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# Perfect Hop Domination in Graphs

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**Abstract**

Let $G = (V(G), E(G))$ be a simple graph. A set $S \subseteq V(G)$ is a *perfect hop dominating set* of $G$ if for every $v \in V(G) \setminus S$, there is exactly one vertex $u \in S$ such that $d_G(u, v) = 2$. The smallest cardinality of a perfect hop dominating set of $G$ is called the *perfect hop domination number* of $G$, denoted by $\gamma_{ph}(G)$. In this paper, we characterize the perfect hop dominating set in the join, corona, lexicographic and Cartesian product of two graphs and determine their corresponding perfect hop domination number.

**Keywords:** perfect hop domination, perfect point-wise non-domination, total perfect hop domination, perfect total $(1,2)^*$-domination and distance 2-domination in graphs

# 1 Introduction

Let $G = (V(G), E(G))$ be a simple graph. The *open neighborhood* of a vertex $v$ of $G$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of $v$, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. A subset $S$ of $V(G)$ is a *hop dominating set* [3] (resp.*perfect hop dominating set*) of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ (resp. there is exactly one vertex $u \in S$) such that $d_G(u,v) = 2$. The smallest cardinality of a hop dominating (resp. perfect hop dominating) set of $G$, denoted by $\gamma_h(G)$ (resp. $\gamma_{ph}(G)$) is the *hop domination number* (resp. *perfect hop domination number*) of $G$. A hop dominating set (resp. perfect hop dominating set) $S$ of $G$ with $|S| = \gamma_h(G)$ (resp. $|S| = \gamma_{ph}(G)$) is called a $\gamma_h$-set (resp. $\gamma_{ph}$-set) of $G$. A set $S \subseteq V(G)$ is a *total perfect hop dominating set* of $G$ if for every $v \in V(G)$, there is exactly one vertex $u \in S$ such that $d_G(u,v) = 2$. The smallest cardinality of a total perfect hop dominating set of $G$, denoted by $\gamma_{tph}(G)$ is called the *total perfect hop domination number* of $G$. Any total perfect hop dominating set of $G$ with $|S| = \gamma_{tph}(G)$ is called a $\gamma_{tph}$-set. A set $S \subseteq V(G)$ is a *perfect total* $(1,2)^*$*-dominating set* of $G$ if for every $w \in V(G)$, there is exactly one vertex $x \in S$ such that $wx \in E(G)$ and for every $u \in V(G) \setminus S$, there is exactly one vertex $v \in S$ such that $d_G(u,v) = 2$. The smallest cardinality of a perfect total $(1,2)^*$-dominating set of $G$ is called the *perfect total* $(1,2)^*$*-domination number* of $G$, denoted by $\gamma^{*pt}_{1,2}(G)$. A perfect total $(1,2)^*$-dominating set $S$ of $G$ with $|S| = \gamma^{*pt}_{1,2}(G)$ is called a $\gamma^{*pt}_{1,2}$-set of $G$. A set $S \subseteq V(G)$ is a *perfect distance* 2*-dominating set* of $G$ if for every $u \in V(G) \setminus S$, there is exactly one vertex $v \in S$ such that $d_G(u,v) \leq 2$. The smallest cardinality of a perfect distance 2-dominating set of $G$ is called the *perfect distance* 2*-domination number* of $G$, denoted by $\gamma_{2p}(G)$. A perfect distance 2-dominating set $S$ of $G$ with $|S| = \gamma_{2p}(G)$ is called a $\gamma_{2p}$-set of $G$. The *open hop neighborhood* set $N_G(v,2)$ of a vertex $v$ of $V(G)$ is the set of all vertices of $G$ which are at a distance equal to 2 from $v$ in $G$. For other terms not define here, refer to [2].

# 2 Results

**Proposition 2.1** *Let $G$ be a connected graph. Then $1 \leq \gamma_{ph}(G) \leq |V(G)|$. Moreover,*

(i) $\gamma_{ph}(G) = 1$ *if and only if $G$ is a trivial graph.*

(ii) *$\gamma_{ph}(G) = 2$ if and only if there exist vertices $a$ and $b$ of $G$ such that $N_G(a) \cap N_G(b) = \emptyset$ and $N_G(a,2) \cap N_G(b,2) = \emptyset$.*

**Example 2.2** *$\gamma_{ph}(P_n) = 2$ for $n = 2,3,4,5$, $\gamma_{ph}(C_n) = 2$ for $n = 4$, $\gamma_{ph}(K_{m,n}) = 2$ for $n, m \geq 2$ and $\gamma_{ph}(K_m) = m$ for $m \geq 1$.*

A set $S \subseteq V(G)$ is a *perfect point-wise non-dominating set* of $G$ if for every $v \in V(G) \setminus S$, there is exactly one vertex $u \in S$ such that $v \notin N_G(u)$. The smallest cardinality of a perfect point-wise non-dominating set of $G$, denoted by $ppnd(G)$ is called the *perfect point-wise non-domination number* of $G$. Any perfect point-wise non-dominating set $S$ of $G$ with $|S| = ppnd(G)$ is called a *ppnd*-set.

**Remark 2.3** *Let $G$ be any graph. Then $ppnd(G) = 1$ if and only if $G$ has an isolated vertex.*

**Example 2.4** *For any positive integers $m$ and $n$,*

(i) $ppnd(K_m) = m$;

(ii) $ppnd(K_{m,n}) = 2$;

(iii) $ppnd(P_n) = \begin{cases} 2, & \text{if } 2 \leq n \leq 6 \\ n, & \text{if } n \geq 7 \end{cases}$

(iv) $ppnd(C_n) = \begin{cases} 2, & \text{if } n = 4,6 \\ 3, & \text{if } n = 3,5 \\ n, & \text{if } n \geq 7 \end{cases}$

# 3 Join of Graphs

The *join* $G+H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G)$ and $v \in V(H)\}$.

