Research on Independence of Random Variables

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Abstract

Taking two-dimensional random variable as the research object, we dig the profound connotation of independence and reveal the true nature of it. Then, from the point of view of entropy in information theory, we explore the relationship between each component of two-dimensional random variable and the equivalence between independence and non-correlation of them.

Keywords: independence, generalized correlation, two-dimensional normal distribution, entropy

Introduction

Independence of random variables is an important part of probability theory. For example, the independence of samples is the condition of many fundamental theories in the mathematical statistics. In this paper, we do a research on independence of random variables.

1 Research on Independence of Two-Dimensional Random Variable

Definition 2.1. Let X and Y be random variables taking values in the probability
space $(\Omega, \mathcal{F}, P)$. Let $f(X)$ and $g(Y)$ from $(\mathbb{R}, \mathcal{B})$ into itself be the real measurable functions which are the non-degenerate random variables. The covariance of $f(X)$ and $g(Y)$ is denoted by the symbol $\text{Cov}(f(X), g(Y))$. We call $\rho_{fg} = \frac{\text{Cov}(f(X), g(Y))}{\sqrt{D(f(X))\sqrt{D(g(Y))}}}$ the correlation coefficient of $f(X)$ and $g(Y)$.

Then we say that $f(X)$ and $g(Y)$ are generalized correlate (resp. generalized non-correlate) if $\rho_{fg} \neq 0$ (resp. $\rho_{fg} = 0$) holds. In particular, it is called a complete generalized correlation when $\rho_{fg} = 1$ holds. [6]

Generalized correlation reflects the relationship among all real measurable functions defined on random variables $X$ and $Y$. Then we will use this definition to explore the true nature of independence.

**Theorem 2.1.** Let $X$ and $Y$ be the random variables. $f(X)$ and $g(Y)$ are the real measurable functions mentioned in definition 2.1, then we have the equivalence

$$F(x, y) = F_X(x)F_Y(y) \Leftrightarrow \rho_{fg} = 0$$

(1)

where $F(x, y)$ is the joint distribution function of $(X, Y)$ and $F_X(x)$ (resp. $F_Y(y)$) the distribution function of $X$ (resp. $Y$).

**Proof.** $\Leftarrow$ We know the condition $\rho_{fg} = 0$, then we have

$$E[f(X)g(Y)] = E[f(X)]E[g(Y)]$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space with any set $A \in \mathcal{F}$ and $B \in \mathcal{F}$.

Assume

\[
\begin{cases}
  f(X) = I_A(X) \\
  g(Y) = I_B(Y)
\end{cases}
\]
where \( I_A(X) \) (resp. \( I_B(Y) \)) denotes the indicator function on the set \( A \) (resp. \( B \)). Then we have
\[
E[f(X)g(Y)] = E[I_A(X)I_B(Y)] = P(AB)
\]
\[
E[f(X)]E[g(Y)] = E[I_A(X)]E[I_B(Y)] = P(A)P(B)
\]
Clearly, \( F(x,y) = F_X(x)F_Y(y) \).

"\( \Rightarrow \)" It can be proved that if \( X \) and \( Y \) are independent, then all real measurable functions defined on them are independent of each other. Clearly, \( \rho_{fg} = 0 \).

From the proof of Theorem 2.1, we can reveal the true nature of independence between random variables. That is, the independence of random variables means that occurrence of \( A \) does not change the probability that \( B \) also occurs if \( X \) and \( Y \) are defined on \( A \) and \( B \), respectively. Based on the theory in this section, a distinction on independence and functional relationship of random variables should be made.[2]

We just know that two random variables are not independent if they are relevant. Next, we will explore the true nature of this conclusion through lemma 1.

**Lemma 1.** The random variables are not independent if they are relevant.

**Proof.** Let \( X \) and \( Y \) be the random variables defined on the probability space \( (\Omega, F, P) \). We know that \( X \) and \( Y \) are relevant, then we have
\[
E[X,Y] \neq E[X]E[Y]
\]
For any \( A, B \in F \), we assume
\[
\begin{cases}
X = I_A(X) \\
Y = I_B(Y)
\end{cases}
\]
where \( I_A(X) \) (resp. \( I_B(Y) \)) denote the indicator on the set \( A \) (resp. \( B \)). Then we have
\[
E[XY] = E[I_A(X)I_B(Y)] = P(AB)
\]
\[
E[X]E[Y] = E[I_A(X)]E[I_B(Y)] = P(A)P(B)
\]
Clearly, $P(AB) \neq P(A)P(B)$, that is, $X$ and $Y$ are not independent.

2 Research on Independent and Irrelevant Problems of Normal Distribution

Before we explore the question, we make a distinction as the Remark 1 follows.

