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On Weighted Gâteaux Differentiability and Some Applications

A. Benbrik, I. Hadi, S. Lahrech, Y. Yousfi

Departement of Mathematics and Computer Sciences
Faculty of Sciences, Mohamed First University, Oujda, Morocco

Abstract

In this paper, we define a new notion of what we called weighted Gâteaux differentiability and some equivalences with the classical Gâteaux differentiability. Aiming to use these results for the resolution of optimization problems and differential equations on some evolutive geometrical domains.

Mathematics Subject Classifications: 49J50, 46G05, 58C20

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1 Introduction and preliminaries

In the study of several mathematical models on evolutive geometrical domains, we need to preserve some conditions as the derivative or some other properties. But generally this is not the case. Which lead to find a suitable operators which represent the domain deformation. An extensive literature deal with the Gâteau differentiability when the strong derivative not exist as [1], [2], [6], [7]. Thus we introduced the notion of weighted Gâteaux differentiability with respect to some operators.
For this, we need to recall the following definitions and introduce some notions.

Definition 1.1 *Let $(E, \|.\|_E)$ be a Banach space. We consider a closed operator $A : D(A) \subset E \longmapsto E$. We assume that the kernel of A that we note*

$N(A)$ has a topological supplement L in E. So $R(A) = A(D(A) \cap L)$ and let $\|Av\|_{R(A)} = \|Av\|_E + \|v\|_E$ whenever $v \in D(A) \cap L$.
The norm $\|.\|_R(A)$ is called "Range Topology".

Remark 1.1 *1. $\|.\|_R(A)$ is finer than $\|.\|_E$ in $R(A)$ and whenever the operator A is closed $(R(A), \|.\|_R(A))$ is a Banach space.*

2. Let $\|v\|_1 = \|Av\|_E + \|v\|_E$ whenever $v \in D(A) \cap L$. So $\|v\|_1 = \|Av\|_{R(A)}$ whenever $v \in D(A) \cap L$. Thus $D(A) \cap L, \|.\|_1$ is a Banach space.

In particular case, when the operator A is further more injective. We have that $\|v\|_2 = \|Av\|_E$ is a norm.

Throughout this paper, E is assumed a *Banach* space. We consider a closed operator $A : D(A) \subset E \longmapsto E$. We assume that the kernel of A that we note $N(A)$ has a topological supplement L in E.

Definition 1.2 *Let U be an open subset of E. We define U as a weighted open set with respect to A if: $\forall x_0 \in U$ $\exists \delta > 0$ whenever $h \in D(A)$ that $\|Ah\|_{R(A)} < \delta$ we have $x_0 + h \in U$*

Proposition 1.1 *Let U be an open subset of E. So U is an open set in E if and only if U is a weighted open set regarding A in E.*

Proof.

- $\Rightarrow$ Let $x_0 \in U$ and let $h \in D(A)$ such that $\|Ah\|_{R(A)} \longrightarrow 0$ so $\|Ah\|_E \longrightarrow 0$ and $\|h\|_E \longrightarrow 0$, which means that $h \longrightarrow 0$ in E, thus $x_0 + h \in U$. Then U is a weighted open set regarding A in E.

- $\Leftarrow$ Let $x_0 \in U$. So since A is a closed operator $\|h\|_E \longrightarrow 0$ implies that $\|Ah\|_E \longrightarrow 0$, then $\|h\|_1 \longrightarrow 0$ so $h \longrightarrow 0$ in E. Thus $x_0 + h \in U$ and U is a open set in E.

2 Weighted Gâteaux differentiability

Definition 2.1 *Let U be a weighted open set regarding A in E and let $f : U \longmapsto F$ such that F is a normed space, let $x_0 \in U$. We say that f is weighted Gteaux $differentiable$ regarding A at x_0 if there is a linear operator $A : D(A) \longmapsto F$ verifying the following conditions:*

1. *$N(A) \subset N(L_1)$.*

2. *$\forall v \in D(A)$ we have lim $\dfrac{f(x_0 + tv) - f(x_0)}{t} = L_1(v)$ whenever $t \mapsto 0^+$.*

3. $\forall v \in \mathcal{D}(A)$ *we have* $\| L_1(v) \|_F \leq c \| Av \|_{R(A)}$.
   In this case $(\bigtriangledown_A f)(x_0)(Av) = L_1(v)$ *et ce* $\forall v \in \mathcal{D}(A)$.
   $(\bigtriangledown_A f)(x_0)$ *is a linear continous operator from* $(R(A), \| \,.\, \|_{R(A)})$ *to* $(F, \| \,.\, \|_F)$.

