Strong Convergence Theorem of Split Equality Fixed Point for Nonexpansive Mappings in Banach Spaces

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Abstract

The purpose of this paper is to propose an algorithm to solve the split equality fixed point problem of nonexpansive mappings in $p$-uniformly convex and uniformly smooth Banach spaces. The strong convergence theorem of the iterative scheme proposed in this paper is obtained without the assumption of semi-compactness on the mappings. The results presented in the paper are new and extend some recent corresponding results.

Mathematics Subject Classifications: 47H09, 47J25

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1 Introduction

The split feasibility problem (SEP) was originally put forward in finite-dimensional Hilbert spaces by Censor and Elfving [1] for modeling inverse problems which arise...
from phase retrievals and in medical image reconstruction [2]. Its precise mathematical formulation is as follows: Let $C$ and $Q$ be nonempty closed and convex subsets of real Hilbert spaces $H_1$ and $H_2$ respectively. The split feasibility problem (SEP) is to find a point $x^* \in C$ such that:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q,$$

where $A : H_1 \rightarrow H_2$ is a bounded linear operator. We use $\Omega$ to denote the set of solution of $SEP(1.1)$, that is $\Omega = \{x^* \in C : Ax^* \in Q\}$. The split feasibility problem was studied extensively due to it is an extremely powerful tool in various disciplines such as signal processing, image restoration, computer tomograph and radiation therapy treatment planning, for details see [3-5]. Recently, Byrne developed split feasibility problem (1.1) in the setting of infinite-dimensional Hilbert spaces, see also [6-8] and the references therein.

If $C$ and $Q$ are the sets of fixed points of two nonlinear mappings, respectively, and $C$ and $Q$ are nonempty closed convex subsets, then the problem (1.1) is said to be a split common fixed point problem for the two nonlinear mappings. That is, the split common fixed point problem (SCEP) for mappings $S$ and $T$ is to find a point $x^* \in H_1$ with the property:

$$x^* \in F(S) \quad \text{and} \quad Ax^* \in F(T),$$

where $S : H_1 \rightarrow H_1$, $T : H_2 \rightarrow H_2$ are two nonlinear operators, $F(S)$ and $F(T)$ denote the fixed point sets of $S$ and $T$, respectively.

In [9], Moudafi proposed split equality problem (SEP) which is formulated as finding points $x^*$ and $y^*$ with the property:

$$x^* \in C \quad \text{and} \quad y^* \in Q, \quad \text{such that} \quad Ax^* = By^*,$$

where $C$ and $Q$ are two nonempty closed convex subsets of $H_1$ and $H_2$, respectively, $A : H_1 \rightarrow H_3$, $B : H_2 \rightarrow H_3$ are two bounded linear operators and $H_1$, $H_2$, $H_3$ be real Hilbert spaces.

When $C := F(S)$, $Q := F(T)$ (where $S : H_1 \rightarrow H_3$, $T : H_2 \rightarrow H_3$ be two nonlinear operators with nonempty fixed point sets $F(S)$ and $F(T)$), then split equality problem (1.3) is called split equality fixed point problem (SEFP), that is:

$$Finding \quad x^* \in F(S) \quad \text{and} \quad y^* \in F(T), \quad \text{such that} \quad Ax^* = By^*,$$

which is a generalization of the split feasibility problem and the split common fixed point problem. It allows asymmetric and partial relations between the variables $x^*$ and $y^*$. It includes many situations in the fields of science, such as decomposition methods for PDE’s, applications in game theory and in intensity modulated radiation therapy (IMRT) for further details, the interested reader is referred to [3-4,10]. We use $\Omega$ to denoted the set of solutions of the split equality fixed point problem (1.4), that is $\Omega = \{(x^*, y^*) : x^* \in F(s) : y^* \in F(T) \quad \text{such that} \quad Ax^* = By^*\}$. 
Concerning the strong and weak convergence theorems for finding a solution of split feasibility problems have been studied by many authors in Hilbert spaces (see, e.g., [11-13]). There is, however, seldom results in the existing literature on split feasibility problems in the setting of two Banach spaces. In 2015, Takahashi and Yao [14] by using hybrid methods and under suitable conditions, obtained some strong and weak convergence theorems for the split feasibility problem and split common null point problem in the setting of one Hilbert space and one Banach space. Then, Tang et al. [15] proved a weak convergence theorem and a strong convergence theorem for split common fixed point problem involving a quasi-strict pseudo contractive mapping and an asymptotical nonexpansive mapping in the setting of two Banach spaces under the assumption of semi-compacteness on the mappings. Motivated by the works of Tang et al., Ma et al. [16] proposed the following iterative algorithm to approximate a solution of the split equality common fixed point problems of two asymptotically nonexpansive semigroups in the setting of two Banach spaces. For any given \(x_0 \in E_1\) and \(y_0 \in E_2\), the sequence \(\{(x_n, y_n)\}\) is defined as follows:

\[
\begin{align*}
  z_n &\in J_2( Ax_n - By_n) \\
  u_n &\in S(t_n)( x_n - \gamma_n J_1^{-1} A^* z_n) \\
  v_n &\in T(t_n)( y_n + \gamma_n J_2^{-1} B^* z_n) \\
  y_{n+1} &\beta_n y_n + (1 - \beta_n)(y_n + \gamma_n J_2^{-1} B^* z_n) \\
  x_{n+1} &\beta_n x_n + (1 - \beta_n)(x_n - \gamma_n J_1^{-1} A^* z_n),
\end{align*}
\]

Under some suitable conditions, strong and weak convergence theorems are established. In [17], Y. Shehu et al. suggested the following iterative scheme to study split feasibility problems and fixed point problems concerning left Bregman strongly relatively nonexpansive mappings in \(p\)-uniformly convex and uniformly smooth Banach spaces.

\[
\begin{align*}
  x_n &\in \Pi_C J_{E_1}^q [J_{E_2}^p u_n - t_n A^* J_{E_2}^p (A u_n - P_Q(A u_n))] \\
  u_{n+1} &\in \Pi_C J_{E_1}^q [\alpha_n J_{E_1}^p u + (1 - \alpha_n) J_{E_1}^p (T x_n)].
\end{align*}
\]

Very recently, Zhou et al. [18] constructed the following iterative scheme to solve split equality problem and proved the strong convergence in \(p\)-uniformly convex and uniformly smooth Banach spaces. For a fixed \(u \in C\) and a fixed \(v \in Q\), the sequence \(\{(x_n, y_n)\}\) is generated by

\[
\begin{align*}
  u_n &\in J_{E_1}^q [J_{E_1}^p x_n - \lambda_n A^* J_{E_2}^p (A x_n - B y_n)] \\
  v_n &\in J_{E_2}^q [J_{E_2}^p y_n + \lambda_n B^* J_{E_2}^p (A x_n - B y_n)] \\
  x_{n+1} &\in \Pi_C J_{E_1}^q (\beta_n J_{E_1}^p u + (1 - \beta_n) J_{E_1}^p u_n) \\
  y_{n+1} &\in \Pi_Q J_{E_2}^q (\beta_n J_{E_2}^p v + (1 - \beta_n) J_{E_2}^p v_n),
\end{align*}
\]

In this paper, Motivated by works above, we propose the following new iterative algorithm to solve split equality fixed point problem (1.4) involved in left Bregman strongly nonexpansive mappings in \(p\)-uniformly convex and uniformly smooth Banach spaces.
spaces. For a fixed $u \in E_1$ and a fixed $v \in E_2$, the sequence $\{(x_n, y_n)\}$ is generated by
\[
\begin{align*}
  u_n &= \Pi_{C_{E_1}}[J_{E_1}^p x_n - \gamma_n A^* J_{E_1}^p (Ax_n - By_n)] \\
  v_n &= \Pi_{Q_{E_2}}[J_{E_2}^p y_n + \gamma_n B^* J_{E_2}^p (Ax_n - By_n)] \\
  x_{n+1} &= \Pi_{C_{E_1}}(\alpha_n J_{E_1}^p u_n + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_1}^p Su_n)) \\
  y_{n+1} &= \Pi_{Q_{E_2}}(\alpha_n J_{E_2}^p v_n + (1 - \alpha_n)(\beta_n J_{E_2}^p v_n + (1 - \beta_n)J_{E_2}^p Tv_n)),
\end{align*}
\]
(1.8)

The strong convergence theorem of the iterative scheme proposed is obtained without the assumption of semi-compactness on the mappings. The results presented in this paper are new and improve and extend [16-18] results.