**Theorem 3.1** *Let $G$ and $H$ be graphs. Then $S \subseteq V(G+H)$ is a perfect hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$ where $S_G$ and $S_H$ are perfect point-wise non-dominating sets of $G$ and $H$, respectively.*

**Proof**. Let $S \subseteq V(G+H)$ be a perfect hop dominating set of $G+H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. Suppose $S_G = \emptyset$. Then $S = S_H$. Thus, for any vertex $v \in V(G)$, $d_{G+H}(v,x) = 1$ for all $x \in S$ contrary to our assumption that $S$ is a perfect hop dominating set of $G+H$. Thus, $S_G \neq \emptyset$. Similarly,

$S_H \neq \emptyset$. Suppose $u \in V(G) \setminus S_G$. Then $u \in V(G+H) \setminus S$. Since $S$ is a perfect hop dominating set of $G+H$, there is exactly one vertex $y \in S$ such that $d_{G+H}(u,y) = 2$. By definition of $G+H$, $y \notin S_H$. Thus, $y \in S_G$ and $uy \notin E(G)$. Hence, $S_G$ is a perfect point-wise non-dominating set of $G$. Similarly, $S_H$ is a perfect point-wise non-dominating set of $H$. Conversely, suppose that $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are perfect point-wise non-dominating sets of $G$ and $H$ respectively. Let $S = S_G \cup S_H$ and $v \in V(G+H) \setminus S$. If $v \in V(G) \setminus S_G$, then since $S_G$ is a perfect point-wise non-dominating set of $G$, there is exactly one vertex $x \in S_G$ such that $vx \notin E(G)$. Hence, by definition of $G+H$, $d_{G+H}(v,x) = 2$. Therefore $S$ is a perfect hop dominating set of $G+H$. Similarly if $v \in V(H) \setminus S_H$, then $S$ is a perfect hop dominating set of $G+H$. $\square$

**Corollary 3.2** *Let $G$ and $H$ be graphs. Then*

$$\gamma_{ph}(G+H) = ppnd(G) + ppnd(H).$$

*In particular, for any positive integers $m$ and $n$,*

(i) $\gamma_{ph}(F_n) = \begin{cases} 2, & \text{if } n = 1 \\ 3, & \text{if } 2 \leq n \leq 6 \\ 1+n, & \text{if } n \geq 7 \end{cases}$

(ii) $\gamma_{ph}(W_n) = \begin{cases} 3, & \text{if } n = 4,6 \\ 4, & \text{if } n = 3,5 \\ 1+n, & n \geq 7 \end{cases}$

(iii) $\gamma_{ph}(K_{m,n}) = 2$

# 4 Corona of Graphs

The *corona* $G \circ H$ of two graphs $G$ and $H$ is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the *ith* vertex of $G$ to every vertex in the *ith* copy of $H$. For every $v \in V(G)$, denote by $H^v$ the copy of $H$ whose vertices are attached one by one to the vertex $v$. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle v \rangle + H^v = v + H^v$. We define the following sets for a nonempty set $A \subseteq V(G)$.

$$A_1 = \{x \in A : |N_G(x) \cap A| = 1\},$$

$$A_2 = \{x \in A : |N_G(x) \cap A| \geq 2\},$$

$$A^{1c} = \{x \in V(G) \setminus A : |N_G(x) \cap A| = 1\},$$

$$A^{2c} = \{x \in V(G) \setminus A : |N_G(x) \cap A| \geq 2\},$$

**Theorem 4.1** *Let $G$ be a nontrivial connected graph and $H$ a graph without isolated vertices. A subset $C$ of $V(G \circ H)$ is a perfect hop dominating set of $G \circ H$ if and only if*

$$C = A \cup \left( \bigcup_{v \in A_1} S_v^1 \right) \cup \left( \bigcup_{v \in A_2} S_v^2 \right) \cup \left( \bigcup_{u \in A^{1c}} D_u^1 \right) \cup \left( \bigcup_{u \in A^{2c}} D_u^2 \right)$$

*where*

*(i) $A$ is a perfect hop and a total dominating set of $G$.*

*(ii) For each $v \in A_1$, $S_v^1 = \emptyset$ or $S_v^1$ is a perfect $k_1$-dominating set of $H^v$ if $deg_G(v) = 1$ where $k_1 = |S_v^1|$.*

*(iii) $S_v^2 = V(H^v)$ for each $v \in A_2$.*

*(iv) For each $u \in A^{1c}$, $D_u^1 = \emptyset$ or $D_u^1$ is a perfect $k_2$-dominating set of $H^u$ if $deg_G(u) = 1$ where $k_2 = |D_u^1|$.*

*(v) $D_u^2 = V(H^u)$ for each $u \in A^{2c}$.*

**Proof**. Let $C$ be a perfect hop dominating set of $G \circ H$ and $A = V(G) \cap C$. Let $x \in V(G)$. If $x \notin A$, then $x \notin C$. Hence, there exists a unique vertex $v \in C$ such that $d_{G \circ H}(x, v) = 2$. We claim that $v \in A$. Suppose that $v \notin A$. Then there exists a vertex $w \in V(G)$ such that $v \in V(H^w)$ and $xw \in E(G)$. If $|V(G)| = 2$, then $H$ is a trivial graph or $v$ is an isolated vertex of $H$, which is a contradiction to the hypothesis. If $|V(G)| > 2$, then there exist vertices $a \in N_{H^w}(v) \backslash C$ and $b \in A$ such that $d_{G \circ H}(a, b) = 2$. Thus, $wb \in E(G)$ implying that $d_{G \circ H}(x, b) = 2$ . This is a contradiction since $C$ is a perfect hop dominating set of $G \circ H$ and $d_{G \circ H}(x, v) = 2 = d_{G \circ H}(x, b)$ where $v, b \in C$. Hence, $v \in A$. This implies that $A$ is a perfect hop dominating set of $G$. Next, we show that $A$ is a dominating set of $G$, that is, $N_G(x) \cap A \neq \emptyset$. Since $d_G(x, v) = 2$ and $v \in A$, there exists $u \in V(G)$ such that $u \in N_G(x) \cap N_G(v)$. If $u \in A$, then we are done. Suppose that $u \notin A$. Since $H$ is a nontrivial graph without isolated vertices, there exists a vertex $y \in A$ such that $d_{G \circ H}(y, y') = 2$ for all vertices $y' \in V(H^x)$. This implies that $y \in N_G(x) \cap A$. Thus, $N_G(x) \cap A \neq \emptyset$, implying that $A$ is a dominating set of $G$. Next, suppose that $x \in A$. We claim again that $N_G(x) \cap A \neq \emptyset$. Since $G$ is connected, there exists a vertex $u \in V(G)$ such that $ux \in E(G)$. If $u \in A$, then we are done. Suppose $u \notin A$. If $|V(G)| = 2$, then there exists a unique vertex $y \in V(H^x) \cap C$. Since $u \notin C$, $\{y\}$ is a point wise non-dominating set of $H^x$. By Remark 2.3, $y$ is an isolated vertex of $H$ which is a contradiction to the hypothesis. Thus, $u \in A$. This implies that $N_G(x) \cap A \neq \emptyset$ for the case that $|V(G)| = 2$. Suppose that $|V(G)| > 2$. Then there exists vertex $z \in V(G) \setminus \{x, u\}$ such that either $x \in N_G(z) \cap N_G(u)$ or