Remark 2.1 *The condition* (1) *of the Definition 2.1 is trivial when the operator* A *is injective.*

Theorem 2.1 *Let* U *be an open set (also assumed as a weighted open set regarding* A *in* E *) and let* $f : U \longmapsto F$ *such that* F *is a normed space, let* $x_0 \in U$. *We assume that* f *is Gteaux differentiable at* x_0, *and also that* $(\nabla_A f)(0) = 0$.*thus* f *is weighted Gteaux differentiable regarding* A *at* x_0 *and*

$$(\bigtriangledown_A f)(x_0) \circ A = (\bigtriangledown f)(x_0)$$

Proof.
Let f be Gâteaux differentiable at x_0, so there is a linear continuous operator $(\nabla f)(x_0)$ from $(E, \| \,.\, \|_E)$ to $(F, \| \,.\, \|_F)$ verifying the following condition $\forall v \in E$ we have $lim \ \dfrac{f(x_0+tv)-f(x_0)}{t} = L_1(v)$ whenever $t \mapsto 0^+$. Let us pose $L_1(v) = (\nabla f)(x_0)(v)$ whenever $v \in E$.
So $\| L_1(v) \|_F = \| (\nabla f)(x_0)(v) \|_F \leq c_1 \| v \|_E$ thus $\| L_1(v) \|_F \leq c \| Av \|_{R(A)}$ and $N(A) \subset N(L_1)$ yields from $(\nabla_A f)(x_0)(0) = 0$. Therefore f is weighted Gâteaux differentiable regarding A at x_0

Remark 2.2 1. *If* A *is injective the hypothesis that* $(\nabla_A f)(x_0)(0) = 0$ *could be omitted.*

2. *When* A *is a linear operator* $(\nabla_A f)(x_0)$ *is also linear on* $R(A)$.

3. *if* A *is non bounded with dense domain and* A^* *is surjective, then* A *is injective and* $R(A)$ *is closed. So our main result is obtained.*

Definition 2.2 *Let* E *be a normed space and* $A : D(A) \subset E \longmapsto E$ *a closed linear operator,* U *an open subset of* E *and* $f : U \longrightarrow F$ *where* F *is a normed space. We say that* f *is two times weighted Gteaux Differentiable regarding* A *at* x_0 *in* U *if we have* $L_1 : (R(A), \| \,.\, \|_{R(A)}) \longrightarrow F$ *satisfying the condition of Definition 2.1 and* $B_1 : (R(A) \times R(A), \| \,.\, \|_{R(A)}) \longrightarrow F$ *bilinear continuous such that:*

1. $N(A) \subset N(B_1)$ *for any* $v \in N(A)$ *we have* $B_1(v, v) = 0$

2. $f(x_0+h) = f(x_0) + L_1(h) + \dfrac{B_1(h,h)}{2} + \| h \|_1^2 \, \varepsilon(h)$ *with* $\varepsilon(h) \longrightarrow 0$ *when* $\| h \|_1 \longrightarrow 0$

In this case we set $(\bigtriangledown_A f)(x_0)(Av) = L(v)$ *for any* $v \in D(A)$ *and*
$(\bigtriangledown_A^2 f)(x_0)(Av, Av) = B_1(v; v)$

Problem.
Let's find $x_0 \in U$ (where U is a weighted open subset with respect to A) and $\epsilon < 0$ such that $\mathcal{P}$ is a problem of minimizing $f(x)$ in $B_A(x_0, \epsilon)$ where $B_A(x_0, \epsilon)$ is a weighted ball in U with respect to $A/$

Theorem 2.2 *Let* U *be an open subset of* E *regarding* A *and* $f : U \longrightarrow R$. *If there is an* $x_0 \in U$ *such that* f *is two times weighted Gteaux differentiable with respect to* A *at the point* x_0 *and* $(\bigtriangledown_A f)(x_0) = 0$ *on* $R(A)$ *furthermore* $(\bigtriangledown_A^2 f)(x_0)$ *est definite positive i.e:* $\exists \alpha > 0$, $(\bigtriangledown_A^2 f)(x_0)(Ah, Ah) \leq \alpha \parallel h \parallel_1^2$ *for any* $h \in \mathcal{D}$. *Then there exists* $\epsilon > 0$ *such that* x_0 *is a solution of the problem* $\mathcal{P}$.