## 2 Preliminaries

Let $E$ be a real Banach spaces and let $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. The modulus of convexity $\delta_E : [0, 2] \to [0, 1]$ is defined by
\[
\delta_E(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\}.
\]
$E$ is called uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$; $p$-uniformly convex if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p \epsilon^p$ for any $\epsilon \in (0, 2]$. The modulus of smoothness $\rho_E(\tau) : [0, \infty) \to [0, \infty)$ is defined by
\[
\rho_E(\tau) = \{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\}.
\]
$E$ is called uniformly smooth if $\lim_{n \to \infty} \frac{\rho_E(\tau)}{\tau} = 0$; $E$ is called $q$-uniformly smooth if there is a $C_q > 0$ so that $\rho_E(\tau) \leq C_q \tau^q$ for any $\tau > 0$. It is well know that if $E$ is $p$-uniformly convex and uniformly smooth, then its dual space $E^*$ is $q$-uniformly smooth and uniformly convex. In this situation, the duality mapping $J_{E_1}^p$ is one-to-one, single-valued and satisfies $J_{E_1}^p = (J_{E_1}^{p*})^{-1}$, where $J_{E_1}^{p*}$ is the duality mapping of $E^*$.

**Definition 2.1.** ([19]) The duality mapping $J_{E_1}^p : E \to 2^{E_1^*}$ is defined by
\[
J_{E_1}^p(x) = \{x^* \in E_1^* : \langle x, x^* \rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\}.
\]
The duality mapping $J_{E_1}^p$ is said to be weak-to-weak continuous if
\[
x_n \rightharpoonup x \Rightarrow \langle J_{E_1}^p x_n, y \rangle \to \langle J_{E_1}^p x, y \rangle, \forall y \in E.
\]

**Lemma 2.2.** ([20]) Let $x, y \in E$. If $E$ is $q$-uniformly smooth, then there is a $C_q > 0$ such that
\[
\|x - y\|^q \leq \|x\|^q - q \langle y, J_{E_1}^p(x) \rangle + C_q \|y\|^q.
\]
**Definition 2.3.** Given a Gâteaux differentiable convex function \( f : E \rightarrow \mathbb{R} \). The Bregman distance with respect to \( f \) is defined as:

\[
\Delta_f(x, y) := f(y) - f(x) - \langle f'(x), y - x \rangle, \quad x, y \in E.
\]

It is well known that the duality mapping \( J_p^E \) is the derivative of the function \( f_p^E(x) = \frac{1}{p} \|x\|^p \). For sake of convenience, the \( \Delta_{f_p^E}(x, y) \) denotes by \( \Delta_p(x, y) \), then the Bregman distance with respect to \( f_p^E \) can be written as follows

\[
\Delta_{f_p^E}(x, y) = \frac{1}{q} (\|x\|^p - \|y\|^p) - \langle J_p^E x - J_p^E y, y \rangle.
\]

From the definition of \( \Delta_p(\cdot, \cdot) \), we get

\[
\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_p^E x - J_p^E z \rangle, \quad (2.1)
\]

and

\[
\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_p^E x - J_p^E y \rangle, \quad (2.2)
\]

for any \( x, y, z \in E \).

Likewise the definition of metric projection, the Bregman projection is defined as follows:

\[
\Pi_C x = \arg\min_{y \in C} \Delta_p(x, y), \quad x \in E,
\]

is the unique minimizer of the Bregman distance (see[21]). The Bregman projection can also be characterized by a variational inequality:

\[
\langle J_p^E x - J_p^E(\Pi_C x), z - \Pi_C x \rangle \leq 0, \forall z \in C, \quad (2.3)
\]

from which one has

\[
\Delta_p(\Pi_C x, z) \leq \Delta_p(x, z) - \Delta_p(x, \Pi_C x), \forall z \in C. \quad (2.4)
\]

Let \( C \) be a convex subset of \( \text{int dom} f_p \), where \( f_p = \left( \frac{1}{p} \right) \|x\|^p \), \( 2 \leq p < \infty \) and let \( T \) be a self-mapping of \( C \). A point \( p \in C \) is said to be an asymptotic fixed point of \( T \) if \( C \) contains a sequence \( \{x_n\}_{n=1}^\infty \) which converges weakly to \( p \) and \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \). The set of asymptotic fixed points of \( T \) is denoted by \( \widehat{F}(T) \).

Recalling that the Bregman distance is not symmetric, we define the following operators.

**Definition 2.4.** A mapping \( T : C \rightarrow C \) with a nonempty asymptotic fixed point set is said to be:

(i) left Bregman strongly nonexpansive (see[22]) with respect to a nonempty \( \widehat{F}(T) \) if

\[
\Delta_p(Tx, p) \leq \Delta_p(x, p), \quad \forall x \in C, \quad p \in \widehat{F}(T)
\]
and if whenever \( \{x_n\} \subset C \) is bounded, \( p \in \widehat{F}(T) \) and
\[
\lim_{n \to \infty} (\Delta_p(x_n, p) - \Delta_p(Tx_n, p)) = 0,
\]
it follows that
\[
\lim_{n \to \infty} \Delta_p(x_n, Tx_n) = 0.
\]

According the Martin-Marquez et al.[22], a left Bregman strongly nonexpansive mapping \( T \) with respect to a nonempty \( \widehat{F}(T) \) is called strictly left Bregman strongly nonexpansive mapping.

(ii) An operator \( T : C \to \text{int dom} f \) is said to be: left Bregman firmly nonexpansive (LBFNE) if
\[
\langle J_p^E(Tx) - J_p^E(Ty), Tx - Ty \rangle \leq \langle J_p^E(Tx) - J_p^E(Ty), x - y \rangle
\]
for any \( x, y \in C \), or equivalently,
\[
\Delta_p(Tx, Ty) + \Delta_p(Ty, Tx) + \Delta_p(x, Tx) + \Delta_p(y, Ty) \leq \Delta_p(x, Ty) + \Delta_p(y, Tx).
\]

It is known every left Bregman firmly nonexpansive mapping is left Bregman strongly nonexpansive with respect to \( F(T) = \widehat{F}(T) \).

**Lemma 2.5.** ([23]) Let \( x, y \in E \). If \( E \) is \( p \)-uniformly convex space, the metric and Bregman distance has the following relation:
\[
\tau \|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_p^E x - J_p^E y \rangle.
\]
where \( \tau > 0 \) is some fixed number. Obviously, if \( \{x_n\} \) and \( \{y_n\} \) are both bounded sequences of a \( p \)-uniformly convex and uniformly smooth space \( E \), then \( x_n - y_n \to 0 \) as \( n \to \infty \) implies that \( \Delta_p(x_n, y_n) \to 0 \) as \( n \to \infty \). Following [24,25], the function \( V_p : E^* \times E \to [0, +\infty) \) associated with \( f_p \), which is defined by
\[
V_p(\bar{x}, x) = \frac{1}{q} \|\bar{x}\|^q - \langle \bar{x}, x \rangle + \frac{1}{p} \|x\|^p, \forall x \in E, \bar{x} \in E^*.
\]
Then \( V_p \) is nonnegative and
\[
V_p(\bar{x}, x) = \Delta_p(J_{E^*}^p(\bar{x}), x) = \Delta_p(J_E^q(\bar{x}), x), \tag{2.5}
\]
for all \( x \in E \) and \( \bar{x} \in E^* \). Moreover, by the subdifferential inequality,
\[
V_p(\bar{x}, x) + \langle \bar{y}, J_{E^*}^*(\bar{x}) - x \rangle \leq V_p(\bar{x} + \bar{y}, x). \tag{2.6}
\]
for all \( x \in E \) and \( \bar{x}, \bar{y} \in E^* \)(see [16], [22]). In addition, \( V_p \) is convex in the first variable. Thus, for all \( z \in E \)
\[
\Delta_p(J_{E^*}^q(\sum_{i=1}^N t_i J_{E^*}^p x_i), z) \leq \sum_{i=1}^N t_i \Delta_p(x_i, z). \tag{2.7}
\]
Lemma 2.6. ([20]) Let \( \{a_n\} \) be a sequence of nonnegative real numbers satisfying the following inequality:
\[
a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 1,
\]
where, \( \{\alpha_n\}, \{\sigma_n\}, \{\gamma_n\} \) satisfy the following conditions: (i) \( \{\alpha_n\} \subset [0, 1] \), \( \sum \alpha_n = \infty \); (ii) \( \limsup \sigma_n < 0 \); (iii) \( \gamma_n \geq 0 \) (\( n \geq 1 \)), \( \sum \gamma_n < \infty \). Then \( a_n \to 0 \) as \( n \to \infty \).

