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Study of Memory Effects in an Inventory Model Using Fractional Calculus

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Abstract

In this paper, we have studied some memory dependent Economic order quantity models as the memory effect has an important role to handle the business policy of the inventory system. Demand of a company product depends on many factors like behavior of staff, environment of the shop, product quality etc which are the main reasons of memory effect on the system. Once the customers gain some poor experience, further they will never purchase products from those companies or shop. So inclusion of memory effect in the inventory model is necessary to handle practical business policy. One of the best way of inclusion of memory effect in the EOQ model is the use of fractional calculus as fractional derivative is defined in terms of integration where the limits of integration are the initial state and current state. Three fractional order models have been developed considering (i) only the rate of change of the inventory level of fractional order α (ii) demand rate as a fractional polynomial of degree $2\alpha,$ where α is the rate of change of the inventory level (iii) demand rate as a fractional polynomial of degree $2m$, where m may be different from the order of the rate of change of the inventory level. Here fractional order is physically treated as an index of memory. To develop the models here Caputo type fractional derivative has been applied. Due to solve those problems, we have used primal geometric programming method and finally some numerical examples are cited to establish the memory effect. Our investigation establishes the existence of memory effect on inventory management through fractional formulation of EOQ models which can never be obtained from classical calculus.

Keywords: Fractional order derivative; Memory dependent derivative; Fractional Laplace transform method; Classical inventory model; Fractional order inventory models

1. Introduction

The idea of fractional calculus was initiated in 1695[1-2].Most of the suggestions for interpretation of fractional calculus are a little bit abstract. They do not give any physical intuition but one of the most useful interpretations is, fractional calculus has power to remember previous effects of the input in order to determine the current value of the output. Such type systems are called memory systems. In classical calculus, a system at each time *t* depends only on the input at that time because the derivatives of integer orders are determined by the property of differentiable functions of time only, and they are differentiable infinitely small neighborhood of the measured point of the time. In the last thirty years, fractional calculus exhibited a remarkable progress in several fields of science such as mechanics, chemistry, biology [29], economics [21, 31], control theory, physics [27-28], signal and image processing [4-8] etc. But it is less explored in the field of inventory management in operations research.

Inventory is stock of goods or resources used in organizations whose models are developed to minimize the total average cost. Haris [23] was the first person who developed the economic order quantity (EOQ) inventory model. The authors in (10-14, 24, 25 and reference there in) used their effort to derive the EOQ models using different criteria of demand, shortage etc using integer order calculus. Why have we considered the inventory system as a memory affected system?

In reality the demand rate is not always same; it changes with respect to time as well as environment; depends on the position of the company, political and social conditions. For example the position of the company or shop near the main road or connection of the company or shop with public etc increase the demand. On the other hand if an object gets popularity in the market then it's demand will increase or if it gets bad impression then it's demand will decrease. In some sense demand of any object is not same in all shop it depends on dealing of the shopkeeper with the customer. This means the selling of any product depends on the quality as well as the shopkeeper's attitude. Another type of memory exists here, which is memory due to holding cost or carrying cost. It is considered due to bad or good dealings of the transportation system. The effects of bad service always has a bad impact on the business. Bad service of the transportation driver may damage the commodity and reduce the profit. For this case owner of the company or shop should think to improve the transportation system to make his business profitable.

Thus, such systems depend on the history of the process not only on the current state of the process, i.e., a memory effect will influence the inventory system. Hence, inventory system is a memory dependent system. Due to the above reasons, we want to enjoy the facility of fractional derivative for considering memory effect of the inventory systems. In this paper we have not only considered the fractional model but also different ways of generalization through fractional calculus have been considered.

Das and Roy [15] introduce fractional order inventory model with constant demand and no shortage but there is no suggestion for application of memory effect on the inventory model. Owing to the power of memory effect of fractional calculus, in this paper we want to fractionalize a classical inventory model with quadratic demand rate into different approach and study their macroscopic behavior for long to low range memory. Furthermore, in the proposed memory less inventory model, quadratic demand is polynomial of highest degree two whereas in case of fractional inventory models, quadratic type demand rate may be fractional polynomials. In the proposed traditional classical inventory model i.e. memory less inventory model, governing differential equation are of first order and holding costs are first order integral of inventory level. In this paper, three types of fractionalization have been considered to observe memory effect considering fractional order inventory models. In the first model, only the rate of change of the inventory level is fractional. In the promising second and third model both the demand rate and the change of the inventory level are fractional. The fractional order of the demand rate and change of the inventory level are same in the second model and different in the third model. To formulate the fractional order inventory models, we have used the memory dependent kernel as described in [29].The fractional differential equations have been solved using fractional Laplace transform [3].We have then calculated associated fractional order holding costs using fractional integration. Analytical results of fractional order inventory model have been introduced by the primal geometric programming method [19].

Our analysis establishes that the shopkeeper or company should be alert to improve about his business policy such as attitude of public dealing, environment of shop or quality of their product. The memory parameter corresponding fractional rate of change of inventory level plays more important role to change the economic condition of the business compared to the memory parameter corresponding carrying cost.

Organization of the paper is maintained as follows: In section 2, we have presented a brief review of fractional calculus, In section 3, a classical inventory model and the corresponding fractional inventory models have been developed and analyzed respectively, in section-4, numerical examples have been cited to illustrate the models and in section 5, the paper ends with some concluding remarks and scope of future research.

2. Review of Fractional Calculus

There are many definitions of fractional derivative. Each of them has own physical interpretation. In this section, we shall introduce two important definitions the Riemann-Liouville and the Caputo definition. Also the fractional Laplace transform method will be introduced as it will be used to develop this paper.

2.1 Riemann-Liouville(R-L) fractional derivative

If $f : \mathrm{R} \rightarrow \mathrm{R}$ is a continuous function then the left Riemann-Liouville(R-L) derivative is denoted and defined in the following form as,

$$ {}_aD_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^m\int_a^x(x-\xi)^{(m-\alpha-1)}f(\xi)d\xi \quad \text{where } m\le\alpha\le m+1 \qquad (1) $$

This R-L definition suffers from some pitfalls. The Riemann-Liouville(R-L) fractional derivative of any constant is non-zero. Fractional Laplace transform of the Riemann-Liouville(R-L) type fractional differential equations involve fractional initial conditions.

2.2 Caputo fractional derivative

M.Caputo [21] eliminated the two difficulties of Riemann-Liouville definition. For any m times differentiable function $f(x)$, the Caputo fractional order derivative of $\alpha-th$ order is denoted and defined in the following form

$$ {}_a^CD_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(m-\alpha)}\int_a^x(x-\xi)^{m-\alpha-1}f^m(\xi)d\xi \quad \text{where } m-1\le\alpha\le m \qquad (2) $$

Caputo fractional derivative of any constant is zero and Caputo fractional order differential equations do not involve initial conditions with fractional derivative. Initial conditions in this case are same as of the classical differential equations. One disadvantage of Caputo fractional derivative is that it is defined for differentiable functions.

2.3 Fractional Laplace transforms Method

The Laplace transform of the function $f(t)$ is defined as

$$ F(s)=L(f(t))=\int_0^{\infty}e^{-st}f(t)dt \quad \text{where s>0 and } s \text{ is the transform parameter} \qquad (3a) $$

The Laplace transformation of n^{th} order derivative is defined as

$$ L\left(f^n(t)\right)=s^nF(s)-\sum_{k=0}^{n-1}s^{n-k-1}f^k(0) \qquad (3b) $$

where $f^n(t)$ denotes *n-th* order derivative of the function *f* with respect to *t* and for non – integer m it is defined in generalized form as,

$$ L\left(f^m(t)\right)=s^mF(s)-\sum_{k=0}^{n-1}s^kf^{m-k-1}(0) \qquad (3c) $$

where *m* is the largest integer such that $(n-1)<m\le n.$

2.4 Memory dependent derivative

The derivative of any function $f(x)$ using the memory kernel can be written in the following form [30, 29]

$$ D_{\xi}\left(f(x)\right)=\frac{1}{\xi}\int_{t-\xi}^{t}K(t-s)f'(s)ds \qquad (4a) $$

where ξ is the time delay and the kernel function $K(t-\xi)$ is differentiable with respect to *t* and *s*. For integer order derivative the kernel is considered as $K(t-t')=\delta(t-t')$ which is a Dirac-delta function or impulse function. It gives the memory less derivative. To derive the concept of memory effect using definition (4a) we consider the kernel as a power function in the form $K(t-s)=\frac{(t-s)^{m-\alpha}}{\Gamma(m-\alpha)}$ and the fractional derivative expressed in the following form

$$\int_a^t K(t-s)f^m(s)ds=\int_a^t \frac{(t-s)^{m-\alpha}}{\Gamma(m-\alpha)} f^m(s)ds = D_a^\alpha\left(f(x)\right) \tag{4b}$$

where f^m denotes the *m-th* order derivative of *f*, which has specific physical meaning. The integer order derivative is a local property but the $\alpha-th$ order fractional derivative is a no local property. The total effects of the commonly used $\alpha-th$ order derivative on the interval *[a, t]* describes the variation of a system in which the instantaneous change rate depends on the past state, is called the ‘‘memory effect’’.

3. Model formulations

In this section, the classical order inventory model and three fractional order inventory models have been analytically developed. The symbols have been used to develop these models, are listed in the table-1.

$(i)R:$ Demand rate	$(ii)Q:$ Total order quantity
$(iii)M:$ Per unit cost	$(iv)C_1:$ Inventory holding cost per unit
$(v)C_3:$ Ordering cost or setup cost	$(vi)I(t):$ Stock level or inventory level
$(vii)T:$ Ordering interval.	$(viii)HOC:$ Inventory holding cost per cycle for the classical inventory model.
$(ix)T^*:$ Optimal ordering interval	$(x)TOC^{av}:$ Total average cost during the total time interval
$(xi)TOC^*:$ Minimized total average cost during the total time interval $[0,T]$ for classical model.	$(xii)(B,.),(\Gamma,.)$ Beta function and gamma function respectively.

Table-1: Different symbols and items for the EOQ models.

