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Interior Domination in Graphs Under Some Binary Operations[1]

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Abstract

Let G be a simple graph. A set $D \subseteq V(G)$ is an interior dominating set of G if D is a dominating set of G and every vertex $v \in D$ is an interior vertex of G. The minimum cardinality of an interior dominating set of G, denoted by $\gamma_{Id}(G)$, is called an interior domination number of G.

In this paper, we characterize the interior dominating sets of the join, corona, lexicographic and Cartesian products of graphs and determine the corresponding interior domination number of these graphs.

Mathematics Subject Classification: 05C69

Keywords: interior domination, join, corona, lexicographic product, Cartesian product

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1 Introduction

Let $G = (V(G), E(G))$ be a simple graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhod* of v is the set $N_G[v] = N_G(v) \cup \{u\}$. The degree $deg_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v, that is, $deg_G(v) = |N_G(v)|$.

Let x and z be two distinct vertices in G. A vertex y distinct from x and z is said to *lie between* x and z if $d_G(x, z) = d_G(x, y) + d_G(y, z)$. A vertex v is an *interior vertex* of G if for every vertex u distinct from v, there exists a vertex w such that v lies between u and w. The set of all interior vertices of G is called the *interior set* of G and is denoted by $Int(G)$. A vertex u in a connected graph G is called an *extreme vertex* if $\langle N_G(u)\rangle$ is a complete graph. The set of all extreme vertices of G is denoted by $Ext(G)$.

The following concepts are defined in [5].

A set $D \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if for every vertex $u \in V(G) \setminus D$ (resp. for every $u \in V(G)$), there exists $v \in D$ such that $uv \in E(G)$. The minimum cardinality of a dominating set (resp. *total dominating set*) of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the *domination number* (resp. *total domination number*) of G.

A set $D \subseteq V(G)$ is an *interior dominating set* [5] of G if D is a dominating set of G and every vertex $v \in D$ is an interior vertex of G. The minimum cardinality of an interior dominating set of G is called the *interior domination number* and is denoted by $\gamma_{Id}(G)$. A γ_{Id}-set is an interior dominating set of G with cardinality $\gamma_{Id}(G)$. A total dominating set $S \subseteq V(G)$ is called an *interior total dominating set* of G if every vertex $v \in S$ is an interior vertex of G. The *interior total domination number* of G, denoted by $\gamma_{It}(G)$, is the smallest cardinality of an interior total dominating set of G. An interior total dominating set S of G with $|S| = \gamma_{It}(G)$ is referred to as a γ_{It}-set of G.

Domination provides several applications both in the position and protection strategies [1, 4]. In a given network or graph, determining the exact location of an intruder and requiring that each node where there is no monitor in it is connected to at least one monitoring device becomes the problem of locating dominating set. As mentioned in [5], the concept of interior set of vertices of a graph G has applications in locating dominating set. Thus, it is necessary to determine the collection of interior vertices where to place monitoring devices so that if there is an object at any vertex in the network or graph, it can be detected and its position is uniquely identified.

This new variant of domination was introduced in 2016 by A. Anto Kinsley and Caroline Selvaraj in [5].

2 Realization Problem

In this section, the parameter γ_{Id} is compared to the known parameter γ. Since every interior dominating set of a connected graph G is a dominating set of G, $\gamma(G) \leq \gamma_{Id}(G)$.

Theorem 2.1 *Given any positive integer $n \geq 3$, there exist connected graphs G and H each of order $4n$ such that*

(*i*) $\gamma(G) = \gamma_{Id}(G) = n$ *and*

(*ii*) $\gamma(H) = n$ *and* $\gamma_{Id}(H) = n + 1$

Proof. (*i*) Let G be the graph shown in Figure 1. It is clear that the set $A = \{x_i : i = 1, 2, ..., n\}$ is both a γ-set and γ_{Id}-set of G. It follows that $\gamma_{Id}(G) = \gamma(G) = |A| = n$.

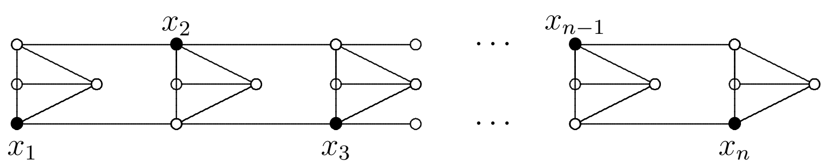

Figure 1: A graph G with $\gamma(G) = \gamma_{Id}(G)$

(*ii*) Let H be the graph shown in Figure 2. Observe that the set $A = \{x_i : i = 1, 2, ..., n\}$ is a γ-set and set $B = \{y_i : i = 1, 2, ..., n + 1\}$ is a γ_{Id}-set of G. Hence, $\gamma(G) = |A| = n$ and $\gamma_{Id}(G) = |B| = n + 1$.