Proof.
Let h tends to 0 regarding A i.e : $h \in \mathcal{D}(A)$ and $\parallel h \parallel_1 \longrightarrow 0$. We have $f(x_0 + h) = f(x_0) + [(\bigtriangledown_A f)(x_0)](Ah) + \dfrac{[(\bigtriangledown_A^2 f)(x_0)](Ah, Ah)}{2} + \parallel h \parallel_1^2 \varepsilon(h)$ but $(\bigtriangledown_A f)(x_0) = 0$ on $R(A)$ and $[(\bigtriangledown_A^2 f)(x_0)](Ah, Ah) \geq \parallel h \parallel_1^2$ thus $f(x_0 + h) - f(x_0) \geq (\frac{\epsilon}{2} + \alpha) \parallel h \parallel_1^2$ and for h sufficiently small $\epsilon(h) = 0$.
So we deduce that $f(x_0 + h) \geq f(x_0)$ for h sufficiently small.

3 Weighted Convexity

Definition 3.1 *Let* E *a normed space and let* $A : \mathcal{D}(A) \subset E \longrightarrow E$ *a closed operator. Let* U *a weighted open subset of* E *with the respect of* A. U *is weighted convex with the respect of* A *if* $\forall x, y \in U : \forall \lambda \in [0, 1]$ *such that* $y + \lambda A(x - y) \in U$

Definition 3.2 *Let* U *weighted convex and* $f : U \longrightarrow R$. *We say that* f *is weighted convex if*
$\forall x, y \in U, \forall \lambda \in [0, 1]$ *we have* $f(y + \lambda A(x - y)) \leq \lambda f(x) + (1 - \lambda) f(y)$

Theorem 3.1 *Let* E *a normed space and let* $A : \mathcal{D}(A) \subset E \longrightarrow E$ *a closed operator. Let* U *a weighted open subset of* E *with the respect of* A *(also supposed weighted convex regarding* A*). Let* $f : U \longrightarrow R$ *such that* f *has a weighted directional derivative regarding* A. *Then*

$$f \text{ is weighted convex regarding } A \Leftrightarrow \forall x, y \in U \text{ such that } x - y \in \mathcal{D}(\mathcal{A}) \text{ and } < (\bigtriangledown_A f)(y), (x - y) > \leq f(x) - f(y)$$

Proof.
$\Rightarrow\}$ Let f weighted convex regarding A. So $\forall x, y \in U, \forall \lambda \in [0,1]$ we have $f(y + \lambda A(x-y)) \leq \lambda f(x) + (1-\lambda) f(y)$, it implies that $\frac{f(y+\lambda A(x-y)) - f(y)}{\lambda} \leq f(x) - f(y)$. Then by applying the limit we find that $< (\bigtriangledown_A f)(y), (x-y) > \leq f(x) - f(y)$

Theorem 3.2 *Let E a normed space and let $A : \mathcal{D}(A) \subset E \longrightarrow E$ a closed operator. Let U a weighted open subset of E with the respect of A (also supposed weighted convex regarding A). Let $f : U \longrightarrow R$ such that f has a weighted directional derivative regarding A at all points following a random direction. Then f is weighted convex regarding $A \Leftrightarrow \forall x, y \in U$ with $x - y \in \mathcal{D}(\mathcal{A})$ and:*

$$< (\bigtriangledown_A f)(x) - (\bigtriangledown_A f)(y), (x-y) > \leq 0 \Leftrightarrow f \text{ is weighted monotone regarding } A$$

Proof.
Using the theorem 3.1, we have that $< (\bigtriangledown_A f)(y), (x-y) > \leq f(x) - f(y)$ and also that $< (\bigtriangledown_A f)(x), (y-x) > \leq f(y) - f(x)$.
So $< (\bigtriangledown_A f)(y), (x-y) > - < (\bigtriangledown_A f)(x), (y-x) > \leq 0$. $(\bigtriangledown_A f)$ is linear then $< (\bigtriangledown_A f)(y), (x-y) > - < (\bigtriangledown_A f)(x), (x-y) > \leq 0$.
Thus $< (\bigtriangledown_A f)(y) - (\bigtriangledown_A f)(x), (x-y) > \leq 0$

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