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# On the Existence of Positive Solutions of Nonlinear Elliptic Equations with a p-Laplace Operator Involving Critical Sobolev Exponents

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**Abstract**

Let $\Omega$ be bounded domain in $\mathbb{R}^N$, $N \geq p+1$ we will study the existence of a solution $u$ for the following non linear elliptic equation

$$\begin{cases} -\Delta_p u = u^q + f(x,u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \partial\Omega; \end{cases} \tag{0.1}$$

where q is the critical Sobolev embedding, i.e $q+1 = \frac{Np}{N-p}$, $f(x,0) = 0$ and $f(x,u)$ is a lower-order perturbation of $u$.

**Keywords:** Existence of positive solutions; Elliptic equations; Critical Sobolev exponents

# 1 Introduction

By the variational formulation solution of the problem (0.1) correspond to critical points of the functional

$$\Psi(u) = \frac{1}{p}\int |\nabla u|^p - \frac{1}{q+1}\int |u|^{q+1} - \int F(x,u)$$

where $F(x,u) = \int_0^1 f(x,t)dt$ and $q+1 = \frac{Np}{N-p}$ is the limiting to Sobolev exponent for the embedding $W_0^{1+p}(\Omega) \subset L^{q+1}(\Omega)$ which is not compact. So the functional $\Psi$ will not satisfy the Palais Smail condition. Hence as in H.Brezis and L.Nirenberg [5] for the linear case we use the result of Ambrosetti and Robinowiz without the Palais Smail condition to claim our main results.

# 2 Existence of Positive Solutions for $-\Delta_p u = u^q + \lambda|u|^{p-2}u$ in $\Omega$, $u = 0$ in $\partial\Omega$ with $q+1 = \frac{Np}{N-p}$

First we will deal with the typical case when $f(x,u) = \lambda|u|^{p-2}u$. with $\lambda$ is a read constant. Indeed : Let $\Omega \subset \mathbb{R}^N$, $N \geq p+1$, be a bounded domain. Consider the following problem:

$$\begin{cases} -\Delta_p u = u^q + \lambda|u|^{p-2}u & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \partial\Omega; \end{cases} \tag{2.1}$$

here $q+1 = \frac{Np}{N-p}$, $\lambda$ is real constant and $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$.
Our main result shows the existence of a solution $u$ satisfying (2.1). We treat separately the cases $N < p^2$ and $N \geq p^2$ . We denote by $\lambda_1$ the first eigenvalue of $-\Delta_p$ with Dirichlet conditions.

## 2.1 The case $N \geq p^2$

**Theorem 2.1.** *Assume $N \geq p^2$. Then for every $\lambda \in (0, \lambda_1)$ there exists a solution of* (2.1).

**Remark 2.1.** *There is no solution of* (2.1) *when $\lambda \geq \lambda_1$ . In fact, let $v$ be the eigenfunction of $-\Delta_p$ corresponding to $\lambda_1$ with $u_1 > 0$ in $\Omega$. Suppose $u$ is a solution of* (2.1). *Then we have*

$$\begin{cases} -\Delta_p u = u^q + \lambda|u|^{p-2}u & in \quad \Omega \\ -\Delta_p v = \lambda_1|v|^{p-2}v & in \quad \Omega \end{cases}$$

and we can rewrite the above problem as:

$$\begin{cases} -\Delta_p u = \lambda m|u|^{p-2}u & \text{in} \quad \Omega \\ -\Delta_p v = \lambda_1|v|^{p-2}v & \text{in} \quad \Omega \end{cases}$$

with the weight $m = 1 + \frac{u^q}{\lambda u^p 1}$.
Let $u$ (respectively $v$) be the first eigenfunction associated to $\lambda_1(\Omega, m)$ (respectively, $\lambda_1(\Omega, m_0)$). It is well known that $u$ and $v$ don't change sign in $\Omega$ (see [3]), so $(u,v) \in D_I$ and $I(u,v) \geq 0$ where

$$D_I = \{(u_1, u_2) \in W_0^{1,p}(\Omega) : \ u_i \text{ and } \frac{u_i}{u_j} \in L^\infty(\Omega) \text{ for } i,j = 1,2\},$$

and

$$I(u,v) =< -\Delta_p u, \frac{u^p - v^p}{u^{p-1}} > + < -\Delta_p v, \frac{v^p - u^p}{v^{p-1}} >$$

with

$$< -\Delta_p u, v >= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx \quad \text{for all} \quad u, v \in W_0^{1,p}(\Omega).$$