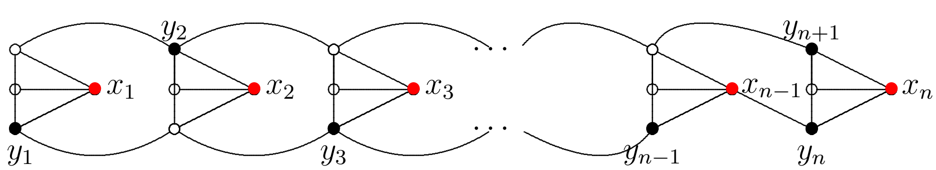

Figure 2: A graph H with $\gamma(H) = n$ and $\gamma_{Id}(H) = n + 1$

This proves the assertion. □

3 Preliminary Results

Theorem 3.1 [5] A dominating set $D \subseteq V(G)$ is an *interior dominating set* if and only if for every $v \in D$, $|N_G(v)| \geq 2$ and for all $x \in N_G(v)$, there exists $y \in N_G(v)$ such that $d_G(x, y) = d_G(x, v) + d_G(v, y)$, that is, there exists $y \in N_G(v) \setminus N_G[x]$ or $d_G(x, y) = 2$.

Example 3.2 Consider the graphs G and H in Figure 3. Then the set $S = \{a, e\}$ is a dominating set of G which is not an interior dominating set of G while $T = \{4, 5\} = Int(H)$ is a γ_{Id}-set of H. Observe that vertex e is the only interior vertex of G and $\{e\}$ is not a dominating set of G.

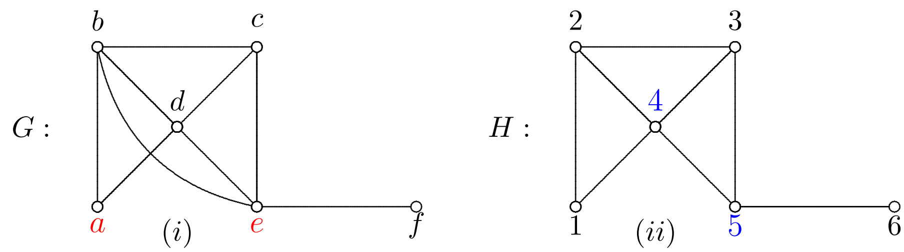

Figure 3: A graph G with interior vertex e and $\gamma_{Id}(H) = 2$

An interior dominating set of a connected nontrivial graph does not always exist as illustrated in Figure 3(i). Hence, it is assumed that all connected nontrivial graphs considered in this paper have nonempty interior sets which are dominating.

As an immediate consequence of the definition of interior vertex and Theorem 3.1, we have the following remarks.

Remark 3.3 *The interior set of the complete graph K_n and the fan F_3 are empty.*

Remark 3.4 *Let G be a connected graph. Then $Int(G) \cap Ext(G) = \varnothing$ and $1 \leq \gamma_{Id}(G) \leq |Int(G)|$.*

The following result characterizes all connected graphs G of order $n \geq 3$ that attain the lower bound of $\gamma_{Id}(G)$.

Theorem 3.5 *Let G be a connected graph of order $n \geq 3$. Then $\gamma_{Id}(G) = 1$ if and only if $G = K_1 + H$, for any graph H with $\gamma(H) \neq 1$.*

Proof. Suppose that $\gamma_{Id}(G) = 1$. Let $S = \{x\}$ be a γ_{Id}-set in G and $H = \langle V(G) \setminus \{x\}\rangle$. Then $G = \langle\{x\}\rangle + \langle V(G) \setminus \{x\}\rangle = K_1 + H$. We claim that $\gamma(H) \neq 1$. Suppose that $\gamma(H) = 1$. Then there exists $a \in V(H)$ such that $deg_H(a) = |V(H)| - 1$. Since $a \in N_G(x)$, x is not an interior vertex of G. This is a contradiction since $S = \{x\}$ is a γ_{Id}-set in G. Thus, $\gamma(H) \neq 1$.

Conversely, suppose that (i) holds. Let $K_1 = \langle\{x\}\rangle$ and $y \in V(H)$. Since $\gamma(H) \neq 1$ and $|V(G)| \geq 3$, there exists $w \in V(H) \setminus \{y\}$ such that $d_G(y, w) = 2$. Thus, x is an interior vertex of G. Hence, $\gamma_{Id}(G) = 1$. □

4 Interior Domination in the Join of Graphs

The *join* $G+H$ of two graphs G and H is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Theorem 4.1 *Let G and H be connected graphs. If $\gamma_{Id}(G) = 1$ and $\gamma_{Id}(H) = 1$, then $G+H$ has no interior dominating set.*

Proof. Let $D_1 = \{v\}$ and $D_2 = \{u\}$ be the γ_{Id}-sets of G and H, respectively. Suppose $G+H$ has an interior dominating set S and $S \subset V(G)$. Let $x \in S$. Since $u \in N_{G+H}(x)$, there exists $y \in N_{G+H}(x)$ such that $d_{G+H}(u,y) = 2$. If $y \in V(H)$, then $y \notin N_H(u)$ which is a contradiction since D_2 is a dominating set of H. On the other hand, if $y \in V(G)$, then $y \notin N_{G+H}(u)$ which is again a contradiction to the definition of $G+H$. Similarly, if $S \subset V(H)$ or $S = S_G \cup S_H$, where $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$, then we get a contradiction. Therefore, $G+H$ does not have an interior dominating set. □

The converse of Theorem 4.1 is not true. To see this, consider the graphs G and $G+H$ in Figure 4. Let $H = K_2$. Then it can be verified that the interior dominating set of $G+H$ does not exists. Furthermore, $Int(G) = \varnothing$ and $Int(K_2) = \varnothing$.