Remark 1. The joint distribution of two normal distribution random variables don't necessarily follow the normal distribution problem. Random variables are independent of each other, certainly not relevant; the other hand, if two random variables are irrelevant, we can't prove that the two are independent of each other.

**Definition 3.1** Let $X$ be a continuous random variable with the density function $f_X(x)$. Then we call

$$H(X) = -\int_{-\infty}^{\infty} f_X(x) \log_2 f_X(x) dx$$

(2)

the entropy of the random variable $X$. Similarly,

$$H(X,Y) = -\iint f(x,y) \log_2 f(x,y) dxdy$$

(3)

is called the joint entropy of $(X,Y)$.

**Theorem 3.1.** Let $(X,Y)$ be two-dimensional normal random variable. Then the relation of $X$ and $Y$ is either relevant or there is no functional relation between the real measurable function $f(X)$ and $g(Y)$ defined on them, respectively.

**Proof.** Random variables $X$ and $Y$ are either relevant or irrelevant. We know that they are not independent if they are relevant. Then we have

$$\rho_{fg} = 0$$

Clearly, $f(X)$ and $g(Y)$ defined on them have no relation.

Theorem 3.1 shows that each component of two-dimensional normal random variable are either relevant or independent.
**Definition 3.2.** Let \((X,Y)\) be the two-dimensional random variable with the density function \(p(x,y)\). \(\mu_i\) (resp. \(\mu_2\)) is the mean of \(X\) (resp. \(Y\)) and \(\sigma_i\) (resp. \(\sigma_2\)) the standard deviation of \(X\) (resp. \(Y\)). Put \(X^* = \frac{X - \mu_1}{\sigma_1}\); \(Y^* = \frac{Y - \mu_2}{\sigma_2}\), then we call \((X^*, Y^*)\) the standardized random variable of \((X,Y)\) with density function \(f(x,y)\). We know

\[
f(x,y) = \sigma_1\sigma_2 p\left(\sigma_1 x + \mu_1, \sigma_2 y + \mu_2\right)
\]

Then we have

\[
H\left(X^*, Y^*\right) = -\iint f(x,y) \log_2 f(x,y) \, dx \, dy
\]

\[
= -\iint \sigma_1\sigma_2 p\left(\sigma_1 x + \mu_1, \sigma_2 y + \mu_2\right) \log_2 \sigma_1\sigma_2 p\left(\sigma_1 x + \mu_1, \sigma_2 y + \mu_2\right) \, dx \, dy
\]

\[
= -\iint p(s,t) \log_2 \sigma_1\sigma_2 ds \, dt - \iint p(s,t) \log_2 p(s,t) \, ds \, dt
\]

\[
= -\log_2 \sigma_1\sigma_2 + H\left(X,Y\right)
\]

Hence, \(H\left(X,Y\right) = H\left(X^*, Y^*\right) + \log_2 \sigma_1\sigma_2\). We call \(H\left(X^*, Y^*\right)\) the standard entropy of \((X,Y)\). [4]

Under Definition 3.2, by example 1, we will explore the relationship between independence and entropy of two-dimensional normal distribution random variable.

**Example 1.** Let \((X,Y)\) be the two-dimensional random variable with normal distribution \(N\left(\mu_1, \mu_2, \sigma_1, \sigma_2, 0\right)\). Put \(X^* = \frac{X - \mu_1}{\sigma_1}\); \(Y^* = \frac{Y - \mu_2}{\sigma_2}\). Suppose that \(p(x,y)\) is the density function of \((X^*, Y^*)\), then we have

\[
H\left(X,Y\right) = H\left(X^*, Y^*\right) + \log_2 \sigma_1\sigma_2
\] (4)
where
\[
H(X^*, Y^*) = -\int\int f(x, y) \log_2 f(x, y) \, dx \, dy
\]
\[-\log_2 \left( \frac{1}{2\pi\sqrt{1-\rho^2}} + \int\int f(x, y) \log_2 \exp \left[ -\frac{1}{2(1-\rho^2)} (x^2 - 2\rho xy + y^2) \right] \, dx \, dy \right)
\]
\[-\log_2 \left( \frac{1}{2\pi\sqrt{1-\rho^2}} \right)
\]
Hence,
\[
H(X, Y) = \log_2 \frac{1}{2\pi\sqrt{1-\rho^2}} + \log_2 \sigma_1 \sigma_2
\]
It is obvious that when the mean and variance are known, the smaller the correlation coefficient \(\rho\) is, the larger the entropy \(H(X, Y)\). And when \(\rho = 0\) holds, we have the maximum \(H(X, Y)\) and \(H(X, Y) = H(X) + H(Y)\) holds. Hence, \(X\) and \(Y\) are independent of each other. From the point of entropy, this example reveal that the independence and non-correlation among the components of two-dimensional normal random variables are equivalent.

**References**


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