial derivations only. Unfortunately, if we homogenize once, the derivations space is still trivial but a second homogenization will lead to the desired conclusion.

Since the aim of this work is to give one counterexample, we will study one case for which 0 is a center but the same process will give analogous results for all other cases and this will be mentioned in the third paragraph.
It is also worth mentioning that the homogenization process is an efficient way of obtaining examples of quadratic systems with non isolated periodic orbit of fixed period \( \tau \) in dimension \( n+1 \) from quadratic systems not necessarily admitting non isolated periodic orbits of fixed period \( \tau > 0 \) in dimension \( n \).

## 2 The Counterexample

Consider in \( \mathbb{R}^2 \) the system of differential equations

\[
\begin{cases}
\dot{x} = -y - xy \\
\dot{y} = x + xy
\end{cases}
\]  

(2)

and let \( \mathcal{A} \) be the corresponding non-associative algebra.

It is well known that this system has the first integral

\[
H(x, y) = x - \ln(x + 1) + y - \ln(y + 1)
\]

and that the stationary point \( (0, 0) \) is a center (see [5]). We have the following results

**Theorem 2.1.** Consider the system (2), then we have

1. The derivation space of the algebra \( \mathcal{A} \) is trivial, \( \text{Der} \mathcal{A} = \{0\} \).
2. The homogenized differential system in \( \mathbb{R}^3 \) writes

\[
\begin{cases}
\dot{x} = -u_1 y - xy \\
\dot{y} = u_1 x + xy \\
u_1 = 0
\end{cases}
\]  

(3)

and the derivation space of its corresponding algebra is \( \{0\} \).

3. The homogenized differential system obtained from (3) writes

\[
\begin{cases}
\dot{x} = -u_1 y - xy \\
\dot{y} = u_1 x + xy \\
u_1 = 0 \\
u_2 = 0
\end{cases}
\]  

(4)

and if \( \mathcal{A}_1 \) is its corresponding algebra then its derivation space is given by

\[
\text{Der} \mathcal{A}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \end{pmatrix}, a, b \in \mathbb{R} \right\}.
\]
Proof. Straightforward computations lead to the results. □

In order to obtain examples of quadratic systems with non isolated periodic orbits of fixed period $\tau > 0$, we state the following result:

**Theorem 2.2.** Let $X$ be the quadratic differential equation in $\mathbb{R}^2$ given by (2) and $Y$ be the homogeneous quadratic system in $\mathbb{R}^3$ obtained from $X$ using the homogenization process. Let $O_\tau$ be a periodic orbit for $X$ of period $\tau$ surrounding the origin and $P_\tau \subset \mathbb{R}^3$ be the set of periodic orbits for $Y$ of period $\tau$ then any element of $P_\tau$ is not isolated in $P_\tau$.

Proof. If we let $(x, y)$ be the coordinates in $\mathbb{R}^2$ and $(x, y, u_1)$ be the coordinates in $\mathbb{R}^3$, then if we restrict $Y$ to the plane $u_1 = 1$, we obtain the vector field $X$. Thus if $X$ has a periodic orbit $O_\tau$ of period $\tau > 0$ then $Y$ has also a periodic orbit $\gamma_\tau$ of period $\tau$ included in the plane $u_1 = 1$. On the other hand, for homogeneous quadratic vector fields on $\mathbb{R}^n$, if $\Phi((\cdot, P)$ is the solution through $P \in \mathbb{R}^n$, then we have

$$\Phi(t, aP) = a\Phi(at, P)$$

for any scalar $a$ when the right hand side is defined.

This justifies that the set $a\gamma_\tau = \{aP, P \in \gamma_\tau\}$ is also a periodic trajectory for $Y$ with period $\frac{\tau}{|a|}$ for $a \neq 0$. In fact, we have a cone of periodic orbits for (4).