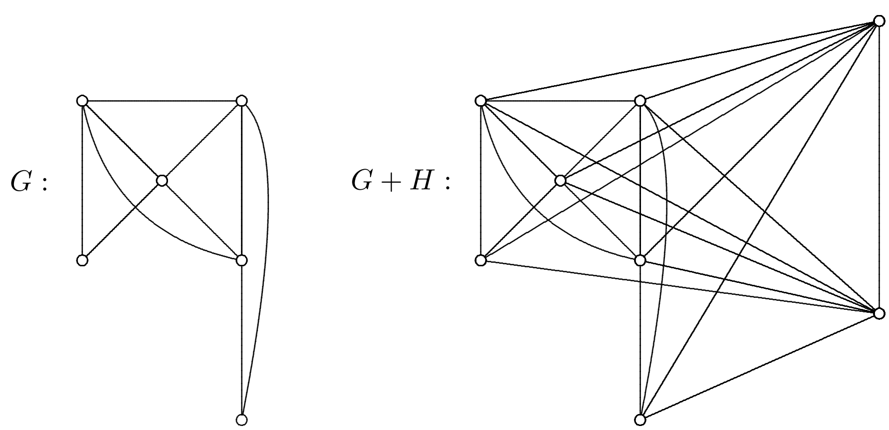

Figure 4: The interior dominating set of $G+H$ does not exist.

Lemma 4.2 *Let G and H be connected graphs such that $\gamma(G) \neq 1$ or $\gamma(H) \neq 1$. Then $Int(G+H) = Int(G) \cup Int(H)$.*

Proof. Let $x \in Int(G+H)$. Then either $x \in V(G)$ or $x \in V(H)$. Suppose $x \in V(G)$. Let $y \in N_G(x)$. Then $y \in N_{G+H}(x)$. Since $x \in Int(G+H)$, there exists $z \in N_{G+H}(x)$ such that $d_{G+H}(y,z) = 2$. Hence, $x \in Int(G)$. Similarly, if $x \in V(H)$, then $x \in Int(H)$.

Let $x \in Int(G) \cup Int(H)$. Then $x \in Int(G)$ or $x \in Int(H)$. Suppose $x \in Int(G)$. Let $y \in N_{G+H}(x)$. If $y \in V(G)$, then we are done. If $y \in V(H)$, then a vertex $z \in V(H)$ exists such that $z \notin N_H(y)$ and $z \in N_{G+H}(x)$ since $\gamma(H) \neq 1$. Hence, $x \in Int(G+H)$. Similarly, if $x \in Int(H)$, then $x \in Int(G+H)$. Therefore $Int(G+H) = Int(G) \cup Int(H)$. □

Theorem 4.3 *Let G and H be connected graphs such that $G+H$ has an interior dominating set. Then $S \subseteq V(G+H)$ is an interior dominating set of $G+H$ if and only if at least one of the following is true:*

(i) $S \subseteq V(G)$, *S is an interior dominating set of G and* $\gamma(H) \neq 1$.

(ii) $S \subseteq V(H)$, *S is an interior dominating set of H and* $\gamma(G) \neq 1$.

(iii) $S \cap V(G)$ *and* $S \cap V(H)$ *are nonempty subsets of the interior sets of G and H, respectively, and* $\gamma(G) \neq 1$ *and* $\gamma(H) \neq 1$.

Proof. Let $S \subseteq V(G+H)$ be an interior dominating set of $G+H$. Suppose $S \cap V(H) = \varnothing$ or $S \cap V(G) = \varnothing$. Then $S \subseteq V(G)$ or $S \subseteq V(H)$. Suppose $S \subseteq V(G)$. Since S is an interior dominating set of $G+H$, by Lemma 4.2, S is an interior dominating set of G. Now, suppose $\gamma(H) = 1$. Let $A = \{u\}$ be a γ-set of H. Let $x \in S \subseteq V(G)$. Since $u \in N_{G+H}(x)$, there exists $y \in N_{G+H}(x)$ such that $d_{G+H}(u, y) = 2$. This is a contradiction to the definition of $G+H$ or A as a γ-set of H. Hence, (i) holds. Similarly, if $S \subseteq V(H)$, then (ii) holds. Now, suppose $S \cap V(G) \neq \varnothing$ and $S \cap V(H) \neq \varnothing$. Since $S \cap V(G) \subseteq S$ and $S \cap V(H) \subseteq S$, by Lemma 4.2, $\varnothing \neq S \cap V(G) \subset Int(G)$ and $\varnothing \neq S \cap V(H) \subset Int(H)$. Suppose $\gamma(G) = 1$ or $\gamma(H) = 1$. Let $\gamma(G) = 1$ and $D = \{v\}$ be a γ-set of G. Let $x \in S \cap V(H)$. Since $v \in N_{G+H}(x)$ and $S \cap V(H) \subset Int(G+H)$, there exists $y \in N_{G+H}(x)$ such that $d_{G+H}(v, y) = 2$. This is a contradiction to the definition of $G+H$ or D as a γ-set of G. Similarly, we get a contradiction if $\gamma(H) = 1$. Therefore $\gamma(G) \neq 1$ and $\gamma(H) \neq 1$. It follows that (iii) holds.