Lemma 2.7. ([26]) Let \( \{a_n\} \) be a sequence of real numbers such that there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that \( a_{n_i} < a_{n_{i+1}} \) for all \( i \in N \). Then there exists a nondecreasing sequence \( \{m_k\} \subset N \) such that \( m_k \to \infty \) and the following properties are satisfied by all (sufficiently large) numbers \( k \in N \):
\[
a_{m_k} \leq a_{m_{k+1}} \quad \text{and} \quad a_k \leq a_{m_k+1}.
\]
In fact, \( m_k = \max\{j \leq k : a_j < a_{j+1}\} \).

3 Main Results

Theorem 3.1. Let \( E_1, E_2 \) and \( E_3 \) be three \( p \)-uniformly convex and uniformly smooth Banach spaces. Let \( C \) and \( Q \) be nonempty closed and convex subsets of \( E_1 \) and \( E_2 \), respectively. Let \( A : E_1 \to E_3, B : E_2 \to E_3 \) be two bounded linear operators and \( A^* : E_3^* \to E_1^* \), \( B^* : E_3^* \to E_2^* \) be the adjoint operators of \( A \) and \( B \), respectively. Let \( S \) be a left Bregman strongly nonexpansive mapping of \( C \) into itself such that \( F(S) = \overline{F}(S) \), \( T \) be a left Bregman strongly nonexpansive mapping of \( Q \) into itself such that \( F(T) = \overline{F}(T) \). For a fixed \( u \in E_1 \) and a fixed \( v \in E_2 \), the sequence \( \{(x_n, y_n)\} \) is generated by
\[
\begin{align*}
  u_n &= \Pi_{C} J_{E_1}^{\alpha_n \gamma_n [J_{E_1}^{\alpha_n \gamma_n (A x_n - B y_n)}]} x_n - \gamma_n A^* J_{E_3}^{\alpha_n \gamma_n (A x_n - B y_n)} (A x_n - B y_n) \\
  v_n &= \Pi_{Q} J_{E_2}^{\alpha_n \gamma_n [J_{E_2}^{\alpha_n \gamma_n (A x_n - B y_n)}]} y_n + \gamma_n B^* J_{E_3}^{\alpha_n \gamma_n (A x_n - B y_n)} (A x_n - B y_n) \\
  x_{n+1} &= \Pi_{C} J_{E_1}^{\alpha_n \gamma_n [J_{E_1}^{\alpha_n \gamma_n (A x_n - B y_n)}]} (u_n + (1 - \alpha_n)(\beta_n J_{E_1}^{\alpha_n \gamma_n (A x_n - B y_n)} u_n + (1 - \beta_n) J_{E_1}^{\alpha_n \gamma_n (A x_n - B y_n)} S u_n)) \\
  y_{n+1} &= \Pi_{Q} J_{E_2}^{\alpha_n \gamma_n [J_{E_2}^{\alpha_n \gamma_n (A x_n - B y_n)}]} (v_n + (1 - \alpha_n)(\beta_n J_{E_2}^{\alpha_n \gamma_n (A x_n - B y_n)} v_n + (1 - \beta_n) J_{E_2}^{\alpha_n \gamma_n (A x_n - B y_n)} T v_n)),
\end{align*}
\]  

(3.1)

where the following conditions are satisfied:
(i) \( \{\beta_n\} \subset (0, 1) \); 
(ii) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \); 
(iii) \( 0 < 2L < \gamma_n \leq M \), where \( M = \max\{(\frac{q}{\lambda A})^{\frac{1}{q-1}}, (\frac{q}{\lambda B})^{\frac{1}{q-1}}\} \), and \( L = \max\{\frac{C_3(\lambda A A)}{q}, \frac{C_3(\lambda B B)}{q}\} \).

If the \( \Omega \) is nonempty, then the sequence \( \{(x_n, y_n)\} \) converges strongly to a solution \( SEFP(1.4) \).

Proof. Let \( (x, y) \in \Omega, e_n := Ax_n - By_n, \ v_n := J_{E_1}^{\alpha_n \gamma_n [J_{E_1}^{\alpha_n \gamma_n (A x_n - B y_n)}]} \) and \( \omega_n := J_{E_2}^{\alpha_n \gamma_n [J_{E_2}^{\alpha_n \gamma_n (A x_n - B y_n)}]} (A x_n - B y_n), \forall n \geq 1 \). From (3.1), we have
\[ \Delta_p(u_n, x) \leq \Delta_p(\varepsilon_n, x) \]
\[ = \Delta_p(J_{E_1}^p J_{E_2}^p (Ax_n - By_n), x) \]
\[ = \frac{1}{q} \| J_{E_1}^p x_n - \gamma_n A^* J_{E_3}^p (Ax_n - By_n) \|^q - \langle J_{E_1}^p x_n, x \rangle \]
\[ + \gamma_n \langle J_{E_3}^p (Ax_n - By_n), Ax \rangle + \frac{1}{p} \| x \|^p \]
\[ \leq \frac{1}{q} \| J_{E_1}^p x_n \|^q - \gamma_n \langle Ax_n, J_{E_3}^p e_n \rangle + \frac{C_q(\gamma_n \| A \|)^q}{q} \| J_{E_3}^p e_n \|^q \]
\[ - \langle J_{E_1}^p x_n, x \rangle + \gamma_n \langle J_{E_3}^p e_n, Ax - Ax_n \rangle \]
\[ + \frac{C_q(\gamma_n \| A \|)^q}{q} \| e_n \|^p \]
\[ = \Delta_p(x_n, x) + \gamma_n \langle J_{E_3}^p e_n, Ax - Ax_n \rangle + \frac{C_q(\gamma_n \| A \|)^q}{q} \| e_n \|^p. \] (3.2)

By similar steps as in (3.2), we have

\[ \Delta_p(v_n, y) \leq \Delta_p(\omega_n, y) \leq \Delta_p(y_n, y) + \gamma_n \langle J_{E_3}^p e_n, By_n - By \rangle + \frac{C_q(\gamma_n \| B \|)^q}{q} \| e_n \|^p. \] (3.3)

Adding (3.2) and (3.3), since \( Ax = By \), and noting the assumption on condition (iii), we arrive at

\[ \Delta_p(u_n, x) + \Delta_p(v_n, y) \leq \Delta_p(\varepsilon_n, x) + \Delta_p(\omega_n, y) \leq \Delta_p(x_n, x) + \Delta_p(y_n, y) - \gamma_n \langle J_{E_3}^p e_n, e_n \rangle + 2L \| e_n \|^p \]
\[ = \Delta_p(x_n, x) + \Delta_p(y_n, y) - \gamma_n \| e_n \|^p + 2L \| e_n \|^p \]
\[ = \Delta_p(x_n, x) + \Delta_p(y_n, y) - (\gamma_n - 2L) \| e_n \|^p \]
\[ \leq \Delta_p(x_n, x) + \Delta_p(y_n, y). \] (3.4)