3.2 Assumptions

In this paper, the classical and fractional order EOQ models are developed on the basis of the following assumptions.

(i) Lead time is zero. (ii)Time horizon is infinite. (iii)There is no shortage. (iv)There is no deterioration.

3.3 Formulation and analysis of memory less inventory model

Here, we have first developed a classical inventory model depending on the above assumptions in the following manner [9]. During the total time interval *[0, T]*, the inventory level depletes due to quadratic demand rate $\mathrm{R(t)} = (a + bt + ct^2), \mathrm{a} > 0, \mathrm{b, c} \geq 0.$, where shortage is not allowed. The inventory reaches zero level at time *t=T*. Therefore, inventory level at any time during the time interval *[0,T]* can be represented by the following first order ordinary differential equation as,

$$\frac{dI(t)}{dt} = -(a + bt + ct^2) \tag{5}$$

with boundary conditions $I(0) = Q$ and $I(T) = 0$.Solution of this boundary value problem gives the inventory level $I(t)$ at time *t* in the following form,

$$I(t) = a(T - t) + \frac{b}{2}(T^2 - t^2) + \frac{c}{3}(T^3 - t^3) \tag{6}$$

Corresponding total inventory holding cost over the time interval *[0,T]* is as,

$$HOC(T) = C_1 \int_{t=0}^{T} I(t)dt = C_1(\frac{a}{2}T^2 + \frac{b}{3}T^3 + \frac{c}{4}T^4) \tag{7}$$

Since, total cost $TOC(T)$ at time *t* for the classical inventory system is the sum of the purchasing cost(PC),inventory holding cost $HOC(T)$ and the ordering cost or the setup cost(C_3), total cost $TOC(T)$ will be,

$$TOC(T) = \left(MQ + C_1\left(\frac{a}{2}T^2 + \frac{b}{3}T^3 + \frac{c}{4}T^4 \right) + (C_3) \right) \tag{8}$$

Where $Q = \left(aT + \frac{b}{2}T^2 + \frac{c}{3}T^3 \right) = I(0)$ (8a)

Substituting, the total average cost *Q* (8a) in (8) and dividing by the ordering interval *T* per unit time per cycle in the form,

$$TOC^{av}(T) = \left(M\left(a + \frac{b}{2}T + \frac{c}{3}T^2 \right) + C_1\left(\frac{a}{2}T + \frac{b}{3}T^2 + \frac{c}{4}T^3 \right) + \left(\frac{C_3}{T} \right) \right) \tag{9}$$

Purpose of studying inventory management is to minimize this $TOC^{av}(T)$. Thus our classical EOQ model under consideration can be represented in the following form,

$$\begin{cases} \text{Minimize } TOC^{av}(T) = \frac{1}{T}\left(MQ + HOC(T) + C_3 \right) \\ \qquad\qquad \text{Subject to } T \geq 0 \end{cases} \tag{10}$$

Minimized total average cost and optimal ordering interval are evaluated from (10) in the section-5.

3.4 Fractionalization of the Classical EOQ Model

We shall now develop the different fractional order inventory models considering fractional rate of change of the inventory level.

Model-I

To study the influence of memory effects, first the differential equation (5) is written using the memory kernel function in the following form [29].

$$\frac{dI(t)}{dt} = -\int \mathrm{k}\left(t-t'\right)(a+bt'+c(t')^2)dt' \tag{11}$$

in which $k(t-t')$ plays the role of a time-dependent kernel. For Markov process it is equal to the delta function $\delta(t-t')$ that generates the equation (5).In fact, any arbitrary function can be replaced by a sum of delta functions, thereby leading to a given type of time correlations. This type of kernel promises the existence of scaling features as it is often intrinsic in most natural phenomena. Thus, to generate the fractional order model we consider $k(t-t') = \frac{1}{\Gamma(1-\alpha)}(t-t')^{\alpha-2}$, where $0<\alpha\le 1$ and $\Gamma(\alpha)$ denotes the gamma function. Using the definition of fractional derivative [2], the equation (5) can be written to the form of fractional differential equations with the Caputo-type derivative in the following form as,

$$\frac{dI(t)}{dt} = -{}_0D_t^{-(\alpha-1)}\left((a+bt+ct^2)\right) \tag{12}$$

Now, applying fractional Caputo derivative of order $(\alpha-1)$ on both sides of (12), and using the fact at Caputo fractional order derivative and fractional integral are inverse operators, the following fractional differential equations can be obtained for the model

${}_0^C D_t^\alpha \left(I\left(t\right)\right) = -(a+bt+ct^2)$ or equivalently

$$\frac{d^\alpha\left(\mathrm{I(t)}\right)}{dt^\alpha} = -\left(a+bt+ct^2\right) \quad , \; 0<\alpha\le 1.0,\; 0\le t\le T \tag{13}$$

with boundary conditions $I(T)=0$ and $I(0)=Q$.

Model-II

Now in model-II, we consider the demand rate as polynomial of t^α then the memory dependent EOQ model will be of the following form (here we consider the exponent of *t* same as the order of fractional derivative)

${}_0^C D_t^\alpha \left(I\left(t\right)\right) = -(a+bt^\alpha+ct^{2\alpha})$ or equivalently

$$\frac{d^\alpha\left(\mathrm{I(t)}\right)}{dt^\alpha} = -\left(a+bt^\alpha+ct^{2\alpha}\right) \quad , \; 0<\alpha\le 1.0,\; 0\le t\le T \tag{14}$$

Model-III

In model-III, we consider the demand rate as polynomial of t^m ($0<m\le 1.0$) then the memory dependent EOQ model will be of the following form (here we consider the exponent of *t* in general may be different from the order of the fractional derivative)

${}_0^C D_t^\alpha \left(I(t)\right) = -\left(a+bt^m+ct^{2m}\right)$ or equivalently

$$\frac{d^\alpha I(t)}{dt^\alpha} = -\left(a+bt^m+ct^{2m}\right) \qquad \text{where } 0<\alpha, m\le 1.0, \text{ and } 0\le t\le T \tag{15}$$

with boundary conditions $I(0)=Q$ and $I(T)=0$

3.4.1 Analytic solution of model-I

Here, we consider the fractional order inventory model-I which will be solved by using Laplace transform method with the initial condition, given in the problem. In operator form the fractional differential equation in (13) can be represented as

$$D^\alpha\left(I(t)\right) = -\left(a+bt+ct^2\right) \quad , \quad D^\alpha \equiv \frac{d^\alpha}{dt^\alpha} \tag{16}$$

where the operator D^α stands for the Caputo fractional derivative with the operator ($D^\alpha = {}_0^c D_t^\alpha$).

Using fractional Laplace transform and the corresponding inversion formula on the equation (16) we get the inventory level for this fractional order inventory model at time *t* which can be written as

$$I(t) = \left(Q - \frac{at^\alpha}{\Gamma(1+\alpha)} - \frac{bt^{(\alpha+1)}}{\Gamma(2+\alpha)} - \frac{2c\, t^{(\alpha+2)}}{\Gamma(3+\alpha)}\right) \tag{17}$$

Using the boundary condition $I(T)=0$ on the equation (17), the total order quantity is obtained as

$$Q = \left(\frac{aT^\alpha}{\Gamma(1+\alpha)} + \frac{b\, T^{(\alpha+1)}}{\Gamma(\alpha+2)} + \frac{2c\, T^{(\alpha+2)}}{\Gamma(3+\alpha)}\right) \tag{18}$$

and corresponding the inventory level at time *t* being,

$$I(t) = \left(\frac{a}{\Gamma(1+\alpha)}\left(T^\alpha - t^\alpha\right) + \frac{b}{\Gamma(2+\alpha)}\left(T^{\alpha+1} - t^{\alpha+1}\right) + \frac{2c}{\Gamma(3+\alpha)}\left(T^{\alpha+2} - t^{\alpha+2}\right)\right) \tag{19}$$

For the model (16), the $\beta - \left(0<\beta\le 1\right)$ order total inventory holding cost is denoted as $HOC_{\alpha\beta}(T)$ and defined as

$$HOC_{\alpha,\beta}(T) = C_1\left({}_0D_T^{-\beta}\left(I(t)\right)\right) = \frac{C_1}{\Gamma(\beta)}\int_0^T (T-t)^{(\beta-1)}\left(I(t)\right)dt$$

$$= \frac{C_1 a T^{(\alpha+\beta)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right) + \frac{C_1 b T^{(\alpha+\beta+1)}}{\Gamma(2+\alpha)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+2,\beta)}{\Gamma(\beta)}\right) + \frac{2C_1 c T^{(\alpha+\beta+2)}}{\Gamma(3+\alpha)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(\alpha+3,\beta)}{\Gamma(\beta)}\right) \tag{20}$$

Here, β is another memory parameter corresponding to the carrying cost which is the transportation related cost. Poor transportation service always has a bad impact on the business. To consider the past experience a memory parameter should be taken into account.

Therefore, the total average cost per unit time per cycle of this fractional model is,

$$TOC^{av}_{\alpha,\beta}=\frac{\left(MQ+HOC_{\alpha,\beta}(T)+C_3\right)}{T}=\frac{aMT^{\alpha-1}}{\Gamma(1+\alpha)}.+\frac{bMT^{\alpha}}{\Gamma(2+\alpha)}+\frac{2cMT^{\alpha+1}}{\Gamma(3+\alpha)}+\frac{aC_1T^{\alpha+\beta-1}}{\Gamma(1+\alpha)\Gamma(\beta)}\left(\frac{1}{\beta}-B(\alpha+1,\beta)\right)$$
$$+\frac{bC_1T^{\alpha+\beta}}{\Gamma(2+\alpha)\Gamma(\beta)}\left(\frac{1}{\beta}-B(\alpha+2,\beta)\right)+\frac{2cC_1T^{\alpha+\beta+1}}{\Gamma(3+\alpha)\Gamma(\beta)}\left(\frac{1}{\beta}-B(\alpha+3,\beta)\right)+C_3 \tag{21}$$

Now, we shall consider the following cases to study the behavior of this fractional order inventory model (i)$0<\alpha\le 1.0, 0<\beta\le 1.0$, (ii)$\beta=1.0$ and $0<\alpha\le 1.0$, (iii)$\alpha=1.0$ and $0<\beta\le 1.0$, (iv) $\alpha=1.0, \beta=1.0$.

