On Modules in Which Every Weak Large Closed Submodule is a Direct Summand

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Abstract
The concept of weak large extensions for modules is introduced here as a general case of the notion of large extensions. Some properties of large extensions hold true for weak large extensions, while some others need special types of modules (e.g. nonsingular modules), and special types of rings. Weak large extending modules (modules in which every WL-closed submodule is a direct summand) are also introduced here. We show that, as of the case of extending modules, the second singular submodule of a WL-extending module splits.

Keywords: Injective modules, Large submodules, Extending modules

1 Introduction
Throughout all modules are unital right $R$-modules over an arbitrary associative ring $R$ with unity. We use $L \leq M$ and $N \leq^\oplus M$ to denote that $L$ is a submodule of $M$ and $N$ is a direct summand of $M$, respectively.
The concept of large (or essential) submodules of an $R$-module $M$ plays an important role in module theory. A submodule $N$ of an $R$-module $M$ is a large submodule, or $M$ is a large extension of $N$ (denoted by $N \leq^L M$), if $N \cap K \neq 0$, for every nonzero submodule $K$ of $M$. For $m \in M$, the right ideal $\{ r \in R : mr \in N \}$ of $R$ will be denoted by $m^{-1}N$. The right ideal $\{ r \in R : mr = 0 \}$ of $R$ is denoted by $\text{ann}_R(m)$. The submodule $\{ m \in M : mI = 0, \text{ for some } I \leq^L R \}$ of $M$ is denoted by $Z(M)$, and $Z_2(M)$ is the submodule of $M$ containing $Z(M)$, which satisfies $Z_2(M)/Z(M) = Z(M/Z(M))$. It is clear that $N \leq^L M$ if and only if $m(m^{-1}N) \neq 0$, for each $m \in M\setminus N$. $A$ is a large closed (or simply closed) submodule of $M$ if it has no proper large extensions in $M$. A submodule $A$ is called a complement of a submodule $B$ in $M$, if $A$ is a maximal submodule of $M$ with the property that $A \cap B = 0$. A submodule $C$ is a complement submodule in $M$, if $C$ is a complement of some submodule $B$ of $M$. It is known that complement submodules, and large closed submodules, are in any module $M$, coincide.

A module $M$ is large extending (or simply extending) (or with the condition $(C_1)$) if every submodule is large in a direct summand, of $M$ (or equivalently, for any submodule $A$ of $M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$, and $A \oplus M_2 \leq^L M$. One of the generalizations of the defining condition of a large extending module $M$ is that, for each submodule $A$ of $M$, there exists a decomposition $M = M_1 \oplus M_2$ such that $A \cap M_2 = 0$, and $A \oplus M_2 \leq^L M$. Such modules are called modules with the condition $(C_1^*)$, which are introduced and studied by M. Kamal, and A. Sayed [10]. Many authors have studied large extending modules, and some of their generalizations (c. f. M. Kamal and B. Mueller [7], [8], [9], M. Kamal [5], M. Kamal and O. Elmnophy [6], and E. Akalan, G. Birkenmeier and A. Tercn [1]). The condition $(C_1)$ is one of the defining conditions of continuous (quasi-continuous) modules, which in turn are generalizations of injective (quasi-injective) modules. Many authors have studied them extensively (c. f. Y. Utumi [12], L. Jeremy [4], B. Mueller and S. Rizvi [11], others).