Conversely, let $x \in S$ and $y \in N_{G+H}(x)$. Suppose (i) holds. If $y \in V(G)$, then we are done. If $y \in V(H)$, then there exists $z \in V(H)$ such that $z \notin N_H(y)$ since $\gamma(H) \neq 1$. Thus, $d_{G+H}(y, z) = d_{G+H}(y, x) + d_{G+H}(x, z) = 2$. It follows that $S \subset Int(G+H)$. By [2], S is a dominating set of $G+H$. Therefore S is an interior dominating set of $G+H$. Similarly, if (ii) or (iii) holds, then S is an interior dominating set of $G+H$. □

Corollary 4.4 *Let G and H be connected graphs such that $G+H$ has an interior dominating set. Then*

$$\gamma_{Id}(G+H) = \begin{cases} 1, & \text{if } \gamma_{Id}(G) = 1 \text{ and } \gamma(H) \neq 1 \text{ or } \gamma_{Id}(H) = 1 \text{ and } \gamma(G) \neq 1 \\ 2, & \text{if } \gamma_{Id}(G) \neq 1,\ \gamma_{Id}(H) \neq 1,\ Int(G) \neq \varnothing \text{ and } Int(H) \neq \varnothing \\ \gamma_{Id}(G), & \text{if } Int(H) = \varnothing,\ \gamma(H) \neq 1 \\ \gamma_{Id}(H), & \text{if } Int(G) = \varnothing,\ \gamma(G) \neq 1 \end{cases}$$

Proof. Suppose $\gamma_{Id}(G) = 1$ and $\gamma(H) \neq 1$. Let $S = \{x\} \subseteq V(G)$ be an interior dominating set of G. Then by Theorem 4.3, S is an interior dominating set of $G + H$. Thus, $\gamma_{Id}(G + H) = 1$. The same result holds if $\gamma_{Id}(H) = 1$ and $\gamma(G) \neq 1$.

Next, suppose $\gamma_{Id}(G) \neq 1$, $\gamma_{Id}(H) \neq 1$, $Int(G) \neq \varnothing$ and $Int(H) \neq \varnothing$. Pick $x \in Int(G)$ and $y \in Int(H)$. Set $S = \{x, y\}$. By Theorem 4.3(iii), S is an interior dominating set of $G + H$. Therefore, $\gamma_{Id}(G + H) = 2$.

Now, suppose $Int(H) = \varnothing$, $\gamma(H) \neq 1$. Let S be a γ_{Id}-set of G. Then by Theorem 4.3(i), S is an interior dominating set of $G + H$. Hence, $\gamma_{Id}(G + H) \leq |S| = \gamma_{Id}(G)$. Next, suppose that S be a γ_{Id}-set of $G + H$. Then by Theorem 4.3, S is an interior dominating set of G and $\gamma(H) \neq 1$. Hence, $\gamma_{Id}(G + H) \geq |S| = \gamma_{Id}(G)$. This implies that $\gamma_{Id}(G + H) = \gamma_{Id}(G)$. Similarly, if $Int(G) = \varnothing$ and $\gamma(G) \neq 1$, then $\gamma_{Id}(G + H) = \gamma_{Id}(H)$. □

5 Interior Domination in the Corona of Graphs

The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained from G by taking a copy H^v of H and forming the join $\langle v \rangle + H^v = v + H^v$.

Lemma 5.1 *Let G be a nontrivial connected graph and H be any graph. Then $V(G) = Int(G \circ H)$.*

Proof. Claim 1. $Int(G \circ H) \subset V(G)$

Let $x \in Int(G \circ H)$. Suppose $x \notin V(G)$. Then $x \in V(G \circ H) \setminus V(G)$. Then $x \in V(H^v)$ for some $v \in V(G)$. Since $v \in N_{G \circ H}(x)$ and $vy \in E(G \circ H)$ for every $y \in N_{G \circ H}(x) \setminus \{v\}$, x is not an interior vertex of $G \circ H$. It follows that $x \in V(G)$.

Claim 2. $V(G) \subset Int(G \circ H)$

Let $v \in V(G)$ and $u \in N_{G \circ H}(v)$. Suppose $u \in V(G)$. Then there exists $y \in V(H^v)$ such that $d_{G \circ H}(y, u) = 2$. If $u \notin V(G)$, then $u \in V(H^v)$. Since G is non-trivial and connected, there exists $w \in V(G) \cap N_G(v)$. Thus, $d_{G \circ H}(u, w) = 2$. Hence, $v \in Int(G \circ H)$.

Therefore $V(G) = Int(G \circ H)$. □

Theorem 5.2 *Let G be a nontrivial connected graph and H be any graph. Then $S \subseteq V(G \circ H)$ is an interior dominating set of $G \circ H$ if and only if $S = V(G)$.*

Proof. Suppose $S \subseteq V(G \circ H)$ is an interior dominating set of $G \circ H$. By Lemma 5.1, $S \subseteq V(G)$. Since S is a dominating set of $G \circ H$, $S = V(G)$.