(i) Case-1: $0<\alpha\le 1.0$, $0<\beta\le 1.0$.

Here, the total average cost is

$$\begin{aligned}TOC^{av}_{\alpha,\beta}(T)=&\frac{aMT^{\alpha-1}}{\Gamma(1+\alpha)}+\frac{bMT^{\alpha}}{\Gamma(2+\alpha)}+\frac{2cMT^{\alpha+1}}{\Gamma(3+\alpha)}+\frac{aC_1T^{\alpha+\beta-1}}{\Gamma(1+\alpha)\Gamma(\beta)}\left(\frac{1}{\beta}-B(\alpha+1,\beta)\right)\\&+\frac{bC_1T^{\alpha+\beta}}{\Gamma(2+\alpha)\Gamma(\beta)}\left(\frac{1}{\beta}-B(\alpha+2,\beta)\right)+\frac{2cC_1T^{\alpha+\beta+1}}{\Gamma(3+\alpha)\Gamma(\beta)}\left(\frac{1}{\beta}-B(\alpha+3,\beta)\right)+C_3T^{-1}\end{aligned} \tag{22}$$

To find the minimum value of $TOC^{av}_{\alpha,\beta}(T)$ we propose the corresponding non-linear programming problem in the following form and solve it by primal geometric programming method.

$$\begin{cases}\text{Min } TOC^{av}_{\alpha,\beta}(T)=AT^{(\alpha-1)}+B_1T^{\alpha}+CT^{\alpha+1}+DT^{(\alpha+\beta-1)}+ET^{(\alpha+\beta)}+FT^{\alpha+\beta+1}+GT^{-1}\\ \text{Subject to } T\ge 0\end{cases} \tag{23}$$

$$\text{where, A}=\frac{aM}{\Gamma(1+\alpha)},B_1=\frac{bM}{\Gamma(2+\alpha)},C=\frac{2cM}{\Gamma(3+\alpha)},D=\frac{aC_1}{\Gamma(1+\alpha)\Gamma(\beta)}\left(\frac{1}{\beta}-B(\alpha+1,\beta)\right),$$
$$E=\frac{bC_1}{\Gamma(2+\alpha)\Gamma(\beta)}\left(\frac{1}{\beta}-B(\alpha+2,\beta)\right),F=\frac{2cC_1}{\Gamma(3+\alpha)\Gamma(\beta)}\left(\frac{1}{\beta}-B(\alpha+3,\beta)\right),G=C_3$$

Primal Geometric Programming Method

In this method we have to first find the dual of the primal problem (23). This is

$$\text{Max}d(w)=\left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B_1}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}\left(\frac{D}{w_4}\right)^{w_4}\left(\frac{E}{w_5}\right)^{w_5}\left(\frac{F}{w_6}\right)^{w_6}\left(\frac{G}{w_7}\right)^{w_7} \tag{24}$$

with the orthogonal and normal conditions

$$w_1+w_2+w_3+w_4+w_5+w_6+w_7=1 \tag{25}$$

and

$$(\alpha-1)w_1+\alpha w_2+(\alpha+1)w_3+(\alpha+\beta-1)w_4+(\alpha+\beta)w_5+(\alpha+\beta+1)w_6-w_7=0 \tag{26}$$

The primal-dual relations corresponding to the problem (23, 24) are

$$\left.\begin{aligned} &AT^{(\alpha-1)} = w_1 d(w), B_1T^{(\alpha)} = w_2 d(w), CT^{(\alpha+1)} = w_3 d(w) \\ &DT^{(\alpha+\beta-1)} = w_4 d(w), ET^{(\alpha+\beta)} = w_5 d(w), FT^{(\beta+\alpha+1)} = w_6 d(w), GT^{-1} = w_7 d(w) \end{aligned}\right\} \tag{27}$$

Using the above relations in (27), we can write the following

$$\left.\begin{aligned} &\frac{B_1 w_3}{C w_2} = \left(\frac{A w_2}{B_1 w_1}\right), \left(\frac{C w_4}{D w_3}\right) = \left(\frac{A w_2}{B_1 w_1}\right)^{(\beta-2)}, \left(\frac{D w_5}{E w_4}\right) = \left(\frac{A w_2}{B_1 w_1}\right) \\ &\left(\frac{E w_6}{F w_5}\right) = \left(\frac{A w_2}{B_1 w_1}\right), \frac{F}{G}\left(\frac{A w_2}{B_1 w_1}\right)^{(\beta+\alpha+2)} = \frac{w_6}{w_7} \end{aligned}\right\} \tag{28}$$

along with $T = \left(\dfrac{A w_2}{B_1 w_1}\right)$ (29)

There are seven non-linear equations (25), (26), and the five equations of (28) with seven unknown $w_1, w_2, w_3, w_4, w_5, w_6, w_7$. Optimal values $w_1^*, w_2^*, w_3^*, w_4^*, w_5^*, w_6^*, w_7^*$ are obtained solving these seven nonlinear equations. Then optimal ordering interval $T_{\alpha,\beta}^*$ will be obtained by substituting w_1^*, w_2^* in (29) ,then minimized total average cost $TOC_{\alpha,\beta}^*$ from (23).

(ii) Case-2: $\beta = 1.0$ and $0 < \alpha \le 1.0$

Total average cost in this case is

$$\begin{aligned} TOC_{\alpha,1}^{av}(T) = &\frac{aMT^{(\alpha-1)}}{\Gamma(\alpha+1)} + \frac{bMT^{\alpha}}{\Gamma(\alpha+2)} + \frac{2cMT^{(\alpha+1)}}{\Gamma(\alpha+3)} + \frac{aC_1T^{\alpha}}{\Gamma(\alpha+1)}\left(1 - B(\alpha+1,1)\right) \\ &+ \frac{bC_1T^{1+\alpha}}{\Gamma(\alpha+2)}\left(1 - B(\alpha+2,1)\right) + \frac{2cC_1T^{\alpha+2}}{\Gamma(\alpha+3)}\left(1 - B(\alpha+3,1)\right) + C_3T^{-1} \end{aligned} \tag{30}$$

Hence, the generalized inventory model (23) will be,

$$\begin{cases} \text{Min } TOC_{\alpha,1}^{av}(T) = AT^{(\alpha-1)} + B_1T^{(\alpha)} + CT^{(\alpha+1)} + DT^{(\alpha+2)} + ET^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \tag{31}$$

$$\begin{aligned} \text{where, } &A = \frac{aM}{\Gamma(\alpha+1)}, B_1 = \frac{bM}{\Gamma(\alpha+2)} + \frac{aC_1}{\Gamma(\alpha+1)}\left(1 - B(\alpha+1,1)\right), D = \left(\frac{2cC_1}{\Gamma(\alpha+3)}\left(1 - B(\alpha+3,1)\right)\right), \\ &C = \left(\frac{2cM}{\Gamma(\alpha+3)} + \frac{bC_1}{\Gamma(\alpha+2)}\left(1 - B(\alpha+2,1)\right)\right), E = C_3. \end{aligned} \tag{32}$$

In the similar manner as in case (i) of model-I, primal geometric programming algorithm corresponding to the model (31) will provide the minimized total average cost and optimal ordering interval $TOC_{\alpha,1}^*$, $T_{\alpha,1}^*$.

(iii) Case-3: $\alpha = 1.0$ and $0 < \beta \le 1.0$.
In this case, total average cost per unit time per cycle is,

$$TOC_{1,\beta}^{av} = \frac{aMT^0}{\Gamma(1+\alpha)}.+\frac{bMT}{\Gamma(3)}+\frac{2cMT^2}{\Gamma(4)}+\frac{aC_1T^\beta}{\Gamma(2)\Gamma(\beta)}\left(\frac{1}{\beta}-B(2,\beta)\right)+\frac{bC_1T^{1+\beta}}{\Gamma(3)\Gamma(\beta)}\left(\frac{1}{\beta}-B(3,\beta)\right)$$
$$+\frac{2cC_1T^{\beta+2}}{\Gamma(4)\Gamma(\beta)}\left(\frac{1}{\beta}-B(4,\beta)\right)+C_3T^{-1} \quad (33)$$

In this case, the inventory model (23) will be,

$$\begin{cases} \text{Min } TOC_{1,\beta}^{av}(T) = AT^o + B_1T + CT^2 + DT^\beta + ET^{1+\beta} + FT^{2+\beta} + GT^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \quad (34)$$

$$\text{where}, A = \frac{aM}{\Gamma(2)}, B_1 = \frac{bM}{\Gamma(3)}, C = \frac{2cM}{\Gamma(4)}, D = \frac{aC_1}{\Gamma(2)\Gamma(\beta)}\left(\frac{1}{\beta}-B(2,\beta)\right),$$
$$E = \frac{bC_1}{\Gamma(3)\Gamma(\beta)}\left(\frac{1}{\beta}-B(3,\beta)\right), F = \frac{2cC_1}{\Gamma(4)\Gamma(\beta)}\left(\frac{1}{\beta}-B(4,\beta)\right), G = C_3$$

In the similar manner as in case (i) of model-I, primal geometric programming algorithm will provide the $TOC_{1,\beta}^{*}$ and $T_{1,\beta}^{*}$.