Here we introduce the concept of weak large extensions, and give some results in analogy with large extensions. Accordingly, we introduce the concept of weak large closed submodules (denoted by $WL$-closed). The weak large closed submodules are used to introduce the concept of weak large extending modules (or $WL$-extending modules), modules in which every $WL$-closed submodule is a direct summand. The concept of $WL$-extending modules is a generalization of the concept of extending modules. In M. Kamal and B. Mueller [8], they have shown that a not singular module $M$ is large extending if and only if $M = N \oplus Z_2(M)$, where $N$ and $Z_2(M)$ are large extending, and $Z_2(M)$ is $N$-injective. Here we show (Theorem 2) that a not singular module $M$ is weak large extending if and only if $Z(M) = Z_2(M)$, and $M = N \oplus Z(M)$, where $N$ is nonsingular large extending, and $Z(M)$ is ($WL$-uniform) weak large
extending. M. Kamal and B. Mueller [9], also have shown that a module over a right noetherian ring is extending if and only if it has \((1 - C_1)\) (modules with every uniform large closed submodule is a direct summand) and every local direct summand is a direct summand, and consequently a direct sum of uniform modules. Here we show (Theorem 4.18) that a module \(M\) over a right noetherian ring is weak large extending if and only if 

\[
\text{M is a direct sum of WL-uniform submodules, where the nonsingular part } N \text{ of } M \text{ has } (1 - C_1), \text{ and with every local direct summand is a direct summand of } N.
\]

## 2 Weak Large Submodules of a Module

**Definition 2.1** Let \(N\) be a nonzero submodule of an \(R\)-module \(M\). \(N\) is called a weak large submodule of \(M\) (denoted by \(N \leq WL \ M\)) if the right ideal 

\[
m^{-1}N = \{r \in R : mr \in N\}
\]

is large in \(R\), for each \(m \in M \setminus N\).

**Remark 2.2** 1- It is clear that large submodules of a module \(M\) are weak large submodules of \(M\), while there are weak large submodules which are not large, for example \(M = C(2) \oplus C(3)\) as a \(\mathbb{Z}\)-module, and \(N = C(2)\), we have \(N\) is a weak large submodule of \(M\); while \(N\) is not large in \(M\), where \(C(p^n) = \{a/p^k : a \in \mathbb{Z}, k = 1, 2, ..., n\}/\mathbb{Z}\) is the known cyclic \(\mathbb{Z}\)-module, and \(p\) is prime.

2- If an \(R\)-module \(M\) is nonsingular, then it is clear that every weak large submodule is a large submodule, of \(M\).

**Lemma 2.3** The following are equivalent for a nonzero submodule \(N\) of an \(R\)-module \(M\):

1) \(N\) a weak large submodule of \(M\);

2) For each \(m \in M \setminus N\), each nonzero element \(r \in R\), there exists \(s \in R\) such that \(0 \neq rs \in N\).

**Proof.** It is clear.

**Lemma 2.4** Let \(f : M_1 \rightarrow M_2\) be an \(R\)-module homomorphism. Then the inverse image of a weak large submodule of \(M_2\) is a weak large submodule of \(M_1\).

**Proof.** Let \(K\) be a weak large submodule of \(M_2\) and \(m \in M_1 \setminus f^{-1}(K)\), hence \(f(m) \in M_2 \setminus K\). It is clear to show that \(m^{-1}f^{-1}(K) \geq (f(m))^{-1}K\), and thus, as \((f(m))^{-1}K \leq L R\), \(m^{-1}f^{-1}(K) \leq L R\). Therefore \(f^{-1}(K) \leq WL \ M_1\).

**Corollary 2.5** Let \(f : M_1 \rightarrow M_2\) be an \(R\)-module epimorphism, and \(K\) be a nonzero submodule of \(M_1\). If \(K\) contains \(\ker f\), and has no proper weak large extensions submodules of \(M_1\), then \(f(K)\) has no proper weak large extensions submodules of \(M_2\).
**Proof.** Let \( f(K) \leqWL L \leq M_2 \), hence, by Lemma 2.4, \( K = f^{-1}(f(K)) \leqWL f^{-1}(L) \leq M_1 \), and hence \( f(K) = f f^{-1}(L) = L \).