$u \in N_G(x) \cap N_G(z)$. If $x \in N_G(z) \cap N_G(u)$, then $z \in A$ since $u \notin A$ and $H$ is a graph without isolated vertices. Thus, $z \in N_G(x) \cap A$. The case that $u \in N_G(x) \cap N_G(z)$ and $u \notin A$ will give a contradiction since $H$ must have an isolated vertex or $H$ is a trivial graph. This case forces $u \in A$. Hence, in any case, $N_G(x) \cap A \neq \emptyset$. Therefore $A$ is a total dominating set of $G$. Thus, $(i)$ holds. Let $v \in A_1$. Then $|N_G(v) \cap A| = 1$. If $deg_G(v) \geq 2$, then there exists distinct vertices $z_1, z_2 \in N_G(v)$ such that $z_1 \in A$ and $z_2 \notin A$. We claim that $S_v^1 = C \cap V(H^v) = \emptyset$. Suppose that there is a vertex $u \in S_v^1$. Then $d_{G\circ H}(z_1, z_2) = d_{G\circ H}(u, z_2) = 2$. Since $u, z_1 \in C$ and $z_2 \notin C$, we have a contradiction as $C$ is a perfect hop domination set of $G \circ H$. Hence, $S_v^1 = \emptyset$. Suppose that $deg_G(v) = 1$ and $S_v^1 \subseteq C \cap V(H^v)$. Let $w \in N_G(v) \cap A$. If $u \in V(H^v) \setminus S_v^1$, then $uy \in E(H^v)$ for every $y \in S_v^1$ since $d_{G\circ H}(u, w) = 2$. Hence, $S_v$ is a perfect $k_1$-dominating set of $H^v$ where $k_1 = |S_v|$. Thus, $(ii)$ holds. Let $v \in A_2$. Then $|N_G(v) \cap A| \geq 2$. This implies that there are two distinct vertices $x, y \in N_G(v) \cap A$. We show that $S_v^2 = V(H^v)$. If $S_v^2 \neq V(H^v)$, there is a vertex $u \in V(H^v) \setminus S_v^2$. Hence, $d_{G\circ H}(u, x) = d_{G\circ H}(u, y) = 2$. This is a contradiction, since $u \notin C$, $x, y \in C$ and $C$ is a perfect hop dominating set. Thus, $S_v = V(H^v)$ and $(iii)$ holds. The proofs of $(iv)$ and $(v)$ are similar to the proofs of $(ii)$ and $(iii)$, respectively. For the converse, suppose that

$$C = A \cup \left(\bigcup_{v\in A_1} S_v^1\right) \cup \left(\bigcup_{v\in A_2} S_v^2\right) \cup \left(\bigcup_{u\in A^{1c}} D_u^1\right) \cup \left(\bigcup_{u\in A^{2c}} D_u^2\right)$$

and conditions $(i)$,$(ii)$,$(iii)$,$(iv)$ and $(v)$ hold. Let $x \in V(G \circ H) \setminus C$ and $v \in V(G)$ such that $x \in V(v + H^v)$. Consider the following cases.

Case 1. $x = v$. Then $x \in V(G) \setminus A$. By $(i)$, $A$ is a perfect hop dominating set of $G$. Hence, $|V(G)| > 2$ and there exists a unique vertex $u \in A$ such that $d_G(x, u) = 2$. Let $w \in N_G(x) \cap N_G(u)$. We claim that $deg_G(x) = 1$. If $|V(G)| = 3$, then we are done. Suppose that $|V(G)| = 4$ and there exists a vertex $z \in N_G(x)$. Then $A$ is not a total dominating set of $G$ contradicting $(i)$. Suppose that $|V(G)| \geq 5$ and there exists a vertex $a \in N_G(x)$. Since $A$ is a total dominating set of $G$, there exists a vertex $b \in N_G(a) \cap A$. Then $d_G(x, u) = d_G(x, b) = 2$. This is a contradiction since $A$ is a perfect hop dominating set of $G$ and $b, u \in A$ and $x \notin A$. Thus, $deg_G(x) = 1$. By $(i)$, $w \in A$. Since $u$ is a unique vertex in $A$ such that $d_G(x, u) = 2$, $|N_G(w) \cap A| = 1$. Since $deg_G(w) \geq 2$, by $(ii)$ , $S_w^1 = \emptyset$. Therefore $u$ is a unique vertex in $C$ such that $d_{G\circ H}(x, u) = 2$.

Case 2. $x \neq v$. Then $x \in V(H^v)$. Since $v \in V(G)$, by $(i)$, there exists vertex $y \in N_G(v) \cap A$. Then $d_{G\circ H}(x, y) = 2$. We claim that $y$ is a unique vertex in $C$. Since $x \notin C$ (that is $x \notin S_v^2$ and $x \notin D_v^2$), by $(iii)$ and $(v)$ , $|N_G(v) \cap A| = 1$. If $deg_G(v) \geq 2$, then $S_v^1 = \emptyset$ or $D_v^1 = \emptyset$ by $(ii)$ and $(iv)$. If $deg_G(v) = 1$, then $S_v^1$ is a perfect $k_1$-dominating set of $H^v$ or $D_v^1$ is a perfect

$k_2$-dominating set of $H^v$ where $k_1 = |S_v^1|$ and $k_2 = |D_v^1|$. Therefore $y$ is a unique vertex in $C$ such that $d_{G\circ H}(x,y) = 2$. Accordingly, $C$ is a perfect hop dominating set of $G \circ H$. □

**Corollary 4.2** *Let $G$ be a connected graph where $\gamma(G) = 1$. Then $\gamma_{ph}(G \circ H) = 2$ if and only if $H$ has an isolated vertex or $G$ has a vertex of degree 1.*