Conversely, let $S = V(G)$. Then by Lemma 5.1, S is an interior dominating set of $G \circ H$. □

Corollary 5.3 *Let G be a nontrivial connected graph and H be any graph. Then $\gamma_{Id}(G \circ H) = |V(G)|$.*

6 Interior Domination in the Lexicographic Product of Graphs

The *lexicographic product* of two graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

Lemma 6.1 *Let G and H be nontrivial connected graphs. Then $Int(G[H]) = Int(G) \times Int(H)$.*

Proof. Let $(x, a) \in Int(G[H])$, $y \in N_G(x)$ and $b \in N_H(a)$. Since $(y, b) \in N_{G[H]}((x, a))$, there exists $(z, c) \in N_{G[H]}((x, a))$ such that $d_{G[H]}((z, c), (y, b) = 2$. Hence, $z \in N_G(x)$, $d_G(y, z) = 2$, and choose $c \in N_H(a)$. Thus, $x \in Int(G)$ and $a \in Int(H)$. This implies that $(x, a) \in Int(G) \times Int(H)$.

Let $(x, a) \in Int(G) \times Int(H)$ and $(y, b) \in N_{G[H]}((x, a))$. Then either $y \in N_G(x)$ or $x = y$ and $b \in N_H(a)$. Suppose $y \in N_G(x)$. Since $x \in Int(G)$, there exists $z \in N_G(x)$ such that $d_G(y, z) = 2$. Then $(z, a) \in N_{G[H]}((x, a))$, where $d_{G[H]}((z, a), (y, b) = 2$. Furthermore, there exists $c \in N_H(a)$ such that $d_H(b, c) = 2$ since $a \in Int(H)$. It follows that $(x, a) \in Int(G[H])$.

Therefore $Int(G[H]) = Int(G) \times Int(H)$. □

Theorem 6.2 *Let G and H be nontrivial connected graphs such that $G[H]$ admits an interior dominating set. A subset $C = \bigcup_{x \in S}(\{x\} \times T_x)$ of $V(G[H])$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for every $x \in S$ is an interior dominating set of $G[H]$ if and only if at least one of the following is true:*

(i) S is an interior total dominating set of G and $T_x \subseteq Int(H)$ for every $x \in S$ or

(ii) S is an interior dominating set of G and T_x is an interior dominating set of H for every $x \in S \setminus N_G(S)$.

Proof. Suppose C is an interior dominating set of $G[H]$. By [3], S is a dominating set of G. We show that $S \subseteq Int(G)$. Let $u \in S$, $v \in N_G(u)$ and $p \in T_u$. Then $(u, p) \in C$ and $(v, p) \in N_{G[H]}(u, p)$. Since $C \subseteq Int(G[H])$, there exists $(w, q) \in N_{G[H]}(u, p)$ such that $d_{G[H]}((v, p), (w, q) = 2$. This means that $d_G(v, w) = 2$ or $v = w$ and $d_H(p, q) = 2$. If $d_G(v, w) = 2$, then

$d_{G[H]}((v,p),(w,q) = 2$. If $v = w$ and $d_H(p,q) = 2$, then $d_{G[H]}((v,p),(w,q) = 2$. Hence, $S \subseteq Int(G)$. Next, we show that $T_x \subseteq Int(H)$ for every $x \in S$. Let $z \in T_x$ and $a \in N_H(z)$. Then $(x,z) \in C$ and $(x,a) \in N_{G[H]}((x,z))$. Since $C \subseteq Int(G[H])$, there exists $(l,c) \in N_{G[H]}((x,z))$ such that $d_{G[H]}((x,a),(l,c)) = 2$. This implies that $l = x$ and $cz \in E(H)$. Hence, $d_H(a,c) = 2$. Thus, $T_x \subseteq Int(H)$ for every $x \in S$. If S is a total dominating set of G, then (i) holds. Assume that S is not a total dominating set of G. Then $S \setminus N_G(S) \neq \varnothing$. Let $x \in S \setminus N_G(S)$. By [3], T_x is a dominating set of H. Since $T_x \subseteq Int(H)$, T_x is an interior dominating set of H. Thus, (ii) holds.

Conversely, let $C = \bigcup_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$ satisfying (i) or (ii). By [3], C is a dominating set of $G[H]$. We claim that $C \subseteq Int(G[H])$. Let $(x,a) \in C$ and $(y,b) \in N_{G[H]}((x,a))$. Then $y \in N_G(x)$ or $x = y$ and $b \in N_H(a)$. Suppose (i) holds. If $y \in N_G(x)$, then there exists $z \in N_G(x)$ such that $d_G(y,z) = 2$. This implies that $(z,a) \in N_{G[H]}((x,a))$ and $d_{G[H]}((z,a),(y,b)) = 2$. Suppose $x = y$ and $b \in N_H(a)$. Since $a \in T_x \subseteq Int(H)$ and $b \in N_H(a)$, there exists $c \in V(H) \cap N_H(a)$ such that $d_H(b,c) = 2$. Thus, $(x,c) \in N_{G[H]}((x,a))$ and $d_{G[H]}((x,c),(x,b)) = 2$. Hence, $C \subseteq Int(G[H])$ and C is an interior dominating set of $G[H]$.

Suppose now that (ii) holds. Let $y \in N_G(x)$. Since x is an interior vertex of G, there exists $z \in N_G(x)$ such that $d_G(y,z) = 2$. Then $(z,a) \in N_{G[H]}((x,a))$ and $d_{G[H]}((y,b),(z,a)) = 2$. Suppose that $x = y$ and $b \in N_H(a)$. Since $T_x \subseteq Int(H) \neq \emptyset$ and $a \in T_x$ with $b \in N_H(a)$, there exists $r \in N_H(a)$ such that $d_H(b,r) = 2$. Hence, $(x,r) \in N_{G[H]}((x,a))$ and $d_{G[H]}((x,r),(y,b)) = 2$.