**Corollary 2.6** Let \( f : M_1 \longrightarrow M_2 \) be an \( R \)-module isomorphism, and \( K \) be a nonzero submodule of \( M_1 \). If \( K \) has no proper weak large extensions submodules of \( M_1 \), then \( f(K) \) has no proper weak large extensions submodules of \( M_2 \).

**Lemma 2.7** Let \( M \) be an \( R \)-module, and \( N \) be a nonzero submodule of \( M \). Then \( N \leqWL M \) if and only if \( N \leqWL K \) and \( K \leqWL M \), for every submodule \( K \) of \( M \) containing \( N \).

**Proof.** Let \( N \leqWL M \), and \( N \leq K \leq M \). As \( m^{-1}N \leqL R \) for \( m \in M \setminus N \), and since \( m^{-1}N \leq m^{-1}K \), we have that \( k^{-1}N \leqL R \), for every \( k \in K \setminus N \) and \( m^{-1}K \leqL R \), for every \( m \in M \setminus K \); i.e. \( (N \leqWL K \) and \( K \leqWL M \)). The converse is obvious.

The following Lemma shows that Weak large extensions for \( R \)-modules is transitive if the ring \( R \) has an extra condition on its elements.

**Lemma 2.8** Let \( N \), and \( K \) be nonzero submodules of an \( R \)-module \( M \), with \( N \leqWL K \leqWL M \). If \( \text{ann}_R(r^2) = \text{ann}_R(r) \), for each \( 0 \neq r \in R \), then \( N \leqWL M \).

**Proof.** Let \( m \in M \setminus N \), and \( 0 \neq r \in R \). If \( m \in K \), then, as \( N \leqWL K \), there exists \( a \in R \) such that \( 0 \neq ra \in m^{-1}N \). Now if \( m \notin K \), then, as \( K \leqWL M \), there exists \( s \in R \) such that \( 0 \neq rs \in m^{-1}K \). If \( mrs \in N \), then we are done, otherwise \( 0 \neq mrs \in K \), and \( rs \neq 0 \). As \( N \leqWL K \), there exists \( b \in R \) such that \( 0 \neq rsb \in (mrs)^{-1}N \). Then, by the given assumption on \( R \), \( 0 \neq r(srs)b \in m^{-1}N \). Therefore \( N \leqWL M \).

**Corollary 2.9** Let \( R \) has no right(or left) zero divisors. If \( N \leqWL K \leqWL M \), then \( N \leqWL M \).

**Lemma 2.10** Let \( N \), and \( K \) be nonzero submodules of an \( R \)-module \( M \), with \( N \leq K \leq M \). If \( N \leqL K \leqWL M \), then \( N \leqWL M \).

**Proof.** The proof is clear from the proof of Lemma 2.8.

**Lemma 2.11** Let \( A_i \leq B_i \) \( (i = 1, 2) \) be submodules of an \( R \)-module \( M \), with \( A_1 \cap A_2 \neq 0 \). If \( A_i \leqWL B_i \) \( (i = 1, 2) \), then \( A_1 \cap A_2 \leqWL B_1 \cap B_2 \).

**Proof.** Let \( b \in B_1 \cap B_2 \), as \( b^{-1}A_i \leqL R \) \( (i = 1, 2) \), we have that \( b^{-1}A_1 \cap b^{-1}A_2 \leqL R \). It is clear to show that \( b^{-1}A_1 \cap b^{-1}A_2 \subseteq b^{-1}(A_1 \cap A_2) \) and hence \( b^{-1}(A_1 \cap A_2) \leqL R \). Therefore \( A_1 \cap A_2 \leqWL B_1 \cap B_2 \).
Corollary 2.12 Let $A_i$ $(i = 1, 2)$ be submodules of an $R$-module $M$, with $A_1 \cap A_2 \neq 0$. Then $A_i \leq_{WL} M_i$ $(i = 1, 2)$ if and only if $A_1 \cap A_2 \leq_{WL} M$.

Proof. From Lemma 2.7, and Lemma 2.11.