From (3.1) and (2.7), we have

\[ \Delta_p(x_{n+1}, x) \leq \Delta_p(J_{E_1}^p (\alpha_n J_{E_2}^p u + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_3}^p S u_n)), x) \]
\[ \leq \alpha_n \Delta_p(u, x) + (1 - \alpha_n) \beta_n \Delta_p(u_n, x) + (1 - \alpha_n)(1 - \beta_n) \Delta_p(S u_n, x) \]
\[ \leq \alpha_n \Delta_p(u, x) + (1 - \alpha_n) \beta_n \Delta_p(u_n, x) + (1 - \alpha_n)(1 - \beta_n) \Delta_p(u_n, x) \]
\[ = \alpha_n \Delta_p(u, x) + (1 - \alpha_n) \Delta_p(u_n, x). \] (3.5)

Following similar process as in (3.5), we obtain

\[ \Delta_p(y_{n+1}, y) \leq \alpha_n \Delta_p(v, y) + (1 - \alpha_n) \Delta_p(v_n, y). \] (3.6)

Adding (3.5) and (3.6), we can get

\[ \Delta_p(x_{n+1}, x) + \Delta_p(y_{n+1}, y) \leq (1 - \alpha_n) [\Delta_p(u_n, x) + \Delta_p(v_n, y)] + \alpha_n [\Delta_p(u, x) + \Delta_p(v, y)]. \] (3.7)
Inserting (3.4) into (3.7) yields
\[ \Delta_p(x_{n+1}, x) + \Delta_p(y_{n+1}, y) \leq (1 - \alpha_n)[\Delta_p(x_n, x) + \Delta_p(y_n, y)] + (1 - \alpha_n)(\gamma_n - 2L)\|e_n\|^p + \alpha_n[\Delta_p(u, x) + \Delta_p(v, y)] \]
\[ \leq (1 - \alpha_n)[\Delta_p(x_n, x) + \Delta_p(y_n, y)] + \alpha_n[\Delta_p(u, x) + \Delta_p(v, y)]. \] (3.8)

Now, by setting \( \Gamma_n(x, y) = \Delta_p(x_n, x) + \Delta_p(y_n, y) \), it follows from (3.8) that
\[ \Gamma_{n+1}(x, y) \leq (1 - \alpha_n)\Gamma_n(x, y) + \alpha_n(\Delta_p(u, x) + \Delta_p(v, y)) \]
\[ \leq \max\{\Gamma_n(x, y), (\Delta_p(u, x) + \Delta_p(v, y))\} \]
\[ \vdots \]
\[ \leq \max\{\Gamma_1(x, y), (\Delta_p(u, x) + \Delta_p(v, y))\}. \] (3.9)

Therefore, \( \{\Gamma_n(x, y)\} \) is bounded. Since \( \Delta_p(x_n, x) \leq \Gamma_n(x, y) \), \( \Delta_p(y_n, y) \leq \Gamma_n(x, y) \) and We know that \( \Delta_p(x_n, x) \), \( \Delta_p(y_n, y) \) are bounded. Consequently \( \{x_n\}, \{y_n\}, \{u_n\}, \{v_n\}, \{Su_n\}, \{Tv_n\} \) are bounded.

Now we divide the rest of the proof into two case.

**Case 1.** Let \((\overline{x}, \overline{y}) \in \Omega\). Assume that \( \Gamma_n(\overline{x}, \overline{y}) \) is monotonically decreasing. Obviously \( \{\Gamma_n(\overline{x}, \overline{y})\} \) is convergent as \( n \to \infty \).

From (3.8) and condition (ii) we know that
\[ 0 \leq (1 - \alpha_n)(\gamma_n - 2L)\|e_n\|^p \leq (1 - \alpha_n)[\Delta_p(x_n, \overline{x}) + \Delta_p(y_n, \overline{y})] - [\Delta_p(x_{n+1}, \overline{x}) + \Delta_p(y_{n+1}, \overline{y})] + \alpha_n[\Delta_p(u, \overline{x}) + \Delta_p(v, \overline{y})] \]
\[ = (1 - \alpha_n)\Gamma_n(\overline{x}, \overline{y}) - \Gamma_{n+1}(\overline{x}, \overline{y}) + \alpha_n[\Delta_p(u, \overline{x}) + \Delta_p(v, \overline{y})] \]
\[ \leq \Gamma_n(\overline{x}, \overline{y}) - \Gamma_{n+1}(\overline{x}, \overline{y}) + \alpha_n[\Delta_p(u, \overline{x}) + \Delta_p(v, \overline{y})], \]
so, \( \|e_n\|^p \to 0 \) as \( n \to \infty \), which implies that
\[ \lim_{n \to \infty} \|Ax_n - By_n\| = 0. \] (3.10)

By (3.4) and (3.7), we have
\[ \Delta_p(u_{n+1}, \overline{x}) + \Delta_p(v_{n+1}, \overline{y}) - [\Delta_p(u_n, \overline{x}) + \Delta_p(v_n, \overline{y})] \]
\[ \leq \Delta_p(x_{n+1}, \overline{x}) + \Delta_p(y_{n+1}, \overline{y}) - [\Delta_p(u_n, \overline{x}) + \Delta_p(v_n, \overline{y})] \]
\[ \leq -\alpha_n[\Delta_p(u_n, \overline{x}) + \Delta_p(v_n, \overline{y})] + \alpha_n[\Delta_p(u, \overline{x}) + \Delta_p(v, \overline{y})] \to 0, \quad n \to \infty. \] (3.11)