**Proof**. Let $\gamma_{ph}(G \circ H) = 2$ and $C$ a $\gamma_{ph}$-set of $G \circ H$. Suppose $G$ is a trivial graph. Then $G \circ H = G + H$. By Theorem 3.2, $C = \{v\} \cup S_H$ where $v \in V(G)$ and $S_H$ is a perfect pointwise non-dominating set of $H$. Since $|C| = 2$, $|S_H| = 1$. By Remark 2.3, the vertex $u \in S_H$ is an isolated vertex of $H$. Suppose $G$ is a connected nontrivial graph and $H$ is a graph without isolated vertices. Then by Theorem 4.1,

$$C = A \cup \left(\bigcup_{v\in A_1} S_v^1\right) \cup \left(\bigcup_{v\in A_2} S_v^2\right) \cup \left(\bigcup_{u\in A^{1c}} D_u^1\right) \cup \left(\bigcup_{u\in A^{2c}} D_u^2\right).$$

where $A$ is a perfect hop and a total dominating set of $G$ by $(i)$. Since $A$ is a total dominating set of $G$, $|A| \geq 2$. Note that $|C| = 2$. Thus, $|A| = 2$. Hence, $S_v^i = D_u^i = \emptyset$ for $i = 1, 2$ and for every $v \in A$ and $u \in V(G) \notin A$. Therefore $C = A$. Since $\gamma(G) = 1$, there exists $u \in V(G)$ such that $ux \in E(G)$ for alla $x \in V(G)$. This implies that $C = A = \{u, y\}$ for some $y \in V(G)$. Since $\gamma_{ph}(G) = |C| = 2$, by Proposition 2.1, $N_G \cap N_G(y) = \varnothing$. Since $x \in N_G(u)$ for all $x \notin \{u\}$ and $u \in N_G(y) = u$. It follows that $deg_H(y) = 1$. Therefore $G$ has a vertex of degree one. Therefore $G$ has a vertex of degree one. For the converse, suppose $H$ has an isolated vertex or $G$ has a vertex of degree 1. Let $\{u\}$ be the $\gamma$-set of $G$. Then for all $v \in V(G) \setminus \{u\}$, $uv \in E(G)$. Hence, $\gamma_{ph}(G \circ H) \neq 1$. Consider the following cases:

Case 1. H has an isolated vertex. Then $\{x\}$ is a *ppnd*-set of $H^u$. Consider $C = \{u, x\}$. Let $v \in V(G \circ H) \setminus C$. If $v \in V(G)$, then $d_{G\circ H}(x, v) = 2$ and if $v \notin V(G)$, then there exists $w \in V(G)$ such that $v \in V(H^w)$. This implies that $d_G(u, v) = 2$. Thus, $C$ is a perfect hop dominating set of $G \circ H$. Since $\gamma_{ph}(G \circ H) \neq 1$, $C$ is a $\gamma_{ph}$-set of $G$. Therefore $\gamma_{ph}(G \circ H) = 2$.

Case 2. $G$ has a vertex of degree 1. Let $y \in V(G)$ such that $deg_G(y) = 1$. Consider $C = \{u, y\}$. Clearly $uy \in V(G)$ since $\{u\}$ is a $\gamma$-set of $G$. Let $a \in V(G \circ H) \setminus C$. If $a \in V(G)$, then $au \in E(G)$ and $ay \notin E(G)$ since $deg_G(y) = 1$. Thus, $d_{G\circ H}(a, y) = 2$. Suppose $a \notin V(G)$. Then there exists $b \in V(G)$ such that $a \in V(H^b)$. Since $ab \in E(G \circ H)$ and $bu \in E(G)$, $d_{G\circ H}(a, u) = 2$. Therefore $C$ is a $\gamma_{ph}$-set of $G \circ H$. This implies that $\gamma_{ph}(G \circ H) = 2$. □

**Corollary 4.3** *Let $G$ be a nontrivial connected graph and $H$ a graph without isolated vertices of order $n$. Then $\gamma_{ph}(G \circ H) = min\{|A| + n\,|A_2| + n\,|A^{2c}| : A$ is a perfect hop and a total dominating set of $G\}$.*

**Proof**. Let $C$ be a $\gamma_{ph}$-set of $G \circ H$. Then by Theorem 4.1,

$$C = A \cup \left( \bigcup_{v \in A_1} S_v^1 \right) \cup \left( \bigcup_{v \in A_2} S_v^2 \right) \cup \left( \bigcup_{u \in A^{1c}} D_u^1 \right) \cup \left( \bigcup_{u \in A^{2c}} D_u^2 \right)$$

where $A$ is a perfect hop and a total dominating set of $G$. Let $S_v^1 = \emptyset$ and $D_u^1 = \emptyset$ for every $v \in A_1$, and $u \in A^{1c}$. Thus,

$$|C| = |A| + n\,|A_2| + n\left|A^{2c}\right|.$$

Hence, $\gamma_{ph}(G \circ H) = min\{|A| + n\,|A_2| + n\,|A^{2c}| :$ A is a perfect hop dominating and total dominating set of $G$. □

# 5 Lexicographic Product

The *lexicographic product* of two graphs $G$ and $H$, denoted by $G[H]$, is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u_1, u_2)(v_1, v_2) \in E(G[H])$ if either $u_1 v_1 \in E(G)$ or $u_1 = v_1$ and $u_2 v_2 \in E(H)$.