Lemma 2.13 Let $M = M_1 \oplus M_2$ be an $R$-module, and $A_i$ be a nonzero submodule of $M_i$ $(i = 1, 2)$. Then $A_i \leq_{WL} M_i$ $(i = 1, 2)$ if and only if $A_1 \oplus A_2 \leq_{WL} M$.

Proof. Let $A_i \leq_{WL} M_i$ $(i = 1, 2)$, and $m \in M \setminus A_1 \oplus A_2$; hence $m = m_1 + m_2$, to avoid triviality, $m_1 \in M_1 \setminus A_1$, and $m_2 \in M_2 \setminus A$. As $A_1 \leq_{WL} M_1$, and $A_2 \leq_{WL} M_2$, we have that $m_1^{-1}A_1 \leq^L R$ $(i = 1, 2)$, and hence $m_1^{-1}A_1 \cap m_2^{-1}A_2 \leq^L R$. It is clear that $m_1^{-1}A_1 \cap m_2^{-1}A_2 \leq (m_1 + m_2)^{-1}(A_1 \oplus A_2) \leq R$, and then $m^{-1}(A_1 \oplus A_2) \leq^L R$. For the converse, let $A_1 \oplus A_2 \leq_{WL} M$, and let $m_1 \in M_1 \setminus A_1$, hence $m_1^{-1}(A_1 \oplus A_2) \leq^L R$. But $m_1^{-1}(A_1 \oplus A_2) = m_1^{-1}A_1$, therefore $A_1 \leq_{WL} M_1$. Similarly, for $A_2 \leq_{WL} M_2$.

Lemma 2.14 Let $M$ be an $R$-module, and $A$ be a proper submodule of $B$ in $M$. Then $B \leq_{WL} M$ if and only if $B/A \leq_{WL} M/A$.

Proof. It is clear, as $m^{-1}B = \overline{m}^{-1}B$, for each $m \in M/B$.

3 Weak Large Closed Submodules

Definition 3.1 A submodule $N$ of a module $M$ is called a weak large closed submodule in $M$ (denoted by $N \leq_{WLc} M$), if $N$ has no proper weak large extensions in $M$; i.e. if $N \leq_{WL} K \leq M$, then $N = K$.

Definition 3.2 Let $N$ be a nonzero submodule of an $R$-module $M$. Then the closure of $N$ in $M$ is

$$WL - C_M(N) = \begin{cases} m \in M : m^{-1}N \leq^L R, & N \neq 0, \\ 0, & N = 0. \end{cases}$$

(1)

Lemma 3.3 Let $N$ be a nonzero submodule of an $R$-module $M$. Then $WL - C_M(N)$ is a submodule of $M$, which is the unique maximal weak large extension of $N$ in $M$.

Proof. It is clear that $N$ is contained in $WL - C_M(N)$. We first show that $WL - C_M(N)$ is a submodule of $M$. Let $m_1, m_2 \in WL - C_M(N)$, hence $m_1^{-1}N \leq_{WL} R$, and $m_2^{-1}N \leq_{WL} R$. Since $m_1^{-1}N \cap m_2^{-1}N \leq^L R$, and is contained in $(m_1 - m_2)^{-1}N$, we have that $m_1 - m_2 \in WL - C_M(N)$. Now let $m \in WL - C_M(N)$ and $0 \neq r \in R$, hence, as $m^{-1}N \leq^L R$, we have that $m^{-1}(m^{-1}N) \leq^L R$. It is easy to show that $m^{-1}(m^{-1}N) \leq (mr)^{-1}N$, and thus $(mr)^{-1}N \leq^L R$. Therefore $mr \in WL - C_M(N)$. It is clear that $WL - C_M(N)$ is a weak large extension of $N$, and contains each weak large extension of $N$ in $M$. 
Corollary 3.4 Let $N$ be a nonzero submodule of an $R$-module $M$. Then $N$ is a weak large closed submodule in $M$ if and only if $N = WL - C_M(N)$.