It is clear that we have

$$\lambda_1(\Omega, m) \leq \lambda_1(\Omega, m_0).$$

If we have $\lambda_1(\Omega, m) \leq \lambda_1(\Omega, m_0) = \lambda_1$ , then

$$\begin{aligned} I(u,v) &= \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla(\frac{u^p - v^p}{u^{p-1}})dx + \int_\Omega |\nabla v|^{p-2}\nabla v.\nabla(\frac{v^p - u^p}{v^{p-1}})dx \\ &= \lambda \int_\Omega m|u|^{p-2}u.(\frac{u^p - v^p}{u^{p-1}})dx + \lambda_1 \int_\Omega |v|^{p-2}v.(\frac{v^p - u^p}{v^{p-1}})dx \\ &= \lambda \int_\Omega m(u^p - v^p)dx + \lambda_1 \int_\Omega (v^p - u^p)dx \\ &= \int_\Omega (\lambda m - \lambda_1)(v^p - u^p)dx. \end{aligned}$$

The function :

$$f(\alpha) = I(\alpha u, v),$$

is continuous on $\mathbb{R}$. Without loss of generality, we can assume that $v$ is strictly positive in $\Omega$. Since $\lambda m(x) > \lambda_1$, we have

$$f(0) = -\int_\Omega (\lambda m - \lambda_1)v^P dx < 0,$$

so for $\epsilon = -\frac{1}{2}f(0) > 0$, there exists $\mu > 0$ such that for every $\alpha < \mu$ we have,

$$I(\alpha u, v) = f(\alpha) < f(0) < 0;$$

which is false, since $I(\alpha u, v) \geq 0$ (see [3]) for every $(u, v) \in D_1$ and $\alpha > 0$.

**Remark 2.2.** *There is no solution of* (2.1) *if $\Omega$ is a (smooth) starshaped domain. This follows from Pohozaevs identity, which is : Suppose $g : \mathbb{R} \to \mathbb{R}$ be continuous with primitive $G(u) = \int_0^u g(v)dv$ and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of the equation*

$$\begin{cases} -\Delta_p(u) = g(u) & in \quad \Omega \\ u = 0 & in \quad \partial\Omega \end{cases}$$

*in a domain $\Omega \subset \mathbb{R}^N$. Then*

$$\frac{n-p}{p}\int_\Omega |\nabla u|^p dx - n\int_\Omega G(u)dx + \frac{1}{p}\int_{\partial\Omega} |\frac{\partial u}{\partial \nu}|^p x.\nu ds = 0, \qquad (2.2)$$

*where $\nu$ denotes the exterior unit normal.*
*In particular, when $g(u) = u^q + \lambda u^{p-1}$ we deduce*

$$\lambda \int_\Omega u^p = \frac{1}{p} \int_{\partial\Omega} (x, \nu) |\frac{\partial u}{\partial \nu}|^p.$$