(iv)Case-4: $\beta = 1.0,\ \alpha = 1.0$.
In this case, the total average cost is as,

$$TOC_{1,1}^{av}(T) = \left[\frac{aMT^0}{\Gamma(2)}+\left(\frac{bM}{\Gamma(3)}+\frac{a}{\Gamma(2)}(1-B(2,1))\right)T+\left(\frac{2cM}{\Gamma(4)}+\frac{b}{\Gamma(3)}(1-B(3,1))\right)T^2+\frac{2c}{\Gamma(4)}(1-B(4,1))T^3+C_3T^{-1}\right] \quad (35)$$

In this case, the inventory model (23) is,

$$\begin{cases} \text{Min } TOC_{1,1}^{av}(T) = AT^{(0)} + B_1T + CT^2 + DT^3 + ET^{-1} \\ \text{subject to } T \ge 0 \end{cases} \quad (36)$$

$$\text{where}, A = \frac{aM}{\Gamma(2)} = aM, B_1 = \left(\frac{bM}{\Gamma(3)}+\frac{aC_1}{\Gamma(2)}(1-B(2,1))\right) = \left(\frac{bM}{2}+\frac{aC_1}{2}\right),$$
$$C = \left(\frac{cM}{3}+\frac{bC_1}{3}\right), D = \frac{2cC_1}{\Gamma(4)}(1-B(4,1)) = \frac{cC_1}{4}, E = C_3 .$$

In the similar way as in case (i) of model-I, primal geometric programming algorithm can provide the minimized total average cost $TOC_{1,1}^{*}(T)$ and optimal ordering interval $T_{1,1}^{*}$. Interesting to note that the analytical results of this model coincides with the results of our classical model (9) where $\beta = 1.0,\ \alpha = 1.0$.

3.4.2 Analytic solution of model-II

Here, we consider the fractional order inventory model, described by the equation (14) where the rate of change of the inventory level $I(t)$ is of fractional order α and the demand is a fractional polynomial of highest order 2α.

The fractional order differential equation in this case can be solved by using fractional Laplace transform method. In operator form the equation (14) becomes, $D^{\alpha}\left(I(t)\right)=-\left(a+bt^{\alpha}+ct^{2\alpha}\right)$ (37)

Using fractional Laplace transform method on (47) we get the inventory level at time *t* as,

$$I(t)=\left(Q-\frac{at^{\alpha}}{\Gamma(1+\alpha)}-\frac{b\Gamma(1+\alpha)t^{(2\alpha)}}{\Gamma(1+2\alpha)}-\frac{c\ \Gamma(1+2\alpha)t^{(3\alpha)}}{\Gamma(3\alpha+1)}\right) \tag{38}$$

Total order quantity for this fractional order inventory model is obtained as

$$Q=\left(\frac{aT^{\alpha}}{\Gamma(1+\alpha)}+\frac{b\ \Gamma(1+\alpha)T^{(2\alpha)}}{\Gamma(2\alpha+1)}+\frac{c\ \Gamma(1+2\alpha)T^{(3\alpha)}}{\Gamma(3\alpha+1)}\right) \tag{39}$$

After using the boundary condition $I(T)=0$ in the equation (38), inventory level for this fractional order inventory model can be obtained as

$$I(t)=\left(\frac{a}{\Gamma(1+\alpha)}\left(T^{(\alpha)}-t^{(\alpha)}\right)+\frac{b\ \Gamma(1+\alpha)}{\Gamma(2\alpha+1)}\left(T^{(2\alpha)}-t^{(2\alpha)}\right)+\frac{c\ \Gamma(1+2\alpha)}{\Gamma(3\alpha+1)}\left(T^{(3\alpha)}-t^{(3\alpha)}\right)\right) \tag{40}$$

Memory dependent $\beta-$order $(0<\beta\leq 1)$ total inventory holding cost is denoted as $HOC_{\alpha,\beta}(T)$ and defined as

$$HOC_{\alpha,\beta}(T)=C_1\left({}_0D_T^{-\beta}\left(I(t)\right)\right)$$

where ${}_0D_T^{-\beta}$ is used in Riemann-Liouville sense. Thus

$$\begin{aligned}HOC_{\alpha,\beta}(T)&=\frac{C_1}{\Gamma(\beta)}\int_0^T (T-t)^{(\beta-1)}\left(I(t)\right)dt\\ &=\frac{C_1aT^{(\alpha+\beta)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right)+\frac{C_1b\Gamma(\alpha+1)T^{(2\alpha+\beta)}}{\Gamma(1+2\alpha)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(2\alpha+1,\beta)}{\Gamma(\beta)}\right)\\ &\quad+\frac{C_1c\Gamma(1+2\alpha)T^{(3\alpha+\beta)}}{\Gamma(3\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(3\alpha+1,\beta)}{\Gamma(\beta)}\right)\end{aligned} \tag{41}$$

Therefore, the total average cost is

$$\begin{aligned}TOC_{\alpha,\beta}^{av}(T)=&M\left(\frac{aT^{(\alpha-1)}}{\Gamma(\alpha+1)}+\frac{b\Gamma(\alpha+1)T^{(2\alpha-1)}}{\Gamma(1+2\alpha)}+\frac{c\Gamma(1+2\alpha)T^{(3\alpha-1)}}{\Gamma(3\alpha+1)}\right)+\frac{C_1aT^{(\alpha+\beta-1)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right)\\ &+\frac{C_1b\Gamma(\alpha+1)T^{(2\alpha+\beta-1)}}{\Gamma(1+2\alpha)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(2\alpha+1,\beta)}{\Gamma(\beta)}\right)+\frac{2C_1c\Gamma(1+2\alpha)T^{(3\alpha+\beta-1)}}{\Gamma(3\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(3\alpha+1,\beta)}{\Gamma(\beta)}\right)+C_3T^{-1}\end{aligned} \tag{42}$$

Now, we shall consider four cases to study the behavior of this fractional order model

$(i) 0<\alpha\le 1.0$ and $0<\beta\le 1.0$ $(ii) \beta=1.0$ and $0<\alpha\le 1.0$ (iii) $\alpha=1.0$ and $0<\beta\le 1.0, (iv) \alpha=1.0, \beta=1.0$.

(i)Case-1: $0<\beta\le 1.0,\ 0<\alpha\le 1.0$

To find the minimum of $TOC_{\alpha,\beta}^{av}(T)$ we propose the corresponding non-linear programming problem in the following form and solve it by primal geometric programming method.

$$\begin{cases}\text{Min } TOC_{\alpha,\beta}^{av}(T) = AT^{(\alpha-1)} + B_1T^{2\alpha-1} + CT^{3\alpha-1} + DT^{(\alpha+\beta-1)} + ET^{(2\alpha+\beta-1)} + FT^{3\alpha+\beta-1} + GT^{-1} \\ \text{Subject to } T \ge 0\end{cases} \tag{43}$$

where, $A = \dfrac{aM}{\Gamma(1+\alpha)}, B_1 = \dfrac{bM\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}, C = \dfrac{cM\Gamma(1+2\alpha)}{\Gamma(3\alpha+1)}, D = \dfrac{aC_1}{\Gamma(1+\alpha)\Gamma(\beta)}\left(\dfrac{1}{\beta} - B(\alpha+1,\beta)\right),$

$E = \dfrac{bC_1}{\Gamma(1+2\alpha)\Gamma(\beta)}\left(\dfrac{1}{\beta} - B(2\alpha+1,\beta)\right), F = \dfrac{2cC_1}{\Gamma(3\alpha+1)\Gamma(\beta)}\left(\dfrac{1}{\beta} - B(1+3\alpha,\beta)\right), G = C_3$

Primal Geometric Programming Method

In this method we have to first find the dual of the primal problem (43).This is

$$\text{Max } d(w) = \left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B_1}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}\left(\frac{D}{w_4}\right)^{w_4}\left(\frac{E}{w_5}\right)^{w_5}\left(\frac{F}{w_6}\right)^{w_6}\left(\frac{G}{w_7}\right)^{w_7} \tag{44}$$

with the orthogonal and normal conditions

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 = 1 \tag{45}$$

$$(\alpha-1)w_1 + (2\alpha-1)w_2 + (3\alpha-1)w_3 + (\alpha+\beta-1)w_4 + (2\alpha+\beta-1)w_5 + (3\alpha+\beta-1)w_6 - w_7 = 0 \tag{46}$$

Corresponding primal-dual relations are given below as,

$$\left.\begin{aligned} &AT^{\alpha-1} = w_1 d(w), B_1T^{2\alpha-1} = w_2 d(w), CT^{(3\alpha-1)} = w_3 d(w), DT^{(\alpha+\beta-1)} = w_4 d(w) \\ &ET^{(2\alpha+\beta-1)} = w_5 d(w), FT^{(3\alpha+\beta-1)} = w_6 d(w), GT^{-1} = w_7 d(w)\end{aligned}\right\} \tag{47}$$

With the help of the above primal-dual relations (47) we obtain,

$$\left.\left(\frac{B_1w_3}{Cw_2}\right) = \left(\frac{Aw_2}{B_1w_1}\right), \left(\frac{Aw_2}{B_1w_1}\right)^{\frac{2\alpha-\beta}{\alpha}} = \left(\frac{Dw_3}{Cw_4}\right), \left(\frac{Aw_2}{B_1w_1}\right) = \left(\frac{Dw_5}{Ew_4}\right), \left(\frac{Aw_2}{B_1w_1}\right) = \left(\frac{Ew_6}{Fw_5}\right), \left(\frac{Aw_2}{B_1w_1}\right)^{\frac{3\alpha+\beta}{\alpha}} = \left(\frac{Gw_6}{Fw_7}\right)\right\} \tag{48}$$

along with, $T^{\alpha} = \left(\dfrac{Aw_2}{B_1w_1}\right)$ (49)

There are seven non-linear equations (45), (46), and the five equations of (48) with seven unknown $w_1, w_2, w_3, w_4, w_5, w_6, w_7$. Solving these seven nonlinear equations we shall get the optimal values $w_1^*, w_2^*, w_3^*, w_4^*, w_5^*, w_6^*, w_7^*$ and hence optimal ordering interval $T_{\alpha,\beta}^{*}$ and minimized total average cost can be obtained by substituting w_1^{*}, w_2^{*} in (49) and then $TOC_{\alpha,\beta}^{*}$ from (43).