Proof. It is clear.

Remark 3.5 Every weak large closed submodule $N$ of a module $M$ is large closed in $M$; for if $N \leq^{WL} M$, and $N \leq K \leq M$, then $N \leq^{WL} K$, and hence $N = K$. There are large closed submodules of $M$, which are not weak large closed in $M$ (see example in Remark 2.2).

Lemma 3.6 Let $M$ be an $R$-module, $A$ be a nonzero submodule of $M$, and $B$ be a submodule of $M$ containing $A$. Then $WL - C_M(A)$ is contained in $WL - C_M(B)$

Proof. The proof is clear, as $m^{-1}A \leq m^{-1}B$, for each $m \in M$.

The following corollary shows that, without an extra condition on the ring $R$, the property of weak large closed is transitive.

Corollary 3.7 Let $M$ be an $R$-module, and $N, K$ be submodules of $M$, with $N \leq K$. If $N$ is a weak large closed submodule of $K$ and $K$ is a weak large closed submodule of $M$, then $N$ is a weak large closed submodule of $M$.

Proof. Let $m \in WL - C_M(N)$, from lemma 3.6., $m \in WL - C_M(K)$. Since $K \leq^{WL} M$, it follows that $m \in K$ and hence $m \in WL - C_K(N)$. As $N \leq^{WL} K$, we have that $m \in N$. Therefore $N$ is a weak large closed submodule of $M$ by corollary 3.4.

Proposition 3.8 Let $M$ be an $R$-module. Then $A \cap B$ is a weak large closed submodule of $M$, for any two weak large closed submodules $A$ and $B$ of $M$.

Proof. Let $m \in WL - C_M(A \cap B)$. Then by Lemma 3.6., $m \in WL - C_M(A) = A$ and $m \in WL - C_M(B) = B$. Therefore $m \in A \cap B$ and by lemma 3.4 we have that $A \cap B$ is a weak large closed submodule of $M$.

Remark 3.9 In any nonsingular module $M$, if $A$ and $B$ are large closed in $M$, then $A \cap B$ is large closed submodule in $M$. This property is not in general true, for example $M = \mathbb{Z} \oplus \mathbb{Z}/(2\mathbb{Z})$, $A = (1, 0)\mathbb{Z}$, and $B = (1, 1)\mathbb{Z}/(3)$, Example 1.6).

Proposition 3.10 Let $f : M \to N$ be an $R$-module isomorphism. Then the isomorphic image of a weak large closed submodule of $M$ is also a weak large closed submodule of $N$. 
Proof. It is clear from lemma 2.4.

Lemma 3.11 Let $M = M_1 \oplus M_2$ be an $R$-module, and $A_i$ be a nonzero submodule of $M_i$ ($i = 1, 2$). Then $A_i \leq_{WL} M_i$ ($i = 1, 2$) if and only if $A_1 \oplus A_2 \leq_{WL} M$.

Proof. It is enough to show that,

$$WL - C_M(A_1 \oplus A_2) = WL - C_{M_1}(A_1) \oplus WL - C_{M_2}(A_2).$$

This follows from the fact that $(m_1 + m_2)^{-1}(A_1 \oplus A_2) = m_1^{-1}A_1 \cap m_2^{-1}A_2.$

Definition 3.12 Let $A$ be a submodule of an $R$-module $M$. A submodule $B$ of $M$, with $A \cap B = 0$, is called a weak large complement of $A$ in $M$, if $B$ is weak large closed in $M$, and $A \oplus B$ is a weak large submodule of $M$.

4 Weak Large Extending Modules

Definition 4.1 An $R$-module $M$ is called a weak large extending module (denoted by $WLE$-module) if $WL$-$Cl(N)$ is a direct summand for every nonzero submodule $N$ of $M$, or equivalently every weak large closed submodule of $M$ is a direct summand.