*If $\Omega$ is starshaped about the origin we have $(x.\nu) > 0$ a.e. on $\partial\Omega$. When $\lambda < 0$ it follows immediately that $u = 0$. When $\lambda = 0$ we deduce that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$ and then by (2.1) we have*

$$0 = -\int_\Omega \Delta_p u = \int_\Omega u^q,$$

*so $u = 0$.*

To prove theorem 2.1 we need some preliminaries and lemmas. Set

$$S_\lambda = \inf_A \{||\nabla u||_p^p - \lambda ||u||_p^p\} \tag{2.3}$$

with $A = \{u \in W_0^{1,p}, ||u||_{q+1} = 1\}$ and $\lambda \in \mathbb{R}$. Thus

$$S_0 = S = \inf_A \{||\nabla u||_p^p\} \tag{2.4}$$

corresponds to the best constant for the Sobolev embedding $W_0^{1,p} \subset L_\Omega^{q+1}$, $q + 1 = \frac{NP}{N-p}$. We will discuss some remarks concerning the best Sobolev constant $S$ :

*(a)* $S$ is independent of $\Omega$ and depends only on $N$ since the ratio $\frac{||\nabla u||_p}{||u||_{q+1}}$ with $q + 1 = \frac{Np}{N-p}$ is invariant under scaling for any $u \in W_0^{1,p}$ i.e. the ratio $\frac{||\nabla u_k||_p}{||u_k||_{q+1}}$ is independent of any $k \in \mathbb{R}$, where $u_k(x) = u(kx)$.

*(b)*The infimum in (2.4) is never achieved when $\Omega$ is a bounded domain since otherwise it would contradict Pohozaev's result.

*(c)* When $\Omega = \mathbb{R}^N$ , the inmum in (2.4) is achieved by the function (see [9]):

$$U(x) = C(1 + |x|^{\frac{p}{p-1}})^{\frac{-(N-p)}{p}} \tag{2.5}$$

or (after scaling) by any of the functions

$$U(x) = C_\epsilon(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{\frac{-(N-p)}{p}} \tag{2.6}$$

where $C$ and $C_\epsilon$ are normalization constants

**Lemma 2.1.** *We have*

$$S_\lambda < S \textit{ for all } \lambda > 0.$$

**Proof:**
Our proof here is an adaptation of an argument due to H. Brezis L. Niremberg (see[5]) and Aubin (see [2]).
Without loss of generality we may assume that $0 \in \Omega$. We need to estimate the ratio

$$Q_\lambda(u) = \frac{||\nabla u||_p^p - \lambda ||u||_p^p}{||u||_{q+1}^p} \tag{2.7}$$

with

$$u(x) = u_\epsilon(x) = \frac{\varphi(x)}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{\frac{-(N-p)}{p}}} \tag{2.8}$$

where $\varphi \in D_+(\Omega)$ is a fixed cut off function such that $\varphi(x) = 1$ for $x$ in some neighborhood of 0 and

$$D_+(\Omega) = \{\varphi \in C_0^\infty(\Omega) \; with \varphi \geq 0\},$$

We claim that, as $\epsilon \to 0$, we have

$$||\nabla u_\epsilon||_p^p = \frac{K_1}{\epsilon^{\frac{N-p}{p-1}}} + O(1) \tag{2.9}$$

$$||u_\epsilon||_{q+1}^p = \frac{K_2}{\epsilon^{\frac{N-p}{p-1}}} + O(\epsilon) \tag{2.10}$$

$$||u_\epsilon||_p^p = \begin{cases} \frac{K_3}{\epsilon^{\frac{N-p^2}{p-1}}} + O(1) & \text{if} \quad N > p^2 \\ K_3 |log \epsilon| + O(1) & \text{if} \quad N = p^2; \end{cases} \tag{2.11}$$

where $K_1$, $K_2$ and $K_3$ denote positive constants which depend only on the dimension $N$ and $\frac{K_1}{K_2} = S$.
Proof of (2.9): We have

$$\nabla u_\epsilon = \frac{\nabla \varphi(x)}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{\frac{(N-p)}{p}}} - (\frac{N-p}{p-1}) \frac{\varphi(x)|x|^{\frac{2-p}{p-1}} x}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{\frac{N}{p}}}$$