**Theorem 5.1** *Let $G$ and $H$ be nontrivial connected graphs with $\gamma(H) \neq 1$. A subset $C = \bigcup_{x \in S} [\{x\} \times T_x]$ of $V(G[H]$ is a perfect hop dominating set of $G[H]$ if and only if the following conditions hold.*

(i) *$S = V(G)$; and*

(ii) *For each $x \in S$, $T_x = V(H)$ if $N_G(x, 2) \neq \varnothing$ and $T_x$ is a perfect point-wise non-dominating set of $H$ if $N_G(x, 2) = \varnothing$.*

**Proof**. Let $C = \bigcup_{x \in S} [\{x\} \times T_x]$ be a perfect hop dominating set of $G[H]$. Suppose $S \neq V(G)$. Let $u \in V(G) \backslash S$. Then $(u, a) \notin C$ for any $a \in V(H)$. Thus, there exists a unique vertex $(y, b) \in C$ such that $d_{G[H]}((u, a), (y, b)) = 2$. Since $u \notin S$ and $y \in S$, $u \neq y$ and $d_G(u, y) = 2$. This implies that $(y, p) \notin C$ for all $p \in V(H) \backslash \{b\}$. Since $\gamma(H) \neq 1$, choose $q \in V(H) \backslash \{b\}$ such that $q \notin N_H(b)$. Then $d_{G[H]}((y, q), (y, b)) = 2$. Pick any $t \in N_H(b)$. Then there exists $z \in S \backslash \{y\}$ such that $d_G(y, z) = 2$. Let $r \in T_z$. Then $d_{G[H]}((y, q), (z, r)) = 2$, a contradiction to the fact that $C$ is a perfect hop dominating set of $G[H]$. Therefore $S = V(G)$. Let $x \in S$. Suppose that $N_G(x, 2) \neq \varnothing$ and $T_x \neq V(H)$. Let $z \in N_G(x, 2)$, $p \in T_z$ and $a \in V(H) \backslash T_x$. Since $(x, a) \notin C$, there is exactly one vertex $(y, b) \in C$ such that $d_{G[H]}((x, a), (y, b)) = 2$. This implies that $x = y$ and $ab \notin E(H)$ or $d_G(x, y) = 2$. Suppose $x = y$ and $ab \notin E(G)$. Then $d_{G[H]}((x, a), (y, b)) = d_{G[H]}((x, a), (z, p)) = 2$ contrary to our assumption that

$C$ is a perfect hop dominating set of $G[H]$. On the other hand, suppose that $d_G(x, y) = 2$. If $y \neq z$, then $d_{G[H]}((x, a), (y, b)) = d_{G[H]}((x, a), (z, p)) = 2$. If $y = z$, then $b = p$. Since $\gamma(H) \neq 1$, there exists $q \in V(H)\backslash N_H[p]$. Let $w \in T_x$. Then $d_{G[H]}((z, q), (z, p)) = d_{G[H]}((z, q), (x, w)) = 2$. Since $(z, q) \notin C$ because $|T_x| = 1$, it follows that $C$ is not a perfect hop dominating set of $G[H]$ a contradiction to our assumption for $C$. Therefore $T_x = V(H)$. Now, let $N_G(x, 2) = \varnothing$ and $a \in V(H)\backslash T_x$. Then $(x, a) \notin C$ and it follows that there is a unique vertex $(y, b) \in C$ such that $d_{G[H]}((x, a), (y, b)) = 2$. Since $N_G(x, 2) = \varnothing$, $x = y$ and $ab \notin E(H)$. This implies that $T_x$ is a perfect point-wise non-dominating set of $H$. Therefore (i) and (ii) hold. Conversely, assume that $C$ satisfies conditions (i) and (ii). Let $(x, a) \notin C$. Since $S = V(G)$, $a \notin T_x$. If $N_G(x, 2) \neq \varnothing$, then we are done since $T_x = V(H)$. If $N_G(x, 2) = \varnothing$, then by (ii), there exists a unique vertex $b \in T_x$ such that $ab \notin E(H)$. Thus, $(x, b) \in C$ and $d_{G[H]}((x, a), (x, b)) = 2$. Accordingly, $C$ is a perfect hop dominating set of $G[H]$. □

To obtain the perfect hop domination number of $G[H]$ for a nontrivial connected graphs $G$ and $H$ with $\gamma(H) \neq 1$, first we consider the following sets.

$$S_1 = \{x \in V(G) : deg_G(x) < |V(G)| - 1\}$$

$$S_2 = \{x \in V(G) : deg_G(x) = |V(G)| - 1\}$$

**Corollary 5.2** *Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively and $\gamma(H) \neq 1$. Then $\gamma_{ph}(G[H]) = (m-q)n + q(ppnd(H))$ where $q = \big|\{x \in V(G) : deg_G(x) = |V(G)| - 1\}\big|$.*

**Proof**. Let $C = \bigcup_{x \in S}[\{x\} \times T_x]$ be a minimum perfect hop dominating set of $G[H]$. By Theorem 5.1, $S = V(G)$ and $T_x = V(H)$ if $N_G(x, 2) \neq \varnothing$ and $T_x$ is a *ppnd*-set of $H$ if $N_G(x, 2) = \varnothing$. Then
$\gamma_{ph}(G[H]) = |C| = \sum_{x \in S_1} |T_x| + \sum_{x \in S_2} |T_x| = |S_1| \cdot |V(H)| + |S_2| \cdot ppnd(H) =$

$$(m - |S_2|) \cdot n + |S_2| \cdot ppnd(H) = (m - q) \cdot n + q \cdot ppnd(H).$$

where $q = \big|\{x \in V(G) : deg_G(x) = |V(G)| - 1\}\big|$. □

If $G$ is a complete graph, then $S_1 = \varnothing$ and $S_2 = V(G)$. Thus, the following result follows immediately from Corollary 5.2.

**Corollary 5.3** *Let $H$ be a nontrivial connected graph with $\gamma(H) \neq 1$ and $m \geq 2$. Then $\gamma_{ph}(K_m[H]) = m(ppnd(H))$.*

The next result is a characterization of a perfect hop dominating set in $G[H]$ for nontrivial connected graphs $G$ and $H$ with $\gamma(H) = 1$.

**Theorem 5.4** *Let $G$ be a nontrivial connected graph whose total perfect hop dominating set exists and $H$ a nontrivial connected graph with $\gamma(H) = 1$. A nonempty proper subset $C = \bigcup_{x\in S}[\{x\} \times T_x]$ of $V(G[H])$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$, is a perfect hop dominating set of $G[H]$ if and only if $S$ is a total perfect hop dominating set of $G$ and $T_x$ is a $\gamma$-set of $H$ for every $x \in S$.*