Accordingly, C is an interior dominating set in $G[H]$. □

Corollary 6.3 *Let G and H be nontrivial connected graphs such that $G[H]$ admits an interior dominating set and $\gamma_{Id}(H) = 1$. Then a subset $C = \bigcup_{x\in S}(\{x\} \times T_x)$ of $V(G[H])$ is a minimum interior dominating set of $G[H]$ if and only if S is a minimum interior dominating set of G, T_x is a γ_{Id}-set for each $x \in S \setminus N_G(S)$, $T_x \subseteq Int(H)$ and $|T_x| = 1$ for all $x \in S$.*

Proof. Suppose $C = \bigcup_{x\in S}(\{x\} \times T_x)$ is a minimum interior dominating set of $G[H]$. By Theorem 6.2, S is an interior dominating set of G and $T_x \subseteq Int(H)$ for all $x \in S$, where T_x is a γ_{Id}-set for each $x \in S \setminus N_G(S)$. Suppose $|T_z| \geq 2$ for some $z \in S$. Let $\{a\}$ be a γ_{Id}-set of H. Define $D_x = \{a\}$ for all $x \in S$. Then $C_1 = \bigcup_{x\in S}(\{x\} \times D_x)$ is an interior dominating set by Theorem 6.2(ii). Moreover,

$$|C_1| = |S| < \sum_{x\in D_1} |T_x| + \sum_{x\in D_2} |T_x| = |C|,$$

where $D_1 = S \cap N_G(S)$ and $D_2 = S \setminus N_G(S)$. This contradicts the fact that C is a minimum interior dominating set of $G[H]$. Therefore, $|T_x| = 1$ for all

$x \in S$. Consequently, $|C| = |S|$.

Let S_1 be an interior dominating set of G. Set $M_x = \{a\}$, where $\{a\}$ is a γ_{Id}-set of H. Then $C_2 = \bigcup_{x\in S_1}(\{x\} \times M_x)$ is an interior dominating set of $G[H]$ by Theorem 6.2(ii). Moreover, $|S| = |C| \leq |C_2| = |S_1|$. This implies that S is a minimum interior dominating set of G.

For the converse, suppose that $C = \bigcup_{x\in S}(\{x\} \times T_x)$ and S is a minimum interior dominating set of G, T_x is a γ_{Id}-set for every $x \in S \setminus N_G(S)$, $T_x \subseteq Int(H)$ and $|T_x| = 1$ for all $x \in S$. By Theorem 6.2, C is an interior dominating set of $G[H]$. If $C_1 = \bigcup_{x\in S_1}(\{x\}\times L_x)$ is an interior dominating set in $G[H]$, then by Theorem 6.2, S_1 is an interior dominating set of G. Let $D_1 = S_1 \cap N_G(S_1)$ and $D_2 = S_1 \setminus N_G(S_1)$. Then

$$|C| = |S| \leq |S_1| = |D_1| + |D_2| \leq \sum_{x\in D_1} |L_x| + \sum_{x\in D_2} |L_x| = |C_1|.$$

This implies that C is a minimum interior dominating set of $G[H]$. □

The following result follows from Corollary 6.3.

Corollary 6.4 *Let G and H be nontrivial connected graphs with $\gamma_{Id}(H) = 1$. Then $\gamma_{Id}(G[H]) = \gamma_{Id}(G)$.*

We shall need the following lemma.

Lemma 6.5 *Let G be a connected graph where $Int(G)$ a total dominating set. If S is an interior dominating set of G, then $\gamma_{It}(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|$. In particular, $\gamma_{It}(G) \leq 2\gamma_{Id}(G)$.*

Proof. Let S be an interior dominating set of G. If $S = Int(G)$ or S is an interior total dominating set, then we are done. So suppose S is not a total dominating set and $S \neq Int(G)$. Then $S \setminus N_G(S) \neq \emptyset$ and for each $y \in S \setminus N_G(S)$, choose an interior vertex $v_y \in V(G)$ such that $yv_y \in E(G)$. Let $T^* = \{v_y : y \in S \setminus N_G(S)\}$. Then $|T^*| \leq |S \setminus N_G(S)|$. Since S is an interior dominating set and $T^* \subseteq N_G(S)$, $T = S \cup T^*$ is an interior total dominating set of G. Thus,

$$\begin{aligned}\gamma_{It}(G) \leq |T| = |S \cup T^*| &= |S \cap N_G(S)| + |S \setminus N_G(S)| + |T^*| \\ &\leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|\end{aligned}$$

If in particular, S is a minimum interior dominating set of G, then

$$\gamma_{It}(G) \leq |T| = |S| + |T^*| \leq 2|S| = 2\gamma_{Id}(G).$$

This proves the assertion. □

Corollary 6.6 *Let G and H be nontrivial connected graphs with $Int(G)$ a total dominating set and $\gamma_{Id}(H) = 2$. Then a subset $C = \bigcup_{x\in S}(\{x\} \times T_x)$ of $V(G[H])$ is a minimum interior dominating set of $G[H]$ if and only if either* (*i*) *S is a minimum interior total dominating set of G and $|T_x| = 1$, where $T_x \subseteq Int(H)$ for all $x \in S$ or*