Since $\varphi = 1$ near 0, it follows that

$$\begin{aligned} \int_\Omega |\nabla u_\epsilon|^p dx &= (\frac{N-p}{p-1})^p \int_\Omega \frac{|x|^{\frac{p}{p-1}}}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^N} + O(1) \\ &= (\frac{N-p}{p-1})^p \int_{\mathbb{R}^N} \frac{|x|^{\frac{p}{p-1}}}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^N} + O(1) \\ &= \frac{K_1}{\epsilon^{(\frac{N-p}{p-1})}} + O(1) \end{aligned}$$

where

$$K_1 = (\frac{N-p}{p-1})^p \int_{\mathbb{R}^N} \frac{|x|^{\frac{p}{p-1}}}{(1+|x|^{\frac{p}{p-1}})^N} = ||\nabla U||_p^p$$

Proof of (2.10): We have

$$\begin{aligned}
\int_\Omega |u_\epsilon|^{q+1} dx &= \int_\Omega \frac{\varphi(x)^{q+1}}{(1+|x|^{\frac{p}{p-1}})^N} dx \\
&= \int_\Omega \frac{\varphi(x)^{q+1}-1}{(1+|x|^{\frac{p}{p-1}})^N} dx + \int_\Omega \frac{1}{(1+|x|^{\frac{p}{p-1}})^N} dx \\
&= O(1) + \int_\Omega \frac{1}{(1+|x|^{\frac{p}{p-1}})^N} dx \\
&= \frac{K_2'}{\epsilon^{\frac{N}{p-1}}} + O(1)
\end{aligned}$$

where

$$K_2' = \int_{\mathbb{R}^N} \frac{dx}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^N} = ||U||_{q+1}^{q+1}$$

so(2.9) follows with $K_2 = ||U||_{q+1}^p$, where $U$ defined in (2.4). Note $\frac{K_1}{K_2} = S$.
Proof of (2.11): We have

$$\begin{aligned}
\int_\Omega |u_\epsilon|^p dx &= \int_\Omega \frac{\varphi(x)^p - 1}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{N-p}} dx + \int_\Omega \frac{1}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{N-p}} dx \\
&= O(1) + \int_\Omega \frac{1}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{N-p}} dx
\end{aligned}$$

When $N > p^2$ we have

$$\int_\Omega \frac{1}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{N-p}} dx = \int_{\mathbb{R}^N} \frac{1}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{N-p}} dx + O(1)$$

so (2.11) follows with

$$K_3 = \int_{\mathbb{R}^N} \frac{1}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{N-p}} dx + O(1)$$

When $N = p^2$, we have for a constant R,

$$\begin{aligned}
\int_\Omega \frac{dx}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{N-p}} &\leq \int_{|x|\leq R} \frac{dx}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{N-p}} \\
&\leq \omega \int_0^R \frac{r^{p^2-1} dr}{(\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}})^{N-p}} \\
&\leq \omega |log \epsilon| + O(1),
\end{aligned}$$

where $\omega$ is the area of $S^{p^2-1}$ ; thus (2.11) follows with $K_3 = \omega$. Combining(2.9),(2.10) , and (2.11), we obtain

$$Q_\lambda(u_\epsilon) = \begin{cases} S + O(\epsilon^{\frac{N-p}{p-1}}) - \lambda \frac{K_3}{K_2}\epsilon^p & \text{if} \quad N > p^2 \\ S + O(\epsilon^p) - \lambda \frac{K_3}{K_2}\epsilon^p |log\epsilon| & \text{if} \quad N = p^2; \end{cases} \tag{2.12}$$

In all cases we deduce that $Q_\lambda(u_\epsilon) < S$ provided $\epsilon > 0$ is small enough. ∎

**Lemma 2.2.** *If* $S < S$, *the infimum in* (2.3) *is achieved. See(E.Lieb* [10], [11]) and (H. Brezis -L. Nirenberg [5]).