**Proof**. Let $C = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, be a perfect hop dominating set of $G[H]$. We claim that $S$ is a total perfect hop dominating set of $G$. Let $u \in V(G)$. If $u \notin S$, then $(u, a) \notin C$ for any $a \in V(H)$. Thus, there is exactly one vertex $(v, b) \in C$ such that $d_{G[H]}((u, a), (v, b)) = 2$. Since $u \notin S$ and $v \in S$, $u \neq v$ and $d_G(u, v) = 2$. Suppose $u \in S$. Since $G$ has a total perfect hop dominating set, $N_G(u, 2) \neq \varnothing$. Let $z \in N_G(u, 2)$. If $z \in S$, then we are done. So suppose that $z \notin S$. Then $|T_u| = 1$, say $T_u = \{p\}$ for some $p \in V(H)$ because $C$ is a perfect hop dominating set of $G[H]$. Let $a \in N_H(p)$. Then there exists a unique $(w, b) \in C \cap N_{G[H]}((u, a), 2)$. Since $b \neq p$, $u \neq w$. Thus, $w \in S \cap N_G(u, 2)$. Hence, $N_G(u, 2) \cap S \neq \varnothing$. Therefore $S$ is a total perfect hop dominating set of $G$. Now, let $x \in S$. Since $S$ is a total perfect hop dominating set of $G$, $|T_x| = 1$, say $T_x = \{a\}$. Let $p \in V(H)\backslash T_x$. Suppose $p \notin N_H(a)$. Then $d_{G[H]}((x, p), (x, a)) = 2$. Since $S$ is a total perfect hop dominating set of $G$, there exists a unique $y \in N_G(x, 2)\cap S$. Pick any $c \in T_y$. Then $(y, c) \neq (x, a)$ but $d_{G[H]}((x, p), (y, c)) = 2$. This implies that $C$ is not a perfect hop dominating set of $G[H]$, a contradiction. Therefore, $T_x$ is a $\gamma$-set of $H$. Conversely, suppose that $S$ is a total perfect hop dominating set of $G$ and $T_x$ is a $\gamma$-set of $H$. Let $(x, a) \notin C$. Then either $x \notin S$ or $x \in S$ and $a \notin T_x$. If $x \notin S$, then a unique vertex $y \in S$ exists such that $d_G(x, y) = 2$. Since $T_y$ is a $\gamma$-set of $H$ for every $y \in S$, a unique vertex $b \in T_y$ exists such that for all $p \in V(H)\backslash\{b\}$, $p \in N_H(b)$. Then $(y, b) \in C$ and $d_{G[H]}((x, a), (y, b)) = 2$. Suppose $x \in S$ and $a \notin T_x$. Then there is exactly one vertex $z \in S$ such that $d_G(x, z) = 2$. Since $T_z$ is a $\gamma$-set of $H$ for every $z \in S$, a unique vertex $c \in T_z$ exists. Hence, $(z, c) \in C$ and $d_{G[H]}((x, a), (z, c)) = 2$. Thus, $C$ is a perfect hop dominating set of $G[H]$. □

**Corollary 5.5** *Let $G$ be a nontrivial connected graph whose total perfect hop dominating set exists and $H$ a nontrivial connected graph with $\gamma(H) = 1$. Then $\gamma_{ph}(G[H]) = \gamma_{tph}(G)$.*

**Proof**. Let $C = \bigcup_{x\in S}[\{x\} \times T_x]$ be a minimum perfect hop dominating set of $G[H]$. Then by Theorem 5.4, $S$ is a minimum total perfect hop dominating set of $G$ and $T_x = \{a\}$ where $a \in V(H)$ such that $deg_H(a) = |V(H)| - 1$. Therefore $\gamma_{ph}(G[H]) = |C| = \sum_{x\in S} |T_x| = |S| = \gamma_{tph}(G)$. □

# 6 Cartesian Product

The *Cartesian product* of two graphs $G$ and $H$, denoted by $G \square H$, is the graph with vertex-set $V(G \square H) = V(G) \times V(H)$ and edge-set $E(G \square H)$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G \square H)$ if and only if either $u_1u_2 \in E(G)$ and $v_1 = v_2$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

**Theorem 6.1** *Let $G$ and $H$ be complete graphs such that $|V(G)| \geq 3$ and $|V(H)| \geq 4$. Then $S$ is a perfect hop dominating set of $G\square H$ if and only if $S = V(G\square H)$.*

**Proof**. Clearly, if $S = V(G\square H)$, then $S$ is a perfect hop dominating set of $G\square H$. Let $S$ be a perfect hop dominating set of $G\square H$. Suppose $S \neq V(G\square H)$. Let $(x, y) \in V(G\square H)\backslash S$. Then there exists a unique vertex $(z, a) \in S$ such that $d_{G\square H}((x, y),(z, a)) = 2$. Thus, $z \neq x$ and $a \neq y$ since $G$ and $H$ are both complete. Now, since $|V(G)| \geq 3$ and $|V(H)| \geq 4$, there exist vertices $w \in V(G)\backslash\{x, z\}$ and $b, c \in V(H)\backslash\{a, y\}$ with $b \neq c$. Hence, $(w, a) \notin S$, $(w, b) \notin S$, $(w, c) \notin S$, $(z, b) \notin S$ and $(z, c) \notin S$. Since $d_{G\square H}((w, c),(z, a)) = d_{G\square H}((w, c),(z, y)) = d_{G\square H}((w, c),(x, b)) = d_{G\square H}((w, c),(x, a)) = 2$, $(z, y) \notin S$, $(x, b) \notin S$ and $(x, a) \notin S$. Similarly, since

$$d_{G\square H}((x, b),(z, a)) = d_{G\square H}((x, b),(w, y)) = 2,$$

$(w, y) \notin S$. This further implies that $(x, c) \notin S$. If $|V(G)| = 3$ and $|V(H)| = 4$, then $N_{G\square H}((z, b), 2) \cap S = N_{G\square H}((z, y), 2) \cap S = N_{G\square H}((z, c), 2) \cap S = N_{G\square H}((x, a), 2) \cap S = \varnothing$, a contradiction to our assumption that $S$ is a perfect hop dominating set of $G\square H$. If $|V(G)| > 3$ and $|V(H)| > 4$, then there exist $v \in V(G)\backslash\{x, w, z\}$ and $e \in V(H)\backslash\{a, b, c, y\}$. If $(v, e) \in S$, then again $S$ is not a perfect hop dominating set of $G\square H$. Hence, $(v, e) \notin S$. Therefore in any case, $S$ is not a perfect hop dominating set of $G\square H$, a contradiction to our assumption. Accordingly, $S = V(G\square H)$. □

**Corollary 6.2** *Let $G$ and $H$ be complete graphs such that $|V(G)| \geq 3$ and $|V(H)| \geq 4$. Then $\gamma_{ph}(G\square H) = |V(G)| \cdot |V(H)|$.*

**Proof**. Let $S$ be a minimum perfect hop dominating set of $G\square H$. Then by Theorem 6.1, $S = V(G\square H)$. Therefore,

$$\gamma_{ph}(G\square H) = |S| = |V(G\square H)| = |V(G)| \cdot |V(H)|.$$

□

Corollary 6.2 is not true if $|V(G)| < 3$ and $|V(H)| < 4$ since $\gamma_{ph}(K_2\square K_2) = \gamma_{ph}(K_2\square K_3) = 2$ and $\gamma_{ph}(K_3\square K_3) = 3$.