Let \( s_n := J_{E_1}^p(\beta_n J_{E_1}^p u_n + (1 - \beta_n) J_{E_1}^p Su_n) \), \( l_n := J_{E_2}^p(\beta_n J_{E_2}^p v_n + (1 - \beta_n) J_{E_2}^p Tv_n) \),
then
\[ x_{n+1} = \Pi_{C \cap J_{E_1}^p(\alpha_n J_{E_1}^p u_n + (1 - \alpha_n) J_{E_1}^p s_n)}, \quad y_{n+1} = \Pi_{C \cap J_{E_2}^p(\alpha_n J_{E_2}^p v_n + (1 - \alpha_n) J_{E_2}^p l_n)}, \]
Further, we obtain that
\[ \Delta_p(x_{n+1}, \overline{x}) + \Delta_p(y_{n+1}, \overline{y}) \]
Therefore, it follows from (3.15), (3.16) that
\[
\Delta_p(J_{E_1}^p (\alpha_n J_{E_1}^p u + (1-\alpha_n) J_{E_1}^p s_n), \bar{x}) + \Delta_p(J_{E_2}^p (\alpha_n J_{E_2}^p v + (1-\alpha_n) J_{E_2}^p l_n), \bar{y}) \\
\leq \alpha_n \Delta_p(u, \bar{x}) + (1-\alpha_n) \Delta_p(s_n, \bar{x}) + \alpha_n \Delta_p(v, \bar{y}) + (1-\alpha_n) \Delta_p(l_n, \bar{y}),
\]
and
\[
\Delta_p(s_n, \bar{x}) = \Delta_p(J_{E_1}^p (\beta_n J_{E_1}^p u_n + (1-\beta_n) J_{E_1}^p S u_n), \bar{x}) \\
\leq \beta_n \Delta_p(u_n, \bar{x}) + (1-\beta_n) \Delta_p(S u_n, \bar{x}) \\
\leq \beta_n \Delta_p(u_n, \bar{x}) + (1-\beta_n) \Delta_p(u_n, \bar{x}) \\
= \Delta_p(u_n, \bar{x}).
\]
Similarly
\[
\Delta_p(l_n, \bar{x}) \leq \Delta_p(v_n, \bar{y}).
\]
from (3.4), (3.11), (3.12), (3.13), (3.14), we have
\[
0 \leq \Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y}) - [\Delta_p(s_n, \bar{x}) + \Delta_p(l_n, \bar{y})] \\
= \Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y}) - [\Delta_p(u_{n+1}, \bar{x}) + \Delta_p(v_{n+1}, \bar{y})] \\
+[\Delta_p(u_{n+1}, \bar{x}) + \Delta_p(v_{n+1}, \bar{y})] - [\Delta_p(s_n, \bar{x}) + \Delta_p(l_n, \bar{y})] \\
\leq \Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y}) - [\Delta_p(u_{n+1}, \bar{x}) + \Delta_p(v_{n+1}, \bar{y})] \\
+[\Delta_p(x_{n+1}, \bar{x}) + \Delta_p(y_{n+1}, \bar{y})] - [\Delta_p(s_n, \bar{x}) + \Delta_p(l_n, \bar{y})] \\
\leq \Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y}) - [\Delta_p(u_{n+1}, \bar{x}) + \Delta_p(v_{n+1}, \bar{y})] \\
+ [\alpha_n \Delta_p(u, \bar{x}) + (1-\alpha_n) \Delta_p(s_n, \bar{x}) + \alpha_n \Delta_p(v, \bar{y}) + (1-\alpha_n) \Delta_p(l_n, \bar{y})] \\
- [\Delta_p(s_n, \bar{x}) + \Delta_p(l_n, \bar{y})] \\
= \Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y}) - [\Delta_p(u_{n+1}, \bar{x}) + \Delta_p(v_{n+1}, \bar{y})] \\
+ \alpha_n \Delta_p(u, \bar{x}) + \Delta_p(v, \bar{y}) - \Delta_p(s_n, \bar{x}) - \Delta_p(l_n, \bar{y}) \\
= - \{\Delta_p(u_{n+1}, \bar{x}) + \Delta_p(v_{n+1}, \bar{y}) - [\Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y})]\} \\
+ \alpha_n [\Delta_p(u, \bar{x}) + \Delta_p(v, \bar{y}) - \Delta_p(s_n, \bar{x}) - \Delta_p(l_n, \bar{y})] \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty.
\]
In addition
\[
\Delta_p(s_n, \bar{x}) + \Delta_p(l_n, \bar{y}) \leq \beta_n [\Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y})] \\
+ (1-\beta_n)[\Delta_p(S u_n, \bar{x}) + \Delta_p(T v_n, \bar{y})] \\
= [\Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y})] - (1-\beta_n)[\Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y})] \\
+ (1-\beta_n)[\Delta_p(S u_n, \bar{x}) + \Delta_p(T v_n, \bar{y})] \\
= \Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y}) - (1-\beta_n)[\Delta_p(S u_n, \bar{x}) \\
- \Delta_p(u_n, \bar{x}) + \Delta_p(T v_n, \bar{y}) - \Delta_p(v_n, \bar{y})].
\]
Therefore, it follows from (3.15), (3.16) that
\[
(1-\beta_n)[\Delta_p(u_n, \bar{x}) - \Delta_p(S u_n, \bar{x}) + \Delta_p(v_n, \bar{y}) - \Delta_p(T v_n, \bar{y})]
\]
\[ \Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y}) - \Delta_p(s_n, \bar{x}) - \Delta_p(l_n, \bar{y}) \to 0 \text{ as } n \to \infty, \]

So
\[ \Delta_p(u_n, \bar{x}) - \Delta_p(Su_n, \bar{x}) + \Delta_p(v_n, \bar{y}) - \Delta_p(Tv_n, \bar{y}) \to 0 \text{ as } n \to \infty. \]

Obvious
\[ \Delta_p(u_n, \bar{x}) \geq 0, \]
and
\[ \Delta_p(v_n, \bar{y}) \geq 0. \]

Hence
\[ \lim_{n \to \infty} [\Delta_p(u_n, \bar{x}) - \Delta_p(Su_n, \bar{x})] = 0, \]
and
\[ \lim_{n \to \infty} [\Delta_p(v_n, \bar{y}) - \Delta_p(Tv_n, \bar{y})] = 0. \]

By definition 2.4, we obtain that
\[ \lim_{n \to \infty} \Delta_p(Su_n, u_n) = 0 \text{ and } \lim_{n \to \infty} \Delta_p(Tv_n, v_n) = 0, \quad (3.17) \]
which implies that
\[ \lim_{n \to \infty} \|Su_n - u_n\| = 0 \text{ and } \lim_{n \to \infty} \|Tv_n - v_n\| = 0. \quad (3.18) \]

Since \( \varepsilon_n := J_{E_1}^q[J_{E_1}^p x_n - \gamma_n A^* J_{E_1}^p (Ax_n - By_n)] \), then we have
\[
0 \leq \| J_{E_1}^p \varepsilon_n - J_{E_1}^p x_n \| \\
\leq \gamma_n \| A^* \| \| J_{E_1}^p (Ax_n - By_n) \| \\
\leq \left( \frac{q}{C_q \| A \|^q} \right)^{1/q - 1} \| A^* \| \| Ax_n - By_n \| \\
\leq M \| A^* \| \| Ax_n - By_n \| \to 0, n \to \infty.
\]

Hence, we obtain
\[ \lim_{n \to \infty} \| J_{E_1}^p \varepsilon_n - J_{E_1}^p x_n \| = 0. \quad (3.19) \]

Since \( J_{E_1}^q \) is also norm-to-norm uniformly continuous on bounded subsets of \( E_1^* \), we have
\[ \lim_{n \to \infty} \| \varepsilon_n - x_n \| = 0. \quad (3.20) \]

Similarly, Since \( \omega_n := J_{E_2}^q[J_{E_2}^p x_n - \gamma_n B^* J_{E_2}^p (Ax_n - By_n)] \), we can get
\[ \lim_{n \to \infty} \| J_{E_2}^p \omega_n - J_{E_2}^p y_n \| = 0. \quad (3.21) \]

Since \( J_{E_2}^q \) is also norm-to-norm uniformly continuous on bounded subsets of \( E_2^* \), we have
\[ \lim_{n \to \infty} \| \omega_n - y_n \| = 0. \quad (3.22) \]
Furthermore, we have from (2.4) that

\[
\Delta_p(\varepsilon_n, u_n) = \Delta_p(\varepsilon_n, \Pi_C\varepsilon_n)
\]

\[
\leq \Delta_p(\varepsilon_n, \bar{x}) - \Delta_p(\Pi_C\varepsilon_n, \bar{x})
\]

\[
= \Delta_p(\varepsilon_n, \bar{x}) - \Delta_p(u_n, \bar{x}),
\]

(3.23)

by the same way,

\[
\Delta_p(\omega_n, v_n) \leq \Delta_p(\omega_n, \bar{y}) - \Delta_p(v_n, \bar{y}).
\]

(3.24)

From (3.23), (3.24), (3.4) and (3.7), we obtain

\[
\Delta_p(\varepsilon_n, u_n) + \Delta_p(\omega_n, v_n)
\]

\[
\leq \Delta_p(\varepsilon_n, \bar{x}) - \Delta_p(u_n, \bar{x}) + [\Delta_p(\omega_n, \bar{y}) - \Delta_p(v_n, \bar{y})]
\]

\[
= [\Delta_p(\varepsilon_n, \bar{x}) + \Delta_p(\omega_n, \bar{y})] - [\Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y})]
\]

\[
\leq [\Delta_p(x_n, \bar{x}) + \Delta_p(y_n, \bar{y})] - [\Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y})]
\]

\[
\leq \alpha_{n-1}[\Delta_p(u, \bar{x}) + \Delta_p(v, \bar{y})]
\]

\[
+(1 - \alpha_{n-1})[\Delta_p(u_{n-1}, \bar{x}) + \Delta_p(v_{n-1}, \bar{y})] - [\Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y})]
\]

\[
\leq \alpha_{n-1}[\Delta_p(u, \bar{x}) + \Delta_p(v, \bar{y})]
\]

\[
+ [\Delta_p(u_{n-1}, \bar{x}) + \Delta_p(v_{n-1}, \bar{y})] - [\Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y})]
\]

\[
= \alpha_{n-1}[\Delta_p(u, \bar{x}) + \Delta_p(v, \bar{y})]
\]

\[
- \{[\Delta_p(u_n, \bar{x}) + \Delta_p(v_n, \bar{y})] - [\Delta_p(u_{n-1}, \bar{x}) + \Delta_p(v_{n-1}, \bar{y})]\}.
\]