**Proof:**
Let $(u_i) \subset W_0^{1,p}$ be a minimizing sequence for (2.3), that is,

$$||u_1||^{q+1} = 1, \tag{2.13}$$

$$||\nabla u_i||_p^p - \lambda ||u_i||_p^p = S_\lambda + o(1) \ as \ i \to \infty. \tag{2.14}$$

Since $u_i$ is bounded in $W_0^{1,p}$ we may extract a subsequence denoted also by $u_i$ such that $u_i$ converges to $u$ weakly in $W_0^{1,p}$, strongly in $L^p$ and a.e. on $\Omega$, with $||u||_{q+1} \leq 1$. Set $v_i = u_i u$, so that $v_j$ converges to 0 weakly in $W_0^{1,p}$ and a.e. on $\Omega$. By (2.4) and (2.13) we have $||\nabla u_i||_p^p \geq S$ . From (2.14) it follows that $\lambda ||u||_p^p \geq SS > 0$ and therefore $u$ is not null. Using (2.14) we get

$$||\nabla u||_p^p + ||\nabla v_i||_p^p - \lambda ||u_i||_p^p = S_\lambda + o(1) asi \to \infty. \tag{2.15}$$

since $v_i$ converges weakly to 0 in $W_0^{1,p}$ . On the other hand, we deduce from a result of Brezis and Lieb [6] that

$$||u + v_i||_{q+1}^{q+1} = ||u||_{q+1}^{q+1} + ||v_i||_{q+1}^{q+1} + o(1)$$

Thus by (2.13) we get

$$1 = ||u||_{q+1}^{q+1} + ||v_i||_{q+1}^{q+1} + o(1)$$

and therefore since $p < q + 1$

$$1 \leq ||u||_{q+1}^p + ||v_i||_{q+1}^p + o(1)$$

which leads to

$$1 \leq ||u||_{q+1}^p + \frac{1}{S} ||\nabla v_i||_p^p + o(1). \tag{2.16}$$

We claim that

$$||\nabla u||_p^p - \lambda ||u||_p^p \leq S_\lambda ||u||_{q+1}^p, \tag{2.17}$$

this will conclude the proof of lemma 2.2 since u is not null. We distinguish two cases:

**(a)** $S > 0$ (i.e., $0 < \; <_1$ ),

**(b)** $S \leq 0$ (i.e., $\geq_1$ ).

In case (a) we deduce from (2.16) that

$$S_\lambda \leq S_\lambda ||u||_{q+1}^p + \frac{S_\lambda}{S} ||\nabla v_i||_p^p + o(1) \tag{2.18}$$

Combining (2.15) and (2.18) we obtain (2.17).
In case (b) we have $S_\lambda \leq S_\lambda ||u||_{q+1}^p$ since $||u||_{q+1} \leq 1$. We deduce, again, (2.17) from (2.15). ∎

**Remark 2.3.** *In their paper (see[5]), H. Brezis and L. Nirenberg mentioned that F. Browder pointed out that this argument proves more i.e. that* $v_i \to 0$ *strongly in* $H_0^1$ *for* $p = 2$. *Here* $v_i \to 0$ *strongly in* $W_0^{1,p}$.

**Proof of Theorem 2.1:**
Let $u \in W_0^{1,p}$ be given by Lemma 2.2 that is,

$$||u||_{q+1} = 1 \; and \; ||\nabla u||_p^p - \lambda ||u||_p^p = S_\lambda$$

We may as well assume that $u \geq 0$ on $\Omega$ (otherwise we replace $u$ by $|u|$). Since $u$ is a minimizer for (2.3) we obtain a Lagrange multiplies $\mu \in \mathbb{R}$ such that

$$-\Delta_p u - \lambda |u|^{p-2} u = u^q \; on \; \Omega$$

In fact, $\mu = S$ , and $S > 0$ since $\lambda < \lambda_1$ . It follows that $ku$ satisfies (1.1) for some appropriate constant $k > 0$ ($k = S_\lambda^{\frac{1}{p-1}}$); note that $u > 0$ on $\Omega$ by the strong maximum principle see (J. L. Vazqez [12]). ∎