(ii)Case-2: $\beta=1.0,\ 0<\alpha\le 1.0$

Now, total average cost is presented as follows

$$TOC_{\alpha,1}^{av}(T)=\frac{aMT^{(\alpha-1)}}{\Gamma(\alpha+1)}+\frac{bMT^{2\alpha-1}}{\Gamma(\alpha+2)}+\frac{cMT^{(3\alpha-1)}\Gamma(1+2\alpha)}{\Gamma(\alpha+3)}+\frac{C_1aT^{\alpha}}{\Gamma(\alpha+1)}\left(1-B(\alpha+1,1)\right)$$
$$+\frac{C_1bT^{2\alpha}\Gamma(\alpha+1)}{\Gamma(1+2\alpha)}\left(1-B(2\alpha+1,1)\right)+\frac{C_1cT^{3\alpha}\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(1-B(3\alpha+1,1)\right)+C_3T^{-1} \tag{50}$$

Therefore, the fractional order inventory model (43) in this case is,

$$\begin{cases}\text{Min } TOC_{\alpha,1}^{av}(T)=AT^{(\alpha-1)}+B_1T^{(2\alpha-1)}+CT^{(3\alpha-1)}+DT^{\alpha}+ET^{2\alpha}+FT^{3\alpha}+GT^{-1}\\ \text{Subject to } T\ge 0\end{cases} \tag{51}$$

$$A=\frac{aM}{\Gamma(\alpha+1)},B_1=\frac{bM}{\Gamma(\alpha+2)},\ C=\frac{cM\Gamma(1+2\alpha)}{\Gamma(\alpha+3)},D=\frac{C_1a}{\Gamma(\alpha+1)}\left(1-B(\alpha+1,1)\right),$$
$$E=\frac{C_1b\Gamma(\alpha+1)}{\Gamma(1+2\alpha)}\left(1-B(2\alpha+1,1)\right),F=\frac{C_1c\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(1-B(3\alpha+1,1)\right),\ G=C_3$$

In the similar manner as in case (i) of model-II, primal geometric programming algorithm can provide $TOC_{\alpha,1}^{*}$ and $T_{\alpha,1}^{*}$.

(iii) Case-3: $\alpha=1.0, 0<\beta\le 1.0$

In this case, total average cost becomes as follows

$$TOC_{1,\beta}^{av}(T)=\frac{aMT^{0}}{\Gamma(2)}+\frac{bM\Gamma(2)T}{\Gamma(3)}+\frac{2cM\Gamma(3)T^{2}}{\Gamma(4)}+\frac{aC_1T^{\beta}}{\Gamma(2)\Gamma(\beta)}\left(\frac{1}{\beta}-B(2,\beta)\right)$$
$$+\frac{bC_1T^{1+\beta}}{\Gamma(3)\Gamma(\beta)}\left(\frac{1}{\beta}-B(3,\beta)\right)+\frac{2cC_1T^{\beta+2}}{\Gamma(4)\Gamma(\beta)}\left(\frac{1}{\beta}-B(4,\beta)\right)+C_3T^{-1} \tag{52}$$

Therefore, the equation (43) reduces as,

$$\begin{cases}\text{Min } TOC_{1,\beta}^{av}(T)=AT^{o}+B_1T+CT^{2}+DT^{\beta}+ET^{1+\beta}+FT^{2+\beta}+GT^{-1}\\ \text{Subject to } T\ge 0\end{cases} \tag{53}$$

Where, $A=\frac{aMT^{0}}{\Gamma(2)},B_1=\frac{bM\Gamma(2)}{\Gamma(3)},C=\frac{cM\Gamma(3)}{\Gamma(4)},D=\frac{aC_1}{\Gamma(2)\Gamma(\beta)}\left(\frac{1}{\beta}-B(2,\beta)\right),$

$E=\frac{bC_1\Gamma(2)}{\Gamma(3)\Gamma(\beta)}\left(\frac{1}{\beta}-B(3,\beta)\right),F=\frac{C_1c\Gamma(3)}{\Gamma(4)\Gamma(\beta)}\left(\frac{1}{\beta}-B(4,\beta)\right),G=C_3$

In the similar way as in case (i) of model-II, the primal geometric programming algorithm can provide the minimum value of the total average cost $TOC^*_{1,\beta}$ and optimal ordering interval and $T^*_{1,\beta}$.

(iv)Case-4: $\alpha = 1.0\,, \beta = 1.0.$

Therefore, the total average cost is as follows

$$TOC^{av}_{1,1}(T) = \left(\begin{array}{l} \frac{aMT^0}{\Gamma(2)} + \left(\frac{bM\Gamma(2)}{\Gamma(3)} + \frac{C_1 a}{\Gamma(2)}\left(1 - B(2,1)\right) \right) T + \left(\frac{cM\Gamma(3)}{\Gamma(4)} + \frac{bC_1\Gamma(2)}{\Gamma(3)}\left(1 - B(3,1)\right) \right) T^2 \\ + \frac{cC_1\Gamma(3)}{\Gamma(4)}\left(1 - B(4,1)\right) T^3 + C_3 T^{-1} \end{array} \right)$$

Therefore, the generalized inventory model (43) becomes as,

$$\begin{cases} \text{Min } TOC^{av}_{1,1}(T) = AT^{(0)} + B_1 T + CT^2 + DT^3 + ET^{-1} \\ \text{Subject to } \mathrm{T} \ge 0 \end{cases} \tag{54}$$

Where, $A = \frac{aM}{\Gamma(2)} = aM\,, B_1 = \left(\frac{bM\Gamma(2)}{\Gamma(3)} + \frac{C_1 a}{\Gamma(2)}\left(1 - B(2,1)\right) \right) = \left(\frac{bM}{2} + \frac{C_1 a}{2} \right),$

$C = \left(\frac{cM\Gamma(3)}{\Gamma(4)} + \frac{bC_1\Gamma(2)}{\Gamma(3)}\left(1 - B(3,1)\right) \right), D = \frac{cC_1\Gamma(3)}{\Gamma(4)}\left(1 - B(4,1)\right) = \frac{cC_1}{4}, E = C_3.$

In the similar way as in case (i) of model-I, primal geometric programming algorithm can provide $TOC^*_{1,1}(T)$ and $T^*_{1,1}$. Interesting to note that the analytical results of this model coincides with the results of our classical model (9) where $\beta = 1.0,\ \alpha = 1.0$.

3.4.3 Analytic solution of model –III

Here, we consider the fractional order inventory model which is described by the equation (15).The fractional order differential equation (15) can be solved by using Laplace transform method with the initial condition, are given in the problem. In operator form the equation (15) becomes,

$$D^{\alpha}\left(I(t)\right) = -\left(a + bt^m + ct^{2m}\right) \tag{55}$$

where α may be different from m, where α is the memory parameter $0 < \alpha \le 1.0$.

Using fractional Laplace transform method on (55) we get,

$$I(t) = \left(Q - \frac{at^{\alpha}}{\Gamma(1+\alpha)} - \frac{b\Gamma(1+\beta)t^{(\alpha+m)}}{\Gamma(\alpha+m+1)} - \frac{c\ \Gamma(1+2m)t^{(\alpha+m)}}{\Gamma(2m+\alpha+1)} \right) \tag{56}$$

after using the boundary condition $I(T) = 0$ in the equation (56), the total order quantity for this type fractional order inventory model can be obtained as,

$$Q = \left(\frac{aT^{\alpha}}{\Gamma(1+\alpha)} + \frac{b\ \Gamma(1+m)T^{(\alpha+m)}}{\Gamma(\alpha+m+1)} + \frac{c\ \Gamma(1+2m)T^{(\alpha+2m)}}{\Gamma(2m+\alpha+1)} \right) \tag{57}$$

Therefore, fractional inventory level can be written as follows

$$I(t)=\left(a\left(\frac{T^{\alpha}}{\Gamma(1+\alpha)}-\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)+b\left(\frac{\Gamma(1+m)\,T^{(\alpha+m)}}{\Gamma(\alpha+m+1)}-\frac{\Gamma(1+\delta)t^{(\alpha+m)}}{\Gamma(\alpha+m+1)}\right)+c\left(\frac{\Gamma(1+2m)\,T^{(\alpha+2m)}}{\Gamma(2m+\alpha+1)}-\frac{\Gamma(1+2m)t^{(\alpha+m)}}{\Gamma(2m+\alpha+1)}\right)\right) \quad (58)$$

Corresponding memory dependent $\beta-\text{th}$ order inventory holding cost is denoted as $HOC_{\alpha,m,\beta}(T)$ and defined as

$$HOC_{\alpha,m,\beta}(T)=C_1\left({}_0D_T^{-\beta}\left(I(t)\right)\right)=\frac{C_1}{\Gamma(\beta)}\int_0^T(T-t)^{(\beta-1)}\left(I(t)\right)dt$$

$$=\frac{C_1aT^{(\alpha+\beta)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right)+\frac{C_1b\,\Gamma(m+1)T^{(m+\alpha+l)}}{\Gamma(\alpha+m)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+m+1,\beta)}{\Gamma(\beta)}\right)$$

$$+\frac{C_1\,c\,\Gamma(1+2m)T^{(\alpha+\beta)}}{\Gamma(\alpha+2m+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+2m+1,\beta)}{\Gamma(\beta)}\right) \quad (59)$$

Therefore, the total average cost is

$$TOC_{\alpha,\mathrm{m},\beta}(T)=\frac{\left(MQ+HOC_{\alpha,\mathrm{m},\beta}(T)+C_3\right)}{T}$$

$$=\frac{1}{T}\left(\begin{array}{l}M\left(\frac{aT^{\alpha}}{\Gamma(1+\alpha)}+\frac{\left(b\Gamma(1+m)T^{(\alpha+m)}\right)}{\Gamma(\alpha+m+1)}+\frac{\left(c\Gamma(1+2m)T^{(\alpha+2m)}\right)}{\Gamma(\alpha+2m+1)}\right)+\frac{C_1aT^{(\alpha+\beta)}}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right)\\+\frac{C_1b\Gamma(m+1)T^{(m+\alpha+\beta)}}{\Gamma(\alpha+m+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+m+1,\beta)}{\Gamma(\beta)}\right)+\frac{C_1c\,\Gamma(2m+1)T^{(2m+\alpha+\beta)}}{\Gamma(\alpha+2m+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+2m+1,\beta)}{\Gamma(\beta)}\right)+C_3\end{array}\right) \quad (60)$$