Following condition (ii) and (3.11), we have

\[
\lim_{n \to \infty} [\Delta_p(\varepsilon_n, u_n) + \Delta_p(\omega_n, v_n)] = 0,
\]

so

\[
\lim_{n \to \infty} \Delta_p(\varepsilon_n, u_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} \Delta_p(\omega_n, v_n) = 0,
\]

(3.25)

which implies that

\[
\lim_{n \to \infty} \|\varepsilon_n - u_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|\omega_n - v_n\| = 0.
\]

(3.26)

Hence

\[
\|x_n - u_n\| \leq \|x_n - \varepsilon_n\| + \|\varepsilon_n - u_n\| \to 0, \quad n \to \infty,
\]

(3.27)

and

\[
\|y_n - v_n\| \leq \|y_n - \omega_n\| + \|\omega_n - v_n\| \to 0, \quad n \to \infty.
\]

(3.28)
Since \( \{ x_n \} \) is bounded, there exists a subsequence \( \{ x_{n_j} \} \) of \( x_n \) that converges weakly to \( x^* \), from (3.27) we get \( u_{n_j} \rightharpoonup x^* \). From (2.3) and Lemma(2.5), we have that

\[
\Delta_p(x^*, \Pi Cx^*) \leq \langle J_{E_1}^p x^* - J_{E_1}^p \Pi Cx^*, x^* - \Pi Cx^* \rangle \\
= \langle J_{E_1}^p x^* - J_{E_1}^p \Pi Cx^*, x^* - x_{n_j} \rangle + \langle J_{E_1}^p x^* - J_{E_1}^p \Pi Cx^*, x_{n_j} - u_{n_j} \rangle \\
+ \langle J_{E_1}^p x^* - J_{E_1}^p \Pi Cx^*, u_{n_j} - \Pi Cx^* \rangle \\
\leq \langle J_{E_1}^p x^* - J_{E_1}^p \Pi Cx^*, x^* - x_{n_j} \rangle + \langle J_{E_1}^p x^* - J_{E_1}^p \Pi Cx^*, x_{n_j} - u_{n_j} \rangle.
\]

As \( j \to \infty \), we have \( \Delta_p(x^*, \Pi Cx^*) = 0 \), i.e. \( x^* \in C \). Similarly, we can obtain \( y^* \in Q \). By (3.18) and \( F(S) = \widehat{F}(S) \), we have \( x^* \in F(S) \). In the same way, By the boundedness of \( y_n \), there exists a subsequence \( \{ y_{n_j} \} \) of \( \{ y_n \} \) that converges weakly to \( y^* \), from (3.28) we get \( v_{n_j} \rightharpoonup y^* \), by (3.18) and \( F(T) = \widehat{F}(T) \), we have \( y^* \in F(T) \). On the other hand, since \( A \) and \( B \) are bounded linear operators, we know that \( Ax_n \rightharpoonup Ax^* \) and \( By_n \rightharpoonup By^* \), also by the weakly lower semi-continuous property of the norm and (3.10), we get

\[
\| Ax^* - By^* \| \leq \liminf_{n \to \infty} \| Ax_n - By_n \| = 0.
\]

So, \( Ax^* = By^* \), this implies \((x^*, y^*) \in \Omega \).

Next, we prove that \( \{ (x_n, y_n) \} \) converges strongly to \((x^*, y^*)\). Now, from (2.6) and (3.1) we have

\[
\Delta_p(x_{n+1}, x^*) \leq \Delta_p(J_{E_1}^p(\alpha_n J_{E_1}^p u + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_1}^p S u_n)), x^*) \\
= V_p(\alpha_n J_{E_1}^p u + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_1}^p S u_n), x^*) \\
\leq V_p(\alpha_n J_{E_1}^p u + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_1}^p S u_n) - \alpha_n(\beta_n J_{E_1}^p u - J_{E_1}^p x^*), x^*) \\
- \alpha_n(\beta_n J_{E_1}^p u_n + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_1}^p S u_n) - x^*) \\
= V_p(\alpha_n J_{E_1}^p u + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_1}^p S u_n), x^*) \\
+ \alpha_n(\beta_n J_{E_1}^p u - J_{E_1}^p x^*, x^*) \\
= \Delta_p(J_{E_1}^p(\alpha_n J_{E_1}^p u + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_1}^p S u_n)), x^*) \\
+ \alpha_n(\beta_n J_{E_1}^p u - J_{E_1}^p x^*, x^*) \\
\leq \alpha_n \Delta_p(x^*, x^*) + (1 - \alpha_n)\beta_n \Delta_p(u_n, x^*) \\
+ (1 - \alpha_n)(1 - \beta_n)\Delta_p(S u_n, x^*) + \alpha_n(\beta_n J_{E_1}^p u - J_{E_1}^p x^*, x^*) \\
\leq (1 - \alpha_n)\Delta_p(u_n, x^*) + \alpha_n(\beta_n J_{E_1}^p u - J_{E_1}^p x^*, x^*),
\]

where \( \rho_n := J_{E_1}^p(\alpha_n J_{E_1}^p u + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_1}^p S u_n)) \). From (3.1), we have

\[
\Delta_p(\rho_n, u_n) \leq \Delta_p(J_{E_1}^p(\alpha_n J_{E_1}^p u + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_1}^p S u_n)), u_n) \\
\leq \alpha_n \Delta_p(u_n, u_n) + (1 - \alpha_n)\beta_n \Delta_p(u_n, u_n) \\
+ (1 - \alpha_n)(1 - \beta_n)\Delta_p(S u_n, u_n) \to 0, \quad n \to \infty,
\]

(3.30)
which implies that $\lim_{n \to \infty} \|\rho_n - u_n\| = 0$, and by (3.27) we have

$$\|\rho_n - x_n\| \leq \|\rho_n - u_n\| + \|u_n - x_n\| \to 0, \text{ as } n \to \infty,$$

so

$$\lim_{n \to \infty} \|\rho_n - x_n\| = 0. \quad (3.31)$$

Since $x_{n_j} \rightharpoonup x^*$, so we have $\rho_{n_j} \rightharpoonup x^*$. Furthermore,

$$\limsup_{n \to \infty} (J_{E_1}^p u - J_{E_1}^p x^*, \rho_n - x^*) = \lim_{j \to \infty} \langle J_{E_1}^p u - J_{E_1}^p x^*, \rho_{n_j} - x^* \rangle \leq 0. \quad (3.32)$$

Submitting (3.2) into (3.29), we obtain

$$\Delta_p(x_{n+1}, x^*) \leq (1 - \alpha_n)\Delta_p(u_n, x^*) + \alpha_n \langle J_{E_1}^p u - J_{E_1}^p x^*, \rho_n - x^* \rangle$$

$$\leq (1 - \alpha_n)\Delta_p(x_n, x^*) + (1 - \alpha_n)\gamma_n \langle J_{E_3}^p e_n, Ax^* - Ax_n \rangle$$

$$+(1 - \alpha_n)\frac{C_q(\lambda_n \|A\|)^q}{q} \|e_n\|^p + \alpha_n \langle J_{E_1}^p u - J_{E_1}^p x^*, \rho_n - x^* \rangle. \quad (3.33)$$

Similary, we have

$$\Delta_p(y_{n+1}, y^*) \leq (1 - \alpha_n)\Delta_p(y_n, y^*) + (1 - \alpha_n)\gamma_n \langle J_{E_1}^p e_n, By_n - By^* \rangle$$

$$+(1 - \alpha_n)\frac{C_q(\gamma_n \|B\|)^q}{q} \|e_n\|^p + \alpha_n \langle J_{E_1}^p u - J_{E_1}^p x^*, \rho_n - x^* \rangle, \quad (3.34)$$

where $\eta_n := J_{E_2}^p(\alpha_n J_{E_2}^p v + (1 - \alpha_n)(\beta_n J_{E_2}^p v_n + (1 - \beta_n)J_{E_2}^p T v_n))$.