Definition 4.2 A nonzero module $M$ is called a weak large uniform module (denoted by $WL$-uniform) if every nonzero submodule of $M$ is weak large in $M$.

Remark 4.3 1- Every large extending module $M$ is a weak large extending modules, as every weak large closed is large closed, in $M$.

2- Every uniform module is $WL$-uniform, while there are $WL$-uniform modules which are not uniform, for example: $M = C(2) \oplus C(3)$ as a $\mathbb{Z}$-module is a $WL$-uniform but not uniform.

3- If $M$ is a non-singular module, then $M$ is $L$-extending, whenever $M$ is $WL$-extending (due to weak large submodules, and large submodules, of $M$, coincide).

Lemma 4.4 Let $M$ be an indecomposable module. Then $M$ is weak large extending if and only if $M$ is Weak large uniform.

Proof. Let $N$ be a nonzero submodule of $M$. Since $M$ is weak large extending, we have that $WL - C_M(N) \leq M$, and since $M$ is an indecomposable module, then $WL - C_M(N) = M$. Therefore $M$ is weak large uniform. The converse is obvious.

Proposition 4.5 Let $M$ be a $WL$-extending module. If $N = WL - C_M(N)$, for each nonzero direct summand of $M$, then either $M$ is $WL$-uniform, or $M$ is nonsingular $L$-extending.
Proof. Let $K$ be a nonzero submodule of $M$, and $L$ be a maximal large extension of $K$ in $M$. Then $L \leq WL - C_M(K) \leq M$, and hence, as $M$ is $WL$-extending, $WL - C(K) \leq M$. Now if $WL - C_M(K) = M$ for each nonzero submodule $K$ of $M$, then $M$ is a $WL$-uniform module. If, for some $0 \neq K$, \( D := WL - C_M(K) \neq WL - C_M(K) \), then $M = D \oplus H$, with $0 \neq H$. By assumption $D = WL - C_M(D)$, and $H = WL - C_M(H)$. As $WL - C_M(D) \cap WL - C_M(H) = 0$, we have that $D$ and $H$ are non-singular, and hence $M$ is a non-singular module. Therefore, $L = D \leq WL - C_M(K)$, i.e. $M$ is a non-singular $L$-extending module.

Lemma 4.6 Isomorphic copy of a $WL$-extending module is $WL$-extending module.

Proof. It is clear from corollary 2.6.

Lemma 4.7 Let $M$ be a module, with $Z(M) \neq 0$, then the weak large closure of $Z(M)$ in $M$ is $Z_2(M)$.

Proof. Let $m \in WL - Cl_M(Z(M))$, we have, as $m^{-1}Z(M) \leq L R$, $\overline{m}$ as an element of $M/Z(M)$, and $\overline{m}(m^{-1}Z(M)) = 0$, that $\overline{m} \in Z(M/Z(M))$, i.e. $m \in Z_2(M)$. Now if $m \in Z_2(M)$, hence $m I \leq Z(M)$, for some right ideal $I \leq L R$. Thus, as $I \leq m^{-1}Z(M) \leq R$, $m \in WL - Cl_M(Z(M))$.

Corollary 4.8 Let $M$ be an module, with $Z(M) \neq 0$, then $Z_2(M)$ is a weak large closed in $M$.

Proposition 4.9 Let $M$ be an module, with $Z(M) \neq 0$. Then $Z(M)$ is contained in $WL - Cl_M(N)$, for any nonzero submodule $N$ of $M$.

Proof. Let $m \in Z(M)$, then $m I = 0$, for some right ideal $I$ of $R$, with $I \leq L R$. Hence $m^{-1}N \leq L R$ (due to $I \leq m^{-1}N$ ); i. e. $m \in WL - Cl_M(N)$.

Corollary 4.10 Let $M$ be a nonzero singular module, then $M$ is a $WL$-uniform module.