(*ii*) *S is an interior dominating set of G such that $|S \cap N_G(S)| + 2|S \setminus N_G(S)| = \gamma_{It}(G)$, $|T_x| = 1$ and $T_x \subseteq Int(H)$ for each $x \in S \cap N_G(S)$, and each T_x is a minimum interior dominating set of H (hence, $|T_x| = 2$) for every $x \in S \setminus N_G(S)$.*

Proof. Suppose $C = \bigcup_{x \in S}(\{x\} \times T_x)$ is a minimum interior dominating set of $G[H]$. Then by Theorem 6.2, S is an interior total dominating set of G and $T_x \subseteq Int(H)$ for every $x \in S$ or S is an interior dominating set of G and T_x is an interior dominating set of H for every $x \in S \setminus N_G(S)$. Suppose first that S is an interior total dominating set of G and $T_x \subseteq Int(H)$ for every $x \in S$. Suppose further that $|T_z| \geq 2$ for some $z \in S$. Let $a \in T_z$ and define $T_z^* = \{a\}$. Then $C^* = [\bigcup_{x \in S \setminus \{z\}}(\{x\} \times T_x)] \bigcup (\{z\} \times T_z^*)$ is an interior dominating set by Theorem 6.2(*i*). This, however, is impossible because $|C^*| < |C|$. Thus, $|T_x| = 1$ for all $x \in S$ and (*i*) holds.

Suppose now that S is not a total dominating set of G. Suppose first that $\gamma_{It}(G) < |S \cap N_G(S)| + 2|S \setminus N_G(S)| \leq |C|$. Choose a minimum interior total dominating set R of G and set $S_x = \{v\}$ for every $x \in R$, where $v \in Int(H)$. Then $Y = \bigcup_{x \in R}(\{x\} \times S_x)$ is an interior total dominating set of $G[H]$ by Theorem 6.2(*i*). It follows that $\gamma_{It}(G) = |R| = |Y| < |C|$, contrary to our assumption of C. Then by [3],

$$\gamma_{It}(G) = |S \cap N_G(S)| + 2|S \setminus N_G(S)|.$$

Next, suppose that there exists $z \in S \cap N_G(S)$ with $|T_z| \geq 2$. Let $a \in T_z \cap Int(H)$ and define $T_z^* = \{a\}$. Then

$$C^* = [\bigcup_{x \in S \setminus \{z\}}(\{x\} \times T_x)] \bigcup (\{z\} \times T_z^*)$$

is an interior dominating set by Theorem 6.2(*ii*). This is not possible because $|C^*| < |C|$. Therefore, $|T_x| = 1$ and $T_x \subseteq Int(H)$ for each $x \in S \cap N_G(S)$.

Finally, suppose that there exists $w \in S \setminus N_G(S)$ such that T_w is not a minimum interior dominating set of H. Since T_w is not a minimum interior dominating set of H, $|T_w| > 2$. Let $L_w = \{a, b\}$, a minimum interior dominating set of H. Then $C_1 = [\bigcup_{x \in S \setminus \{w\}}(\{x\} \times T_x)] \bigcup (\{w\} \times L_w)$ is an interior dominating set by Theorem 6.2(*ii*). Again, this is not possible because $|C_1| < |C|$. Therefore, T_x is a minimum interior dominating set of H for every $x \in S \setminus N_G(S)$.

For the converse, let $C = \bigcup_{x \in S}(\{x\} \times T_x)$. Suppose first that (*i*) holds. Then $|C| = |S| = \gamma_{It}(G)$. Also, by Theorem 6.2(*i*), C is an interior dominating set of $G[H]$. Let $C_1 = \bigcup_{x \in S_1}(\{x\} \times D_x)$ be an interior dominating set of $G[H]$. By Theorem 6.2, S_1 is an interior dominating set of

G. If S_1 is a total dominating set, then $|C| = |S| = \gamma_{It}(G) \leq |S_1| = |C_1|$. If S_1 is not a total dominating set, then D_x is an interior dominating set of H for every $x \in S_1 \setminus N_G(S_1)$. Since $\gamma_{Id}(H) = 2$, $|D_x| \geq 2$ for every $x \in S_1 \setminus N_G(S_1)$. Therefore, by [3],

$$|C| = \gamma_{It}(G) \leq |D_1| + 2|D_2| \leq |C_1|,$$

where $D_1 = S_1 \cap N_G(S_1)$ and $D_2 = S_1 \setminus N_G(S_1)$. This shows that C is a minimum interior dominating set of $G[H]$. If (ii) holds, then a similar argument may be used to show that C is a minimum interior dominating set of $G[H]$. □

Corollary 6.7 *Let G and H be nontrivial connected graphs with $Int(G)$ a total dominating set and $\gamma_{Id}(H) > 2$. Then a subset $C = \bigcup_{x \in S}(\{x\} \times T_x)$ of $V(G[H])$ is a minimum interior dominating set of $G[H]$ if and only if S is a minimum interior total dominating set of G and $|T_x| = 1$ where $T_x \subseteq Int(H)$ for each $x \in S$.*