Adding (3.33) and (3.34), we get

$$\Gamma_{n+1}(x^*, y^*) = \Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)$$

$$\leq (1 - \alpha_n)\Gamma_n(x^*, y^*) + (1 - \alpha_n)(\gamma_n - 2L)\|e_n\|^p$$

$$+ \alpha_n \langle J_{E_1}^p u - J_{E_1}^p x^*, \rho_n - x^* \rangle + \langle J_{E_2}^p v - J_{E_2}^p y^*, \eta_n - y^* \rangle$$

$$\leq (1 - \alpha_n)\Gamma_n(x^*, y^*) + \alpha_n \langle J_{E_1}^p u - J_{E_1}^p x^*, \rho_n - x^* \rangle + \langle J_{E_2}^p v - J_{E_2}^p y^*, \eta_n - y^* \rangle. \quad (3.35)$$

By Lemma 2.6, we know that $\lim_{n \to \infty} \Gamma_n(x^*, y^*) = 0$, which implies

$$\lim_{n \to \infty} [\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)] = 0, \quad (3.36)$$

then

$$\lim_{n \to \infty} \|x_n - x^*\| = 0 \text{ and } \lim_{n \to \infty} \|y_n - y^*\| = 0. \quad (3.37)$$

That is the sequence $\{(x_n, y_n)\}$ converges strongly to a solution $(x^*, y^*) \in \Omega$.

**Case 2.** Assume that $\{\Gamma_n(\bar{x}, \bar{y})\}$ is not a monotonically decreasing sequence. Let $\tau : N \to N$ be a mapping for all $n \geq n_0$ by

$$\tau(n) := \max\{k \in N : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$


Obviously, \( \tau(n) \) is a non-decreasing sequence such that \( \tau(n) \to \infty \) as \( n \to \infty \) and
\[
0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.
\] (3.38)

The same as in case 1, we have the following equalities or inequalities
\[
\lim_{n \to \infty} \|Ax_{\tau(n)} - By_{\tau(n)}\| = 0,
\]
\[
\lim_{n \to \infty} \|A^*J_{E_1}^p(Ax_{\tau(n)} - By_{\tau(n)})\| = 0, \quad \lim_{n \to \infty} \|B^*J_{E_2}^p(Ax_{\tau(n)} - By_{\tau(n)})\| = 0,
\]
\[
\lim_{n \to \infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0, \quad \lim_{n \to \infty} \|y_{\tau(n)+1} - y_{\tau(n)}\| = 0,
\]
\[
\lim_{n \to \infty} \|Su_{\tau(n)} - u_{\tau(n)}\| = 0, \quad \lim_{n \to \infty} \|Tv_{\tau(n)} - v_{\tau(n)}\| = 0
\]
and
\[
\limsup_{n \to \infty} \langle J_{E_1}^p u - J_{E_1}^p \bar{x}, \rho_{\tau(n)} - \bar{x} \rangle \leq 0,
\]
\[
\limsup_{n \to \infty} \langle J_{E_2}^p v - J_{E_2}^p \bar{y}, \eta_{\tau(n)} - \bar{y} \rangle \leq 0.
\]
Since \( \{(x_{\tau(n)}, y_{\tau(n)})\} \) is bounded, there exists a subsequence of \( \{(x_{\tau(n)}, y_{\tau(n)})\} \) which converges weakly to \( (x^*, y^*) \in \Omega \). From (3.35), we know that
\[
\Gamma_{\tau(n)+1}(x^*, y^*) \leq (1 - \alpha_{\tau(n)})\Gamma_{\tau(n)}(x^*, y^*)
\]
\[
+ \alpha_{\tau(n)}\left[\langle J_{E_1}^p u - J_{E_1}^p x^*, \rho_{\tau(n)} - x^* \rangle + \langle J_{E_2}^p v - J_{E_2}^p y^*, \eta_{\tau(n)} - y^* \rangle\right].
\]
From (3.38), we have
\[
\Gamma_{\tau(n)}(x^*, y^*) \leq \Gamma_{\tau(n)+1}(x^*, y^*)
\]
\[
\leq (1 - \alpha_{\tau(n)})\Gamma_{\tau(n)}(x^*, y^*)
\]
\[
+ \alpha_{\tau(n)}\left[\langle J_{E_1}^p u - J_{E_1}^p x^*, \rho_{\tau(n)} - x^* \rangle + \langle J_{E_2}^p v - J_{E_2}^p y^*, \eta_{\tau(n)} - y^* \rangle\right],
\]
i.e.
\[
\Gamma_{\tau(n)}(x^*, y^*) \leq \langle J_{E_1}^p u - J_{E_1}^p x^*, \rho_{\tau(n)} - x^* \rangle + \langle J_{E_2}^p v - J_{E_2}^p y^*, \eta_{\tau(n)} - y^* \rangle.
\]
Then from (3.32) we have \( \limsup_{n \to \infty} \Gamma_{\tau(n)}(x^*, y^*) \leq 0 \). Thus, \( \lim_{n \to \infty} \Delta_p(x_{\tau(n)}, x^*) = 0, \lim_{n \to \infty} \Delta_p(y_{\tau(n)}, y^*) = 0 \). So
\[
\lim_{n \to \infty} \|x_{\tau(n)} - x^*\| = 0, \quad \lim_{n \to \infty} \|y_{\tau(n)} - y^*\| = 0.
\]
It follows from \( \lim_{n \to \infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0 \) and \( \lim_{n \to \infty} \|y_{\tau(n)+1} - y_{\tau(n)}\| = 0 \) that \( \lim_{n \to \infty} \|x_{\tau(n)+1} - x^*\| = 0, \lim_{n \to \infty} \|y_{\tau(n)+1} - y^*\| = 0 \). Now, by Lemma 2.5, we have that
\[
0 \leq \Gamma_{\tau(n)+1}(x^*, y^*) \leq \langle x_{\tau(n)+1} - x^*, J_{E_1}^p x_{\tau(n)+1} - J_{E_1}^p x^* \rangle
\]
\[
+ \langle y_{\tau(n)+1} - y^*, J_{E_1}^p y_{\tau(n)+1} - J_{E_1}^p y^* \rangle
\]
\[
\leq \|x_{\tau(n)+1} - x^*\|\|J_{E_1}^p x_{\tau(n)+1} - J_{E_1}^p x^*\|
\]
\[
+ \|y_{\tau(n)+1} - y^*\|\|J_{E_2}^p y_{\tau(n)+1} - J_{E_2}^p y^*\| \to 0, \quad n \to \infty.
\]
For \( n \geq n_0 \), it is easy to see that \( \Gamma_{\tau(n)}(x^*, y^*) \leq \Gamma_{\tau(n)+1}(x^*, y^*) \) if \( n \neq \tau(n) \) (that is, \( \tau(n) < n \)), because \( \Gamma_j(x^*, y^*) \geq \Gamma_{j+1}(x^*, y^*) \) for \( \tau(n)+1 \leq j \leq n \). As a consequence, we obtain for all \( n \geq n_0 \),

\[
0 \leq \Gamma_n(x^*, y^*) \leq \max\{\Gamma_{\tau(n)}(x^*, y^*), \Gamma_{\tau(n)+1}(x^*, y^*)\} = \Gamma_{\tau(n)+1}(x^*, y^*).
\]

Hence, \( \lim_{n \to \infty} \Gamma_n(x^*, y^*) = 0 \). That is \( \{(x_n, y_n)\} \) converges strongly to \((x^*, y^*)\). This completes the proof.