**Theorem 6.3** *Let $H$ be a connected non-complete graph of order greater than 3 and whose perfect total $(1,2)^*$-dominating set or perfect distance 2-dominating set exists. A nonempty proper subset $C = \bigcup_{x\in S}[\{x\} \times T_x]$ of $V(P_2\Box H)$ where $S \subseteq V(P_2)$ and $T_x \subseteq V(H)$ for each $x \in S$ is a perfect hop dominating set of $P_2\Box H$ if and only if at least one of the following is satisfied.*

(i) *$S = \{x\}$ and $T_x$ is a perfect total $(1,2)^*$-dominating set of $H$.*

(ii) *$S = V(P_2)$, $T_x = T_y$ for $x, y \in S$ and $T_x$ is a perfect distance 2-dominating set in $H$.*

**Proof**. Let $C = \bigcup_{x\in S}[\{x\} \times T_x]$, where $S \subseteq V(P_2)$ and $T_x \subseteq V(H)$ for each $x \in S$, be a perfect hop dominating set of $P_2\Box H$. Let $S = \{x\}$ and $a \in V(H)\backslash T_x$. Then $(x,a) \notin C$. Hence, there exists a unique vertex $(x,b) \in C$ such that $d_{P_2\Box H}((x,a),(x,b)) = 2$. This implies that $d_H(a,b) = 2$. Since $b \in T_x$, $T_x$ is a perfect hop dominating set of $H$. Let $y \in V(P_2)\backslash\{x\}$ and $w \in V(H)$. Then $(y,w) \notin C$. Thus, a unique vertex $(x,c) \in C$ exists such that $d_{P_2\Box H}((y,w),(x,c)) = 2$. Since $y \in N_G(x)$, $w \in N_H(c)$ and $c \in T_x$. Hence, $T_x$ is a perfect total dominating set of $H$. Hence, (i) holds. Suppose $S = V(P_2)$. Let $a \in V(H)\backslash T_x$ where $x \in S$. Then $(x,a) \notin C$. This implies that there exists a unique vertex $(z,q) \in C$ such that $d_{P_2\Box H}((x,a),(z,q)) = 2$. It follows that either $x = z$ and $d_H(a,q) = 2$ or $z \in N_G(x)$ and $a \in N_H(q)$. If $x = z$ and $d_H(a,q) = 2$, then $q \in T_x$. Thus, $T_x$ is a perfect hop dominating set of $H$. If $z \in N_G(x)$ and $a \in N_H(q)$, then $z \in S$ since $S = V(P_2)$ and $q \in T_z$. This implies that $T_x = T_z$ and $T_x$ is a perfect dominating set in $H$. Therefore (ii) holds. Conversely, assume that condition (i) or (ii) holds. Let $(x,a) \notin C$. If (i) holds, then either $x \in S$ and $a \notin T_x$ or $x \notin S$ and $a$ is any vertex of $H$. Let $x \in S$ and $a \notin T_x$. Since $T_x$ is a perfect hop dominating set of $H$, there exists a unique vertex $b \in T_x$ such that $d_H(a,b) = 2$. Thus, $(x,b) \in C$ and $d_{P_2\Box H}((x,a),(x,b)) = 2$. Let $x \notin S$ and $a \in V(H)$. Then $x \in N_G(y)$ for $y \in S$. Since $T_y$ is a perfect total dominating set of $H$, there exists a unique vertex $c \in T_y$ such that $a \in N_H(c)$. Hence, $(y,c) \in C$ and $d_{P_2\Box H}((x,a),(y,c)) = 2$. Suppose (ii) holds. Then $a \notin T_x$. If $T_x$ is a perfect hop dominating set of $H$, then there exists a unique $b \in T_x$ such that $d_H(a,b) = 2$. Hence, $(x,b) \in C$ and $d_{P_2\Box H}((x,a),(x,b) = 2$. If $T_x$ is a perfect dominating set of $H$, then a unique vertex $b \in T_x$ exists such that $a \in N_H(b)$. Since $T_x = T_y$ for $y \in S$, $b \notin T_y$. Hence, $(y,b) \in C$ and $d_{P_2\Box H}((x,a),(y,b)) = 2$. Therefore $C$ is a perfect hop dominating set of $P_2\Box H$. □

**Corollary 6.4** *Let $H$ be a connected non-complete graph of order greater than 3. Then*

$$\gamma_{ph}(P_2\square H) = \begin{cases} min\{\gamma^{*pt}_{1,2}(H), 2\cdot\gamma_{2p}(H)\}, & \text{if } \gamma^{*pt}_{1,2}(H) \text{ exists} \\ 2\cdot\gamma_{2p}(H), & \text{otherwise} \end{cases}$$

**Proof**. Let $C = \bigcup_{x\in S}[\{x\}\times T_x]$ be a minimum perfect hop dominating set of $G\square H$. Then by Theorem 6.3(i), $S = \{x\}$ and $T_x$ is a minimum perfect total $(1,2)^*$-dominating set of $H$ if it exists or by Theorem 6.3(ii), $S = V(P_2)$ and $T_x = T_y$ for all $x, y \in S$ and $T_x$ is a minimum perfect distance 2-dominating set of $H$. Therefore

$$\gamma_{ph}(P_2\square H) = \begin{cases} min\{\gamma^{*pt}_{1,2}(H), 2\cdot\gamma_{2p}(H)\}, & \text{if } \gamma^{*pt}_{1,2}(H) \text{ exists} \\ 2\cdot\gamma_{2p}(H), & \text{otherwise} \end{cases}$$