Proof. Suppose that $C = \bigcup_{x \in S}(\{x\} \times T_x)$ is a minimum interior dominating set of $G[H]$. Suppose that S is an interior dominating set of G but not total dominating. Then by Theorem 6.2(ii), T_x is an interior dominating set of H for every $x \in S \setminus N_G(S)$. Since $\gamma_{Id}(H) > 2$, it follows that $|T_x| > 2$ for every $x \in S \setminus N_G(S)$. Now, by [3], $\gamma_{It}(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|$. Since

$$|C| = \sum_{x \in S \cap N_G(S)} |T_x| + \sum_{x \in S \setminus N_G(S)} |T_x|,$$

it follows that $\gamma_{It}(G) < |C|$. Let S_1 be a minimum interior total dominating set of G and set $Q_x = \{a\}$ for every $x \in S_1$, where $a \in Int(H)$. Put $Q = \bigcup_{x \in S_1}(\{x\} \times Q_x)$. Then Q is an interior dominating set of $G[H]$ by Theorem 6.2. Moreover, $|Q| = |S_1| = \gamma_{It}(G)$. Thus, $|Q| < |C|$, contrary to our assumption of C. Therefore, S is an interior total dominating set of G. Using a similar argument, it can be shown that S is a minimum interior total dominating set of G and $|T_x| = 1$, where $T_x \subseteq Int(H)$ for every $x \in S$.

For the converse, suppose that $C = \bigcup_{x \in S}(\{x\} \times T_x)$ and S is a minimum interior total dominating set of G with $|T_x| = 1$ where $T_x \subseteq Int(H)$ for each $x \in S$. By Theorem 6.2, C is an interior dominating set of $G[H]$. If $C_1 = \bigcup_{x \in S_1}(\{x\} \times L_x)$ is an interior dominating set of $G[H]$, then by Theorem 6.2, S_1 is an interior dominating set of G. Let $D_1 = S_1 \cap N_G(S_1)$ and $D_2 = S_1 \setminus N_G(S_1)$. By Theorem 6.2,

$$|D_1| + 2|D_2| \leq \sum_{x \in D_1} |L_x| + \sum_{x \in D_2} |L_x| = |C_1|.$$

Thus, by Lemma 6.5, $\gamma_{It}(G) \leq |C| \leq |C_1|$. This implies that C is a minimum interior dominating set of $G[H]$. □

The following result gives the interior domination number of the lexicographic product of two connected graphs.

Corollary 6.8 *Let G and H be nontrivial connected graphs with $Int(G)$ a total dominating set and $\gamma_{Id}(H) \geq 2$. Then $\gamma_{Id}(G[H]) = \gamma_{It}(G)$.*

Proof. Let S be a minimum interior total dominating set of G. Pick $a \in Int(H)$ and set $T_x = \{a\}$ and $C = \bigcup_{x \in S}(\{x\} \times T_x)$. By Corollary 6.6 and Corollary 6.7, C is a minimum interior dominating set of $G[H]$. Thus, $\gamma_{Id}(G[H]) = |C| = |S| = \gamma_{It}(G)$. □

7 Interior Domination in the Cartesian Product of Graphs

The *Cartesian product* of two graphs G and H, denoted by $G \Box H$, is the graph with vertex set $V(G \Box H) = V(G) \times V(H)$ and edge set $E(G \Box H)$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G \Box H)$ if and only if either $u_1 u_2 \in E(G)$ and $v_1 = v_2$ or $u_1 = u_2$ and $v_1 v_2 \in E(H)$.

Lemma 7.1 *Let G and H be nontrivial connected graphs. Then every vertex of $G \Box H$ is an interior vertex, that is, $Int(G \Box H) = V(G \Box H)$.*

Proof. Let $(x, a) \in V(G \Box H)$ and $(y, b) \in N_{G[H]}((x, a))$. Then $y \in N_G(x)$ and $a = b$ or $x = y$ and $b \in N_H(a)$. If $y \in N_G(x)$ and $a = b$, then there exists $c \in N_H(a)$ such that $d_{G \Box H}((x, c), (y, b)) = 2$. If $x = y$ and $b \in N_H(a)$, then there exists $z \in N_G(x)$ such that $d_{G \Box H}((y, b), (z, a)) = 2$. Since (x, a) was arbitrarily chosen, every vertex of $G \Box H$ is an interior vertex of $G \Box H$. □

Theorem 7.2 *Let G and H be nontrivial connected graphs. Then $C \subseteq V(G \Box H)$ is an interior dominating set of $G \Box H$ if and only if C is a dominating set of $G \Box H$.*

Proof. Follows from Lemma 7.1. □

Corollary 7.3 *Let G and H be nontrivial connected graphs. Then*

$$\gamma_{Id}(G \Box H) = \gamma(G \Box H).$$

Example 7.4 Consider the graph $G \Box H$ in Figure 5. Then the set of red vertices is a γ-set of $G \Box H$, that is, $\gamma(G \Box H) = 4$. It can also be verified that the set is a γ_{Id}-set. Hence, $\gamma(G \Box H) = 4 = \gamma_{Id}(G \Box H)$.

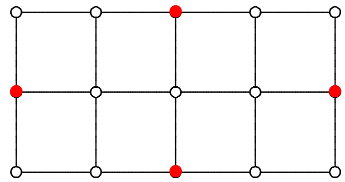

Figure 5: A graph with $\gamma_{Id}(G\Box H) = 4 = \gamma(G\Box H)$

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