Now, we shall consider eight cases to study this type fractional order model as follows

$(i)0<m\le 1.0\ ,0<\alpha\le 1.0,0<\beta\le 1.0,$ $(ii)m=1.0$ *and* $0<\alpha\le 1.0,0<\beta\le 1.0$

$(iii)\ \beta=1.0$ *and* $0<m\le 1.0,0<\alpha\le 1.0,$ $(\mathrm{iv})\alpha=1.0$ *and* $0<m\le 1.0,0<\beta\le 1.0$

$(v)m=\beta=1.0$ *and* $0<\alpha\le 1.0,$ $(vi)m=\alpha=1.0$ *and* $0<\beta\le 1.0,$

$(vii)\alpha=\beta=1.0\ ,0<m\le 1.0,$ $(\mathrm{viii})m=\alpha=\beta=1.0$

(i)Case-1: $0<\alpha\le 1.0\ ,0<m\le 1.0,0<\beta\le 1.0$

To find the minimum value of the total average cost $TOC_{\alpha,\mathrm{m},\beta}^{av}(T)$ we propose the corresponding non-linear programming problem in the following form and solve it using primal geometric programming method, are discussed bellow

$$\text{Min }TOC_{\alpha,m,\beta}^{av}(T)=\begin{cases}\begin{pmatrix}AT^{\alpha-1}+B_1T^{(\alpha+m-1)}+CT^{(\alpha+2m-1)}+DT^{(\alpha+\beta-1)}+ET^{(m+\alpha+\beta-1)}\\+FT^{(2m+\alpha+\beta-1)}+GT^{-1}\end{pmatrix}\\\text{Subject to }T\ge 0\end{cases} \quad (61)$$

where, $A=\frac{\mathrm{Ma}}{\Gamma(\alpha+1)}, B=\frac{\mathrm{Mb}\,\Gamma(m+1)}{\Gamma(m+\alpha+1)}, C=\frac{\mathrm{c\,M}\,\Gamma(2m+1)}{\Gamma(2m+\alpha+1)}, D=\frac{C_1 a}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right),$

$E=\frac{C_1 b\Gamma(m+1)}{\Gamma(m+\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(m+\alpha+1,\beta)}{\Gamma(\beta)}\right), F=\frac{\Gamma(1+2m)C_1 c}{\Gamma(2m+\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(2m+\alpha+1,\beta)}{\Gamma(\beta)}\right), G=C_3.$

Primal Geometric Programming Method

In this method we have to first find the dual of the primal problem (61).This is

$$\text{Max } d(w)=\left(\frac{A}{w_1}\right)^{w_1}\left(\frac{B_1}{w_2}\right)^{w_2}\left(\frac{C}{w_3}\right)^{w_3}\left(\frac{D}{w_4}\right)^{w_4}\left(\frac{E}{w_5}\right)^{w_5}\left(\frac{F}{w_6}\right)^{w_6}\left(\frac{G}{w_7}\right)^{w_7} \tag{62}$$

with the orthogonal and normal conditions are

$$w_1+w_2+w_3+w_4+w_5+w_6+w_7=1 \tag{63}$$

$$(\alpha-1)w_1+(\alpha+m-1)w_2+(\alpha+2m-1)w_3+(\alpha+\beta-1)w_4+(\alpha+m+\beta-1)w_5+(\alpha+2m+\beta-1)w_6-w_7=0 \tag{64}$$

Corresponding, primal- dual relations are given below as

$$\left.\begin{aligned}&AT^{(\alpha-1)}=w_1 d(w), B_1 T^{\alpha+m-1}=w_2 d(w), CT^{(\alpha+2m-1)}=w_3 d(w)\\&CT^{(\alpha+2m-1)}=w_3 d(w), DT^{(\alpha+\beta-1)}=w_4 d(w), ET^{(\alpha+m+\beta-1)}=w_5 d(w), GT^{-1}=w_7 d(w)\end{aligned}\right\} \tag{65}$$

The relations (65) gives the following

$$\frac{B_1 w_3}{Cw_2}=\left(\frac{Aw_2}{B_1 w_1}\right),\left(\frac{Dw_3}{Cw_4}\right)=\left(\frac{Aw_2}{B_1 w_1}\right)^{\left(\frac{2m-\beta}{m}\right)},\left(\frac{Dw_5}{Ew_4}\right)=\left(\frac{Aw_2}{B_1 w_1}\right),\left(\frac{Ew_6}{Fw_5}\right)=\left(\frac{Aw_2}{B_1 w_1}\right),\left(\frac{Aw_2}{B_1 w_1}\right)^{\left(\frac{2m+\beta+\alpha}{m}\right)}=\frac{Gw_6}{Fw_7} \tag{66}$$

along with

$$T^m=\left(\frac{Aw_2}{B_1 w_1}\right) \tag{67}$$

There are seven non-linear equations (63), (64) and the five equations in (66) with seven unknown $w_1, w_2, w_3, w_4, w_5, w_6, w_7$. Solving these seven nonlinear equations we shall get the optimal values $w_1^*, w_2^*, w_3^*, w_4^*, w_5^*, w_6^*, w_7^*$ and hence optimal ordering interval $T_{\alpha,\beta}^*$ and minimized total average cost can be obtained by substituting w_1^*, w_2^* in (67) and then minimized total average cost from (61).

(ii)Case-2: $m=1.0\ ,0<\alpha\le 1.0, 0<\beta\le 1.0.$

Then system (61) reduces to

$$\begin{cases}\text{Min } \mathrm{TOC}_{\alpha,1,\beta}^{av}(T)=AT^{(\alpha-1)}+B_1 T^{(\alpha)}+CT^{(\alpha+1)}+DT^{(\alpha+\beta-1)}+ET^{(\alpha+\beta)}+FT^{(\alpha+\beta+1)}+GT^{-1}\\ \text{Subject to } T\ge 0\end{cases} \tag{68}$$

where, $A=\frac{\mathrm{Ma}}{\Gamma(\alpha+1)}, B_1=\frac{\mathrm{Mb}}{\Gamma(\alpha+2)}, C=\frac{2\mathrm{cM}}{\Gamma(\alpha+3)}, D=\frac{C_1 a}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+1,\beta)}{\Gamma(\beta)}\right),$

$E=\frac{C_1 b}{\Gamma(\alpha+2)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+2,\beta)}{\Gamma(\beta)}\right), F=\frac{2C_1 c}{\Gamma(\alpha+3)}\left(\frac{1}{\Gamma(\beta+1)}-\frac{B(\alpha+3,\beta)}{\Gamma(\beta)}\right), G=C_3$

In the similar manner as in case (i) of model-III, using primal geometric programming algorithm $TOC^{*}_{\alpha,1,\beta}(T)$ and $T^{*}_{\alpha,1,\beta}$ can be found.

(iii) Case-3: $0 < m \le 1.0, 0 < \alpha \le 1.0, \text{and}\ \beta = 1.0.$

For this parametric values the system (61) reduces to

$$\begin{cases} \text{Min TOC}^{av}_{\alpha,\text{m},1}(T) = AT^{(\alpha-1)} + B_1T^{(\alpha+m-1)} + CT^{(\alpha+2m-1)} + DT^{(\alpha)} + ET^{(\alpha+m)} + FT^{(2m+\alpha)} + GT^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \quad (69)$$

where, $\text{A} = \dfrac{\text{Ma}}{\Gamma(\alpha+1)}, B_1 = \dfrac{\text{Mb}\,\Gamma(m+1)}{\Gamma(m+\alpha+1)}, C = \dfrac{\text{cM}\,\Gamma(2m+1)}{\Gamma(2m+\alpha+1)},\ D = \dfrac{C_1a}{\Gamma(\alpha+1)}\left(1 - B(\alpha+1,1)\right),$

$$E = \frac{C_1b\,\Gamma(m+1)}{\Gamma(m+\alpha+1)}\left(1 - B(m+\alpha+1,1)\right), F = \frac{\Gamma(1+2m)C_1c}{\Gamma(2m+\alpha+1)}\left(1 - B(2m+\alpha+1,1)\right),\ G = C_3$$

In the similar way as in case (i) of model-III, primal geometric programming algorithm helps to give the results $TOC^{*}_{\alpha,m,1}(T)$ and $T^{*}_{\alpha,m,1}(T)$.

(iv)Case-4: $\alpha = 1.0\ , 0 < m \le 1.0, 0 < \beta \le 1.0.$

In this case, inventory model (61) can be written as follows

$$\begin{cases} \text{Min TOC}^{av}_{1,\text{m},\beta}(T) = AT^{0} + B_1T^{(\text{m})} + CT^{(2m)} + DT^{(\beta)} + ET^{(m+\beta)} + FT^{(2m+\beta)} + GT^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \quad (70)$$

where $\text{A} = \text{Ma}\ , B_1 = \dfrac{\text{Mb}\,\Gamma(m+1)}{\Gamma(m+2)}, C = \dfrac{\text{cM}\,\Gamma(2m+1)}{\Gamma(2m+2)},\ D = C_1a\left(\dfrac{1}{\Gamma(\beta+1)} - \dfrac{B(2,\beta)}{\Gamma(\beta)}\right)$

$$E = \frac{C_1b\Gamma(m+1)}{\Gamma(m+2)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(m+2,\beta)}{\Gamma(\beta)}\right),\ F = \frac{\Gamma(1+2m)C_1c}{\Gamma(m+2)}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(m+2,\beta)}{\Gamma(\beta)}\right),\ G = C_3$$

Using primal geometric programming algorithm we can find minimized total average cost $TOC^{*}_{1,\text{m},\beta}$ and optimal ordering interval $T^{*}_{1,\text{m},\beta}$ as describe in case-1.