□

**Theorem 6.5** *Let $G$ be a complete graph of order greater than 2 and $H$ a non-trivial connected graph of order greater than 2 whose perfect total $(1,2)^*$-dominating set exists. A nonempty proper subset $C = \bigcup_{x\in S}[\{x\}\times T_x]$ of $V(G\square H)$ where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$ is a perfect hop dominating set of $G\square H$ if and only if $|S| = 1$ and $T_x$ is a perfect total $(1,2)^*$-dominating set of $H$ for each $x \in S$.*

**Proof**. Let $C = \bigcup_{x\in S}[\{x\}\times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$, be a perfect hop dominating set of $G\square H$. We claim that $|S| = 1$. Since $C$ is a nonempty proper subset of $V(G\square H)$, there exists $(x,a) \notin C$. Hence, a unique vertex $(y,b) \in C$ exists such that $d_{G\square H}((x,a),(y,b)) = 2$. This implies that either $x = y$ and $d_H(a,b) = 2$ or $x \neq y$ and $a \in N_H(b)$. Suppose that $x = y$ and $d_H(a,b) = 2$. Then $x \in S$. Let $c \in N_H(a)$. Since

$$d_{G\square H}((x,a),(z,c)) = d_{G\square H}((x,a),(x,b)) = 2$$

for all $z \neq x$, $(z,c) \notin C$. Also, since $d_{G\square H}((z,c),(w,a)) = d_{G\square H}((z,c),(x,b)) = 2$ for all $w \in V(G)\backslash\{x,z\}$, and $c \in N_H(a) \cap N_H(b)$, $(w,a) \notin C$. Similarly, $(v,b) \notin C$ for all $v \neq x$. If $|V(H)| = 3$, then $(x,c) \in C$ for $c \in N_H(a) \cap N_H(b)$. Hence, $S = \{x\}$. Suppose that $|V(H)| \geq 4$. Let $e \in V(H)\backslash\{a,b,c\}$ where $c \in N_H(a) \cap N_H(b)$. Since $H$ is connected, $e \in N_H(a) \cup N_H(b) \cup N_H(c)$. Hence, $(z,e) \notin C$ for all $z \neq x$. Without loss of generality, suppose that $e \in N_H(a)$. Since $d_{G\square H}((z,a),(x,c)) = d_{G\square H}((z,a),(x,e)) = 2$ for all $z \neq x$, it follows that $(x,e) \notin C$. If $ec \notin E(H)$ or $eb \notin E(H)$ but not both, then $S = \{x\}$. Suppose that $ec \in E(H)$ and $eb \in E(H)$. If $N_G(e,2) \cap T_x \neq \varnothing$, then $S = \{x\}$. If $N_H(e,2) \cap T_x = \varnothing$, then there exists a unique $f \in$

$N_H(e)$ such that $(z,f) \in C$. This implies that $z \neq x$ and $z \in S$. This is a contradiction to our assumption that $S$ is a perfect hop dominating set of $G\Box H$ since $d_{G\Box H}((x,a),(z,f)) = d_{G\Box H}((x,a),(x,b)) = 2$. Thus, $S = \{x\}$. On the other hand suppose that $x \neq y$ and $a \in N_H(b)$. Since $d_{G\Box H}((x,a),(y,b)) = d_{G\Box H}((x,a),(z,b)) = 2$ for all $z \neq y$, it follows that $(z,b) \notin C$. Similarly, $(z,c) \notin C$ for all $z \neq x$ and $c \in N_H(a)\backslash\{b\}$. Suppose $(y,a) \in C$. Since $d_{G\Box H}((x,c),(y,a)) = d_{G\Box H}((x,c),(z,a)) = 2$ for all $z \neq x, y$, $(z,a) \notin C$ for all $z \in V(G)\backslash\{x,y\}$. Similarly, using argument above, we can show that $S = \{y\}$. Hence, $|S| = 1$. Next, we claim that $T_x$ is a perfect total $(1,2)^*$-dominating set of $H$. Let $S = \{x\}$ and $a \in V(H)\backslash T_x$. Then $(x,a) \notin C$. This implies that there exists a unique vertex $(x,b) \in C$ such that $d_{G\Box H}((x,a),(x,b)) = 2$. Hence, $d_H(a,b) = 2$ implying that $T_x$ is a perfect hop dominating set $H$. Since $|V(G)| \geq 3$, there exists a vertex $y \in V(G)\backslash\{x\}$. Let $a \in V(H)$. Then $(y,a) \notin C$. Thus, there exists a unique vertex $(x,c) \in C$ such that $d_{G\Box H}((y,a),(x,c))) = 2$. Hence, $a \in N_H(c)$. Since $c \in T_x$, $T_x$ is a perfect total dominating set of $H$. Therefore $T_x$ is a perfect total $(1,2)^*$-dominating set of $H$. Conversely, suppose that $S = \{x\}$ and $T_x$ is a perfect total $(1,2)^*$-dominating set of $H$. Let $(y,a) \notin C$. Then $y = x$ and $a \notin T_x$ or $x \neq y$ and $a \in V(H)$. Suppose $y = x$ and $a \notin T_x$. Since $T_x$ is a perfect hop dominating set of $H$, there exists a unique vertex $b \in T_x$ such that $d_H(a,b) = 2$. Thus, $(x,b) \in C$ and $d_{G\Box H}((y,a),(x,b)) = 2$. On the other hand, if $x \neq y$ and $a \in V(H)$, then there exists a unique vertex $c \in T_x$ such that $a \in N_H(c)$ since $T_x$ is a perfect total dominating set of $H$. Hence, $(x,c) \in C$ and $d_{G\Box H}((y,a),(x,c)) = 2$. Accordingly, $C$ is a perfect hop dominating set of $G\Box H$. □

**Corollary 6.6** *Let $G$ be a complete graph of order greater than 2 and $H$ a connected graph of order greater than 2 whose perfect total $(1,2)^*$-dominating set exists. Then $\gamma_{ph}(G\Box H) = \gamma_{1,2}^{*pt}(H)$*

**Proof**. Let $C = \bigcup_{x\in S}[\{x\} \times T_x]$ be a minimum perfect hop dominating set of $G\Box H$. By Theorem 6.5, $|S| = 1$ and $T_x$ is $\gamma_{1,2}^{*pt}$-set of $H$. Therefore $\gamma_{ph}(G\Box H) = |C| = \sum_{x\in S} |T_x| = |S| \cdot |T_x| = \gamma_{1,2}^{*pt}(H)$. □

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