\section{Application}

\subsection{Application to split variational inclusion problem}

Suppose that \( A : E_1 \to E_3 \), \( B : E_2 \to E_3 \) be two bounded linear operators. Let \( M : E_1 \to 2^{E_1^*} \) and \( N : E_2 \to 2^{E_2^*} \) be maximal monotone operators. Let us consider the following so called split variational inclusion problem:

\[
\text{find } \ x^* \in M^{-1}(0), \ y^* \in N^{-1}(0) \text{ such that } Ax^* = By^*. \tag{4.1}
\]

Given a maximal monotone operator \( M : E_1 \to 2^{E_1^*} \), from [27], we known that the associated resolvent mapping, \( J^M_\mu := (I + \mu M)^{-1} \), \( x \in E_1 \) is left Bregman strongly nonexpansive and \( 0 \in M(x^*) \) if and only if \( x^* \) is a fixed point of \( F(J^M_\mu) \).

Taking \( S = J^M_\mu, T = J^N_\nu \) in Theorem 3.1, we have the following result.

\textbf{Theorem 4.1.} Let \( E_1, E_2 \) and \( E_3 \) be three \( p \)-uniformly convex and uniformly smooth Banach spaces. Let \( A : E_1 \to E_3 \), \( B : E_2 \to E_3 \) be two bounded linear operators and \( A^* : E_3^* \to E_1^* \), \( B^* : E_3^* \to E_2^* \) be the adjoint operators of \( A \) and \( B \), respectively. Let \( M : E_1 \to 2^{E_1^*} \) and \( N : E_2 \to 2^{E_2^*} \) be maximal monotone operators and suppose that the solution set of problem (4.1) is denoted by \( \Omega \neq \emptyset \). The sequence \( \{(x_n, y_n)\} \) is generated by

\[
\begin{aligned}
& u_n = \Pi_{C} J^q_{E_1^*}[J^p_{E_1} x_n - \gamma_n A^* J^p_{E_3}(Ax_n - By_n)] \\
& v_n = \Pi_{Q} J^q_{E_2}[J^p_{E_2} y_n + \gamma_n B^* J^p_{E_3}(Ax_n - By_n)] \\
& x_{n+1} = \Pi_{C} J^q_{E_1^*}(\alpha_n J^p_{E_1} u_n + (1 - \alpha_n)(\beta_n J^p_{E_1} u_n + (1 - \beta_n)J^p_{E_1} J^M_\mu u_n)) \\
& y_{n+1} = \Pi_{Q} J^q_{E_2^*}(\alpha_n J^p_{E_2} v_n + (1 - \alpha_n)(\beta_n J^p_{E_2} v_n + (1 - \beta_n)J^p_{E_2} J^N_\nu v_n)),
\end{aligned}
\tag{4.2}
\]

where the following conditions are satisfied:

(i) \( \{\beta_n\} \subset (0, 1) \);

(ii) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \);

(iii) \( 0 < 2L < \gamma_n \leq R \), where \( R = \max\{\left(\frac{q}{c_0\|A\|_q}\right)^{\frac{1}{q-1}}, \left(\frac{q}{c_0\|B\|_q}\right)^{\frac{1}{q-1}}\} \), and \( L = \max\{\frac{c_0(L_n\|A\|_q)^q}{q}, \frac{c_0(L_n\|B\|_q)^q}{q}\} \).

Then, the sequence \( \{(x_n, y_n)\} \) converges strongly to \((x^*, y^*)\) in the solution set \( \Omega \) of (4.1).
4.2 Application to split Equilibrium problem

Let $E_1$, $E_2$ and $E_3$ be three $p$-uniformly convex and uniformly smooth Banach spaces. Let $C$ and $Q$ be nonempty closed and convex subsets of $E_1$ and $E_2$, respectively. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be two bounded linear operators. Let $F : C \times C \to R$, $H : Q \times Q \to R$ be bi-functions. Let us consider the following so called split Equilibrium problem:

\[
\text{find } x^* \in C, \ y^* \in Q \text{ such that } F(x^*, z) \geq 0, \ H(y^*, \mu) \geq 0, \ \forall z \in C, \ \mu \in Q \text{ and } Ax^* = By^*.
\]  

(4.3)

To solve problem (4.3), we assume the following conditions are satisfied

(A1) $F(x, x) = 0, \forall x \in C$ and $H(x, x) = 0, \forall x \in Q$;

(A2) $F$ and $H$ are monotone, that is, $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$ and $H(x, y) + H(y, x) \leq 0, \forall x, y \in Q$;

(A3) For all $x, y, z \in C$, $\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$ and For all $x, y, z \in Q$, $\lim_{t \to 0} H(tz + (1 - t)x, y) \leq H(x, y)$;

(A4) For each $x \in C$, the function $y \mapsto F(x, y)$ is convex and lower semi-continuous and For each $x \in Q$, the function $y \mapsto H(x, y)$ is convex and lower semi-continuous.

Let us define the resolvent mappings $T^F_\mu, \mu > 0$ and $T^H_\nu, \nu > 0$ as

\[
T^F_\mu(x) = \{ z \in C : F(z, y) + \frac{1}{\mu} \langle y - z, J_{E_1}^p z - J_{E_1}^p x \rangle \geq 0, \forall y \in C \}.
\]

and

\[
T^H_\nu(x) = \{ z \in Q : H(z, y) + \frac{1}{\nu} \langle y - z, J_{E_2}^p z - J_{E_2}^p x \rangle \geq 0, \forall y \in Q \}.
\]

It is well know that the resolvent mapping $T^F_\mu$ and $T^H_\nu$ are left Bregman firmly nonexpansive (hence left Bregman strongly nonexpansive). Furthermore, it is know that if $x^*$, $y^*$ solves problem (4.3), then $x^* = T^F_\mu x^*$ and $y^* = T^H_\nu y^*$.

The following result can be directly obtained from Theorem 3.1.

**Theorem 4.2.** Let $E_1$, $E_2$ and $E_3$ be three $p$-uniformly convex and uniformly smooth Banach spaces. Let $C$ and $Q$ be nonempty closed and convex subsets of $E_1$ and $E_2$, respectively. Suppose that $F : C \times C \to R$, $H : Q \times Q \to R$ be bi-functions. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be two bounded linear operators and $A^* : E_3^* \to E_1^*$, $B^* : E_3^* \to E_2^*$ be the adjoint operators of $A$ and $B$, respectively. Suppose that the solution set of problem (4.3) is denoted by $\Omega \neq \emptyset$. The sequence $\{(x_n, y_n)\}$ is
generated by
\[
\begin{align*}
    u_n &= \Pi_{C}J_{E_1}^q[J_{E_1}^p x_n - \gamma_n A^* J_{E_1}^p (Ax_n - By_n)] \\
    v_n &= \Pi_{Q}J_{E_2}^q[J_{E_2}^p y_n + \gamma_n B^* J_{E_2}^p (Ax_n - By_n)] \\
    x_{n+1} &= \Pi_{C}J_{E_1}^q(\alpha_n J_{E_1}^p u_n + (1 - \alpha_n)(\beta_n J_{E_1}^p u_n + (1 - \beta_n)J_{E_1}^p T_{\mu}^F u_n)) \\
    y_{n+1} &= \Pi_{Q}J_{E_2}^q(\alpha_n J_{E_2}^p v_n + (1 - \alpha_n)(\beta_n J_{E_2}^p v_n + (1 - \beta_n)J_{E_2}^p T_{\nu}^H v_n)),
\end{align*}
\]
where the following conditions are satisfied:
(i) \( \{\beta_n\} \subset (0, 1) \);
(ii) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \);
(iii) \( 0 < 2L \leq \gamma_n \leq N \), where \( N = \max\{\left(\frac{q}{C_q\|A\|q}\right)^{\frac{1}{q-1}}, \left(\frac{q}{C_q\|B\|q}\right)^{\frac{1}{q-1}}\} \), and \( L = \max\{\frac{C_q(\lambda_n\|A\|q)}{q}, \frac{C_q(\lambda_n\|B\|q)}{q}\} \).

Then, the sequence \( \{(x_n, y_n)\} \) converges strongly to \( (x^*, y^*) \) in the solution set \( \Omega \) of (4.4).

References


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