(v) Case-5: $m = \beta = 1.0$ and $0 < \alpha \le 1.0$

Therefore, generalized inventory model (61) reduces as,

$$\begin{cases} \text{Min TOC}^{av}_{1,\alpha,1}(T) = AT^{(\alpha-1)} + B_1T^{(\alpha)} + CT^{(\alpha+1)} + DT^{(\alpha+2)} + ET^{-1} \\ \text{Subject } T \ge 0 \end{cases} \quad (71)$$

where, $\text{A} = \dfrac{\text{Ma}}{\Gamma(\alpha+1)}, B_1 = \dfrac{\text{Mb}}{\Gamma(\alpha+2)}, C = \dfrac{2\text{cM}}{\Gamma(\alpha+3)},\ D = \dfrac{C_1a}{\Gamma(\alpha+1)}\left(1 - B(\alpha+1,1)\right),$

$$E = \frac{C_1b}{\Gamma(\alpha+2)}\left(1 - B(\alpha+2,1)\right),\ F = \frac{2C_1c}{\Gamma(\alpha+3)}\left(1 - B(\alpha+3,1)\right),\ G = C_3$$

In the similar way as in case (i) of model-III, primal geometric programming algorithm can give the minimized total average cost $TOC^{*}_{1,\alpha,1}$ and optimal ordering interval $T^{*}_{1,\alpha,1}$.

(vi) Case-6: $m = 1.0, \alpha = 1.0,\ 0 < \beta \le 1.0$

Therefore, generalized inventory model (61) reduces as,

$$\begin{cases} \text{Min } \mathrm{TOC}_{1,1,\beta}^{av}\left(T\right) = AT^{0} + B_1T + CT^{(2)} + DT^{(\beta)} + ET^{(\beta+1)} + FT^{(\beta+2)} + GT^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \tag{72}$$

$$\text{where}, \mathrm{A} = \mathrm{Ma}, B_1 = \frac{\mathrm{Mb}}{2}, C = \frac{\mathrm{cM}}{3}, D = C_1 a\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(2,\beta)}{\Gamma(\beta)}\right),$$

$$E = \frac{C_1 b}{2}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(3,\beta)}{\Gamma(\beta)}\right), F = \frac{C_1 c}{3}\left(\frac{1}{\Gamma(\beta+1)} - \frac{B(4,\beta)}{\Gamma(\beta)}\right), G = C_3$$

In the similar manner as in case (i) of model-III, primal geometric programming algorithm gives the analytical results of the minimized total average cost $TOC_{1,1,\beta}^{*}$ and optimal ordering interval $T_{1,1,\beta}^{*}$.

(vii)Case-7: $\alpha = \beta = 1.0$ and $0 < m \le 1.0$

In this case, the generalized inventory model (61) becomes as,

$$\begin{cases} MinTOC_{1,\mathrm{m},1}^{av}\left(T\right) = AT^{0} + B_1T^{m} + CT^{2m} + DT + ET^{m+1} + FT^{2m+1} + GT^{-1} \\ \qquad\qquad \text{Subject to } T \ge 0 \end{cases} \tag{73}$$

$$\text{where}, A = \frac{aM}{\Gamma(2)}, B_1 = \frac{bM\Gamma(m+1)}{\Gamma(m+2)}, C = \frac{cM\Gamma(2m+1)}{\Gamma(2m+2)}, D = \frac{C_1 aM}{\Gamma(2)}\left(\frac{1}{\Gamma(2)} - B(2,1)\right)$$

$$E = \frac{C_1 b\Gamma(m+1)}{\Gamma(m+2)}\left(\frac{1}{\Gamma(2)} - B(m+2,1)\right), F = \frac{C_1 c\Gamma(2m+1)}{\Gamma(2m+2)}\left(\frac{1}{\Gamma(2)} - B(2m+2,1)\right), G = C_3$$

In the similar manner as in case (i) primal geometric programming algorithm can give the minimized total average cost $TOC_{1,\mathrm{m},1}^{*}$ and optimal ordering interval $T_{1,\mathrm{m},1}^{*}$.

(viii)Case-8: $\alpha = m = \beta = 1.0$.

In this case, the generalized inventory model (61) becomes as,

$$\begin{cases} \text{Min } \mathrm{TOC}_{1,1,1}^{av}\left(T\right) = AT^{0} + B_1T + CT^{(2)} + DT^{3} + ET^{-1} \\ \text{Subject to } T \ge 0 \end{cases} \tag{74}$$

$$\text{Where}, A = Ma, B_1 = \left(\frac{\mathrm{M}b}{2} + C_1 a\left(1 - B(2,1)\right)\right) = \left(\frac{\mathrm{M}b}{2} + \frac{C_1 a}{2}\right), \ E = C_3$$

$$C = \left(\frac{cM}{3} + \frac{C_1 b}{2}\left(1 - B(3,1)\right)\right) = \left(\frac{cM}{3} + \frac{C_1 b}{3}\right), D = \frac{\Gamma(3)C_1 c}{\Gamma(4)}\left(\frac{1}{\Gamma(2)} - B(4,1)\right) = \frac{C_1 c}{4}$$

In the similar way as in case (i) of model-I, primal geometric programming algorithm can provide the minimized total average cost $TOC_{1,1}^{*}\left(T\right)$ and optimal ordering interval $T_{1,1}^{*}$. Interesting to note that the analytical results of this model coincides with the results of our classical model (9) where $\alpha = m = \beta = 1.0$.

4. Numerical illustrations

(i) To illustrate numerically the developed classical and fractional order inventory model we consider empirical values of the various parameters in proper units as

$a = 40, b = 20,\ c = 2, c_1 = 15, c_3 = 200, U = 300$. The optimal ordering interval, minimized total average cost of the classical inventory model are found 0.2409 units and $1.3643\text{x}10^4$ units respectively using matlab minimization process.

(ii) Here, we provide numerical illustration for the fractional order inventory models considering same parameters as used in classical inventory model.

(a)				(b)				(c)		
α	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$ (**x10^4**)		β	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$ (**x10^4**)		α	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$ (**x10^4**)
0.0951	**3.3023**	**1.3643**		**0.1**	**0.2523**	**1.361**		0.1	3.6215	1.3459
0.1	**3.2830**	**1.3708**		**0.166991**	**0.2507**	**1.3643**		**0.11725**	**3.5369**	**1.3692**
0.2	2.8981	1.5004		0.2	0.2498	1.3653		0.2	3.1492	1.4775
0.3	2.5267	1.6171		0.3	0.2473	1.3673		0.3	2.7123	1.5967
0.4	2.1649	1.7136		0.4	0.2449	1.3683		0.4	2.3007	1.6964
0.5	1.8098	1.7817		**0.5**	**0.2430**	**1.3685**		0.5	1.9072	1.7684
0.6	**1.4597**	**1.8123**		0.6	0.2417	1.3681		**0.6**	**1.5272**	**1.8035**
0.7	1.1145	1.7954		0.7	0.2408	1.3673		0.7	1.1582	1.7915
0.8	0.7778	1.7207		0.8	0.2405	1.3664		0.8	0.8023	1.7214
0.9	0.4669	1.5776		0.9	0.2406	1.3653		0.9	0.4759	1.5817
1.0	**0.2409**	**1.3643**		**1.0**	**0.2409**	**1.3643**		**1.0**	**0.2414**	**1.3692**

Table-2: Optimal ordering interval and minimized total average cost $TOC^*_{\alpha,\beta}$: (a) For $\beta = 1.0$, and α varies from 0.1 to 1.0 as defined in section 3.4.1 (fractional model-**I**) case-**2**, (b)for $\alpha = 1.0$, and β varies from 0.1 to 1.0 as defined insection3.4.1(fractional model-**I**) case-**3**, (c) for $\beta = 0.5$, and α varies from 0.1 to 1.0 as defined in section 3.4.1(fractional model-**I**) case-**1**.

It is clear from the table-2(a) that there is a critical value of the memory parameter (here it is $\alpha = 0.6$), where the minimized total average cost becomes **maximum** and then gradually decreases bellow and above. In such case, low values of α signifies large memory of the inventory problem. Moreover there is another critical point ($\alpha = 0.0951$)at which minimum value of the total average cost becomes equal to the memory less minimized total average cost ($\alpha = 1$) but optimal ordering interval in that case is different. Our analysis shows that for large memory effect, the system needs more time to reach the minimum value of the total average cost taking longer time to sell the optimum lot size compared to the memory less inventory system. Hence, rate of selling for large memory, the system is affected. Therefore, to reach the same profit as in case of memory less system, the shopkeeper should change his business policy such as attitude of public dealing, environment of shop or company, product quality etc. The above described facts happen in real life inventory system which cannot consider in the memory less inventory model.

Initially the business started with reputation with maximum profit minimizing the total average cost. As time goes on, the company starts to lose it's reputation due to various unwanted causes. Accordingly, the company starts to downfall of its business when downfall becomes maximum at $\alpha = 0.6$.Attaining highest at the point, the company changes its business policy and takes care to recover its reputation.

In table-2(b) we presented the optimal ordering interval and minimized total average cost $TOC^*_{\alpha,\beta}$ for $\alpha = 1.0$, and β varies from 0.1 to 1.0. It is clear from the table-2(b) that there is a critical memory value of the memory parameter β (here it is $(\beta = 0.5)$) for which minimized total average cost $TOC^*_{1,\beta}$ becomes **maximum** and then gradually decreases below and above.

When memory parameter $(\alpha = 1.0)$ but another memory parameter(the exponent of holding cost β)varies, optimal ordering interval, minimized total average cost do not carry sensitive difference from large to low memory value of the memory parameter β. For the large memory effect of the memory parameter (β), system does not take significantly more time to reach the minimum value of the total average cost compared to the memory less inventory system. For all memory value of the memory parameter β, minimized total average cost is almost same compared to the memory less system. Practically, β is the memory parameter corresponding inventory holding cost or carrying cost. Here, memory or past experience is considered as bad attitude of the shopkeeper to the transportation driver for the shoes business or cloth business etc. But, in general the transportation driver does not react corresponding bad attitude of the shopkeeper. On the other hand, transportation driver may be bad as a service man i.e. he may not serious his duty. Due to the above reason, the system is affected by the bad service of the transportation but this is not effective too which is also proved from the table-2(b).