Proof. Let $N$ be a nonzero submodule of $M$, hence, form Proposition 4.9, $M = Z(M) \leq WL - Cl_M(N)$. Therefore $N \leq WL - Cl_M(N) = M$.

In the following theorem we show that weak large extending modules are inherited by direct summand.

Lemma 4.11 Let $M = N \oplus L$, and $A$ be a weak large closed submodule of $N$, then $A \oplus L$ is a weak large closed submodule of $M$. 
Remark 4.13 If \( A \oplus L \) be a weak large in a submodule \( K \) of \( M \), and 
\( n + l = k \in K \), with \( n \in N \), and \( l \in L \). We show that \( n^{-1}A \leq L R \). To this end, let \( r \in R \). Since \( k^{-1}(A \oplus L) \leq L R \), it follows that there exists \( s \in R \) such that \( 0 \neq rs \), and \( krs \in A \oplus L \). Hence \( nrs \in A \), i.e. \( n^{-1}A \leq L R \). Since \( A \) be a weak large closed submodule of \( N \), we have that \( n \in A \), and that \( k \in A \oplus L \). Therefore \( K = A \oplus L \), i.e. \( A \oplus L \) is a weak large closed submodule of \( M \).

**Theorem 4.12** If an module \( M \) is a weak large extending, then every direct summand \( N \) of \( M \) is weak large extending.

**Proof.** Let \( M = N \oplus L \) be a weak large extending module, and let \( A \) be a weak large closed submodule of \( N \). By Lemma 4.14, \( A \oplus L \) is a weak large in \( M \). Since \( M \) is weak large extending, we have that \( A \oplus L \leq L R \), and hence \( A \leq L N \).

**Remark 4.13** If \( M \) is a module, with \( Z(M) \neq 0 \), then \( Z(M) \leq L Z_2(M) \).

For, if \( m \in Z_2(M) \setminus Z(M) \), then \( 0 \neq \overline{m} \in Z_2(M)/Z(M) = Z(M/Z(M)) \), and hence \( \overline{mI} = 0 \), for some \( I \leq L R \); i.e. \( 0 \neq mI \leq Z(M) \).

**Lemma 4.14** Let \( M \) be a weak large extending module, with \( Z(M) \neq 0 \). Then \( Z(M) = Z_2(M) \).

**Proof.** By Corollary 4.8, we have that \( Z_2(M) \) is a weak large closed submodule of \( M \), and hence, by the large extending property for \( M \), \( M = N \oplus Z_2(M) \), with \( N \) a nonsingular submodule of \( M \). We show that \( WL-Cl(N) = N \oplus Z(M) \).

It is clear, by Proposition 4.9, \( Z(M) \leq WL-Cl(N) \). Now let \( m = n + z \in N \oplus Z(M) \), where \( n \in N \) and \( z \in Z(M) \). It follows that \( mI = nI \leq N \), for some \( I \leq L R \) (due to \( zI = 0 \)). It follows that \( I \leq m^{-1}N \), and hence \( m^{-1}N \leq L R \), i.e. \( m \in WL-Cl(N) \).

Since \( M \) is a weak large extending module, we have that \( N \oplus Z(M) = WL-Cl(N) \leq L M \). It follows that \( Z(M) \leq L Z_2(M) \), and hence, from Remark 4.13, \( Z(M) = Z_2(M) \).

**Theorem 4.15** Let \( M \) be a non-singular module. Then \( M \) is weak large extending if and only if \( M = N \oplus Z(M) \), with \( Z(M) \) (a WL-uniform submodule) is weak large extending, and \( N \) is non-singular large extending.