Let $O_\tau$ and $\gamma_\tau$ as above, since 0 is a center for $X$, arbitrarily close to $O_\tau$ we can find another periodic orbit $O_{\tau'}$ with $\tau'$ close to $\tau$. This second orbit $O_{\tau'}$ generates a cone of periodic orbits for $Y$. Associated to $O_{\tau'}$, there exists a periodic orbit $\gamma_{\tau'}$ for $Y$ included in the plane $u_1 = 1$ as $\gamma_\tau$ is and with period $\tau'$. If $O_{\tau'}$ is close to $O_\tau$, $\gamma_{\tau'}$ will be close to $\gamma_\tau$. In the cone including $\gamma_{\tau'}$, all orbits for $Y$ are periodic and periods are of the form $\frac{\tau'}{|a|}$ with $a \in \mathbb{R}^*$. For some convenient $a$ close to 1, $\frac{\tau'}{|a|}$ will equal $\tau$ and there exists a periodic orbit close to $\gamma_{\tau'}$ of period $\tau$. As this periodic orbit is arbitrarily close to $\gamma_{\tau}$, $\gamma_\tau$ is not isolated in $P_\tau$. □

We are now able to give the counterexample.

**Theorem 2.3.** Consider the system (4) and let $A_1$ be its corresponding algebra, we have the following results

1. $\text{Der} A_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{pmatrix}, \lambda, \mu \in \mathbb{R} \right\}$. 


2. For any $\tau > 0$, $P_\tau$ is nonempty and any element $\gamma_\tau$ in $P_\tau$ is non isolated in $P_\tau$.

3. For any $P = (x_0, y_0, z_0, t_0) \in \gamma_\tau$, any nontrivial derivation $D \in A_1$ and any nonzero $a \in \mathbb{R}$, the set $\{\exp(ta^{-1}D)P, t \in \mathbb{R}\}$ is not a periodic orbit.

Proof. The first point has been stated in theorem 2 and the second in theorem 3.

Let $P = (x_0, y_0, z_0, t_0)$ be our initial condition. Without loss of generality, we may suppose $z_0 = 1$ and $t_0 = 1$. Since we have performed the homogenization process twice, this orbit is included in the plane defined by $u_1 = u_2 = 1$. On the other hand, for any derivation $D$ and any nonzero real number $a$, the first two rows of the matrix corresponding to $\exp(ta^{-1}D)$ are

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

Thus, for the initial condition $X = (x_0, y_0, 1, 1)$, the first two components of the vector $\exp(ta^{-1}D)P$ are constant. Therefore, the set $\{e^{(ta^{-1}D)}P, t \in \mathbb{R}\}$ can not be a periodic orbit. $\square$

3 Final Remarks

It is well known that the system

\[
\begin{align*}
\dot{x} &= ax - bxy \\
\dot{y} &= -cy + dxy,
\end{align*}
\]

with $a, b, c, d$ all positive real numbers, has the point $(\frac{c}{d}, \frac{a}{b})$ as a center (see [2]).

If we translate $(\frac{c}{d}, \frac{a}{b})$ to $(0, 0)$, we obtain the new quadratic system

\[
\begin{align*}
\dot{x} &= -\frac{bc}{d}y - bxy \\
\dot{y} &= \frac{ad}{b}x + dxy.
\end{align*}
\]

It is possible to generalize to this new system what has been done in paragraph 2.

More generally, we can apply the process described in paragraph 2 to any quadratic system in $\mathbb{R}^2$ having the origin as a center, we recall all possible cases (see [5]).
Theorem 3.1. The equation
\[
\frac{dy}{dx} = -\frac{x + ax^2 + (2b + \alpha)xy + cy^2}{y + bx^2 + (2c + \beta)xy + dy^2}
\]
has a center if and only if one of the following conditions holds
1. \( \alpha = \beta = 0 \).
2. \( a + c = b + d = 0 \).
3. \( a = c = \beta = 0 \) (or \( b = d = \alpha = 0 \)).
4. \( a + c = \beta = \alpha + 5(b + d) = bd + 2d^2 + a^2 = 0 \), but \( b + d \neq 0 \) (or \( b + d = \alpha = \beta + 5(a + c) = ac + 2a^2 + d^2 = 0 \), but \( a + c \neq 0 \)).
5. \( \frac{\alpha}{\beta} = \frac{b+d}{a+c} = k, ak^3 - (3b + \alpha)k^2 + (3c + \beta)k - d = 0 \).

References


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