In table-2(b) , $TOC^*_{\alpha,\beta}$ shows the similar behavior as in table-2(a) and in this case the ordering time interval $T^*_{\alpha,\beta}$ is less which implies for $\alpha = 1.0$ and fractionally varying β lesser time is required to attain minimum value of the total average cost.

It is clear from the table-2(c) that when both the memory parameters α, β are fractional then there is a critical memory value $(\alpha = 0.6 , \beta = 0.5)$of them for which minimized total average cost becomes **maximum** and then gradually decreases below and above. When exponent of holding cost is fractional $\beta = 0.5$ and another memory parameter (α) varies, minimized total average cost $TOC_{0.11725,0.5}$ is same to $TOC_{1.0,0.5}$ but in the optimal ordering interval there is difference. For the large memory effect of the memory parameter (α), system takes more time to reach the minimum value of the total average cost compared to the low memory effect. It is also observed from table-2(c) that for $\beta = 0.5$ the value of $TOC^*_{\alpha,\beta}$ is maximum at $\alpha = 0.7$ in model-II but it attains maximum at $\alpha = 0.6$ in model-I. This fact happens due to consideration of fractional polynomials in the demand rate in model-II.

Table-2(c) shows very minor change in $TOC^*_{\alpha,\beta}$ from that in table-2(a). In this case, $T^*_{\alpha,\beta}$ is slightly higher than in table-2(a) which implies at same range of α but with fractional β optimal ordering interval is slightly higher i.e. to reach the same value of the minimized total average cost, little higher time is required.
The following table-3(a) and 3(b) has been constructed for model-II, where rate of change of inventory level is of fractional order α and the highest degree of the demand polynomial is also fractional order 2α.
For $\alpha = 1.0$ and β is arbitrary $(0 < \beta \le 1.0)$ fractional, the model-II and model-I are identical. The obtained numerical results are same as given in table-2(b).

(a)				(b)		
α	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$		α	$T^*_{\alpha,\beta}$	$TOC^*_{\alpha,\beta}$
0.1	**1.0000x10^4**	**12.2060**		**0.1**	**675.3851**	**427.1055**
0.2	1.0000x10^4	98.7967		0.2	135.0484	1.8042 x10^3
0.3	4.6832x10^3	983.5621		0.3	46.3908	4.6288x10^3
0.4	203.7677	4.9821x10^3		0.4	18.7019	8.8477x10^3
0.5	21.0506	1.1192x10^4		0.5	7.8573	1.3414x10^4
0.6	5.0603	1.5846x10^4		0.6	**7.2745**	**1.3643x104**
0.7	**1.9656**	**1.7787x10^4**		**0.7**	3.4517	1.6771x10^4
0.8	0.9469	1.7591x10^4		0.8	**1.6676**	**1.8113x10^4**
0.9	0.4797	1.6026x10^4		0.9	0.8742	1.7670x10^4
1.0	**0.2430**	**1.3685x10^4**		**1.0**	0.4621	1.6008x10^4

Table-3: Optimal ordering interval and minimized total average cost $TOC^*_{\alpha,\beta}$ (a) for $\beta = 0.5$, and α varies from 0.1 to 1.0 as defined in section 3.4.2(fractional model-**II**) case-**1**,(b) for $\beta = 1.0$, and α varies from 0.1 to 1.0 as defined in section 3.4.1(fractional model-**II**) case-**2**.

It is also clear from the table-3(a), for large memory effect, the memory parameter α and β minimized total average cost and optimal ordering interval is significantly effective. It can be concluded that for this case, the system takes more time to reach minimum value of the total average cost compared to low memory effect. Hence, in this case for large memory value from $\alpha = 0.3, \beta = 0.5$ to $\alpha = 0.1, \beta = 0.5$ business will take long time to reach the minimized total average cost. The long time implies that there may be some demurrage of inventory.
For α from 0.1to1.0 and $\beta = 0.5$, table-3(a) shows that for low value of α ,$TOC^*_{\alpha,\beta}$ is very low compared to table-2(a) but it reaches to its **maximum** value 1.8113x10^4 units at $\alpha = 0.7$. After which, the total minimized cost is same order as in table-2(a)

but in this case before attaining the maximum value of $TOC^*_{\alpha,\beta}$ optimal ordering interval is very high and it is less after attaining the maximum value. In this case, the model will realistic on or after $\alpha = 0.5, \beta = 0.5$.

Here also the minimized total average cost $TOC^*_{\alpha,1}$ is **maximum** at $\alpha = 0.7$ and gradually decreases below and above. Inclusion of memory in carrying cost has an effect which cannot be neglected. There is a memory value of the memory parameter (here it is $\alpha = 0.5385987, \beta = 1.0$) where minimum value of the total average cost becomes equal to the same to $TOC^*_{1,1}$ but the optimal ordering interval is high in the memory dependent case.

It is also clear from the table-3(b) that there is huge difference between $TOC^*_{0.1,1.0}$ and $TOC^*_{1.0,1.0}$ ($TOC^*_{1.0,1.0} >> TOC^*_{0.1,1.0}$)but the optimal ordering interval $T^*_{0.1,1.0}$ is much higher than $T^*_{1.0,1.0}$ .Hence, for large memory effect, the length of the ordering interval is highly large then the process will continue a long time and consequently there may arise significant effect on the inventory demurrage. For the above reason in the real-life application optimal ordering interval $T^*_{\alpha,\beta}$ should belong to $(0.2409, 7.8753)$ and due to the demurrage of inventory there is less significance of $T^*_{\alpha,\beta} \in (18.7019, 675.3851)$.

Another conclusion can be done from the above that for any value of β (the fractional exponent for holding cost) there is a memory value (here it is $\alpha = 0.5385987$) for which the value of $TOC^*_{\alpha,\beta}$ is same to the minimized total average cost $TOC^*_{1,1}$ but there is a significant difference in the values of optimal ordering interval. It is also clear that the memory parameter α plays significant role to show influence of memory or past history of the system compared to the memory parameter β. It actually occurs in reality because shopkeeper's attitude and quality of the product all are considered for the memory parameter α which has very much power to attack the business compared to the bad or good service of the transportation driver.

The table-3(a) and 3(b) shows that the model-II where demand is fractional, is valid for short memory and in this case both table-3(a) and 3(b) shows that the optimal ordering interval is very high for low memory (here it is $\alpha \le 0.5$).

The following table-4 has been constructed for model-III where rate of change of inventory level is of fractional order α and highest degree of the demand polynomial is also m. (where α and m may be different).

For $m = \beta = 1.0$, and α varies from 0.1 to 1.0 the model-III and model-I are identical. The obtained numerical results are same as given in table-2(a).

For $\alpha = m = 1.0$, and β varies from 0.1 to 1.0 model-III and model-I are identical. The obtained numerical results are same as given in table-2(b).

α	$T^*_{\alpha,\mathrm{m},\beta}$	$TOC^*_{\alpha,\mathrm{m},\beta}$
0.1	**287.6893**	**2.1935x10^3**
0.2	143.7971	3.6595x10^3
0.3	75.1066	5.6814x10^3
0.4	39.9421	8.2434 x10^3
0.5	21.0506	1.1192 x10^4
0.6	**10.6353**	**1.4190 x10^4**
0.62797	**8.6677**	**1.4969 x10^4**
0.7	4.9038	1.6697x10^4
0.8	**1.8994**	**1.8004x10^4**
0.9	0.5687	1.7404x10^4
1.0	**0.1969**	**1.4969 x10^4**

Table-4: Optimal ordering interval and minimized total average cost $TOC^*_{\alpha,\mathrm{m},\beta}$ for $m=\beta=0.5$, and α varies from 0.1 to 1.0 as defined in section3.4.3 (fractional model-**III**) case-**1**.

From the table-4(a), it is obvious that minimized total average cost $TOC^*_{\alpha,\mathrm{m},\beta}$ is **maximum** at the memory value $m=\beta=0.5$ and $\alpha=0.8$ and gradually decreases below and above with respect to the memory parameter α .Here, it has observed that for the low value of the memory parameter, minimized total average cost is less compared to the high memory effect.
Again, it is clear from the table-4(a) that there is a memory value $\alpha=0.62797$ for which the minimum value of the total average cost $TOC^*_{0.62797,0.5,0.5}$ is same to $TOC^*_{1,0.5,0.5}$ but there is a significant difference in the values of the optimal ordering interval. Hence, in this case, we can say that the business is highly attacked by the bad past experience of the system compared to the low memory value. To recover this type bad exogenous effect, businessman should be alert in present or future. It is also shown that for large memory effect, the inventory system takes more time to reach the minimum value of the total average cost compared to low memory effect.
The table-4(a) shows that for low value of α, the optimal ordering interval is low compared to the table-3(b).

5. Conclusion

Due to the property of carrying memory, fractional calculus has become an improvement field of research in science and engineering as well as inventory management system also, because it has significant dependence on its past history. Fractional calculus takes an important role in this study. In this paper, we have formulated three different type of generalized inventory model corresponding to a classical inventory model with nonlinear demand rate. Two memory parameters are considered where one corresponds to the fractional rate of change of inventory level and another corresponds to the exponent of the fractional integration to find out carrying cost of the inventory problem. We have also found the analytic solutions for three models for different ranges of α and β $\left(0<\alpha\le 1; 0<\beta\le 1.0\right)$ (smaller values of α and β corresponds to long memory (close to 0.1) and large values (close to 1.0) corresponds to short memory) with solving the fractional differential equation using fractional Laplace transform technique. From the numerical estimation, we have seen that each case of α and β $\left(0<\alpha\le 1,\ \beta=1.0; \alpha=1.0,\ 0<\beta\le 1.0; \alpha \text{ and } \beta \text{ both fractional}\right)$, the minimized total average cost increases to attain a maximum value at $\alpha=0.6$ or $\alpha=0.7$ and then falls down. Short memory works more effectively when demand rate is a fractional polynomial. On the other hand, for quadratic type demand rate long memory dominates. This implies the fractional order inventory model with fractional demand polynomial works effectively for a business which has been newly started. Hence, our second and third model is more effective as well as efficient in this situation. On the other hand, for an already established business with long memory, our first model is more effective. More works with practical information need to be carried out for future aspects.

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