**Proof.** Let \( M \) be weak large extending. By Collorally 4.8, and Lemma 4.14, \( Z(M) = Z_2(M) \) is a weak large closed submodule of \( M \), and hence \( M = N \oplus Z(M) \), with \( N \) is non-singular. By Theorem 4.12, \( Z(M) \), and \( N \) are weak large extending submodules. Then, by Remark 2.2, \( N \) is a large extending module. By Proposition 4.9, \( Z(M) \) is a WL-uniform.

For the converse, Let \( 0 \neq K \) be a weak large closed submodule of \( M \). By Proposition 4.9, \( Z(M) \leq WL-Cl(K) = K \). It follows that \( K = L \oplus Z(M) \),
where \( L =: K \cap N \leq N \). From Remark 2.2, \( K \) is large closed in \( M \), and hence \( L \) is a large closed submodule of \( N \). Since \( N \) is large extending, it follows that \( L \leq \oplus N \), and hence \( K \leq \oplus M \).

**Lemma 4.16** ( [9] , Theorem 8) Let \( M \) be an module over a right noetherian ring \( R \). Then \( M \) is an extending module if and only if \( M \) is direct sum of uniform submodules, has \((1 - C_1)\), and every local summand of \( M \) is a direct summand.

**Theorem 4.17** Every weak large extending \( R \)-module \( M \) over a right noetherian ring \( R \) is a direct sum of \( WL \)-uniform submodules.

**Proof.** Let \( M \) be a not singular weak large extending \( R \)-module, where \( R \) is a right noetherian ring. Then, by Theorem 4.15, \( M = N \oplus Z(M) \), with \( Z(M) \) (a \( WL \)-uniform submodule) is weak large extending, and \( N \) is non-singular large extending. Hence, by Lemma 4.16, \( N \) is a direct sum of uniform submodules, thus \( N \) is a direct sum of \( WL \)-uniform submodules.

**Lemma 4.18** Let \( M \) be a \( R \)-module, with \( Z(M) \neq 0 \). If \( M \) is a \( WL \)-uniform, then \( M = Z(M) \).

**Proof.** Let \( M \neq Z(M) \), and let \( m \in M \setminus Z(M) \). As \( Z(M) \leq WL M \), we have that \( m^{-1}Z(M) \leq L R \), and that \( 0 \neq m(m^{-1}Z(M)) \cap Z(M) \leq mR \cap Z(M) \), which is a contradiction.

**Theorem 4.19** The following are equivalent of a module \( M \) over a right noetherian ring \( R \):

1. \( M \) is a weak large extending module.
2. \( M \) is a direct sum of \( WL \)-uniform submodules, and \( M/Z(M) \) has \((1 - C_1)\), with every local direct summand of \( M/Z(M) \) is a direct summand.

**Proof.** 1) \( \Rightarrow \) 2): By Theorem 4.15, we have that \( M = N \oplus Z(M) \), with \( Z(M) \) (a \( WL \)-uniform submodule) is weak large extending, and \( N \) is non-singular large extending. By Lemma 4.16, It follows that \( M/Z(M) \cong N \) is a direct sum of uniform submodules, has \((1 - C_1)\), and every local summand of \( N \) is a direct summand.

2) \( \Rightarrow \) 1): Let \( M = \oplus_{i \in J} N_i \) be a direct sum of \( WL \)-uniform submodules, with \( M/Z(M) \) has \((1 - C_1)\), and with every local direct summand of \( M/Z(M) \) is a direct summand. By Lemma 4.18, \( M = (\oplus_{i \in J} N_i) \oplus (\oplus_{i \in K} N_i) \), where \( J, K \subseteq I \), \( 0 = Z(N_i) \) for each \( i \in J \), and \( Z(N_i) = N_i \) for each \( i \in K \). It follows that \( M = \oplus_{i \in J} N_i \oplus Z(M) \). By Lemma 4.16, \( \oplus_{i \in J} N_i \) is large extending module, and by Corollary 4.10, \( Z(M) \) is \( WL \)-uniform module. Therefore, by Theorem 4.15, \( M \) is a weak large extending module.
References


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