**Remark 2.4.** *In their paper, H. Brezis and L. Nirenberg gave another proof of Theorem 2.1 which did not involve Lemma 2.2. Here we try to do the same argument.*
*Consider*

$$\mu_r = \inf_{A_r} \{||\nabla u||_p^p - \lambda ||u||_p^p\} \; for \; r < q \tag{2.19}$$

*where* $A_r = \{u \in W_0^{1,p} with ||u||_{r+1} = 1\}$, *Notice* $\lim_{r \to q} \mu_r = S_\lambda$. *Moreover since the embedding* ($W_0^{1,p} \subset L^{r+1}$) *is compact, the infimum in* (2.19) *is achieved at some* $u_r \in W_0^{1,p}$ *such that* $u_r \geq 0$ *on* $\Omega$, $||u||_{r+1} = 1$ *and (see also P. Drabec and S. I. Pohozaev* [8])

$$-\Delta_p u_r - \lambda |u_r|^{p-2} u_r = \mu_r u_r^r. \tag{2.20}$$

It follows that

$$S||u_r||_{q+1}^p - \lambda||u_r||_p^p \leq ||\nabla u_r||_p^p - \lambda||u_r||_p^p = \mu_r, \tag{2.21}$$

As $r \to q$ (through a subsequence), $u_r$ converge weakly in $W_0^{1,p}$ to $u$. Passing to the limit in (2.21) we obtain

$$S = \lambda||u||_p^p \leq S_\lambda$$

and thus (by Lemma 2.1), $u$ is not identicaly null. Finally, we deduce from (2.20) that $u$ satisfies

$$-\Delta_p u - \lambda|u|^{p-2}u = S_\lambda u_q.$$

Note $ku$ with ($k = S_\lambda^{\frac{1}{q-1}}$) as a solution of (2.2).

**Remark 2.5.** *Theorem 2.1 establishes the existence of solutions of the problem 1.1 when $N \geq p^2$ . Now we turn to the question when $p < N < p^2$ .*
*When $N$ is an integer and if we take $1 < p < 2$, we distinguish three cases:*

**(a)** *When $1 < p^2 \leq 2$, there is no integer $N$ such that $p < N < p^2$ i.e. Theorem 2.1 holds for any dimension $N$ when $1 < p^2 \leq 2$.*

**(b)** *When $2 < p^2 \leq 3$, there is one integer $N = 2$ such that $p < N < p^2$ .*

**(c)** *When $3 < p^2 \leq 4$, there are two integers $N = 2, 3$ such that $p < N < p^2$ . When $p > 2$, there are several integers such that $p < N < p^2$ .*

## 2.2 The case $p < N < p^2$

**Theorem 2.2.** *Assume $\Omega$ is a ball. There exists a solution of (2.2) if and only if $\lambda \in (\frac{1}{2^p}\lambda_1, \lambda_1)$.*

**Lemma 2.3.** *We have*

$$S_\lambda < S \ forall \lambda > \frac{1}{2^p}\lambda_1$$

**Proof:**
As before, we will estimate the ratio

$$Q_\lambda(u) = \frac{||\nabla u||_p^p - \lambda||u||_p^p}{||u||_{q+1}^p}$$

with

$$u(x) = u_r(r) = \frac{\varphi(r)}{(\epsilon^{\frac{p}{p-1}} + |r|^{\frac{p}{p-1}})^{\frac{N-p}{p}}} \; r = |x|, \; \epsilon > 0$$

where $\varphi$ is a fixed smooth function such that $\varphi(0) = 1$, $\varphi(1) = 0$, $\varphi^{'}(0) = 0$ and $\varphi^{'}(r) \geq 0$ for all $r \in (0, 1)$. We claim that , as $\epsilon 0$, we have

$$||\nabla u_\epsilon||_p^p = \frac{K_1}{\epsilon^{\frac{N-p}{p-1}}} + \omega \int_0^1 r^\alpha |\varphi^{'}(r)|^p dr + O(\epsilon^{\frac{-N-p}{p-1}}) \tag{2.22}$$

$$||u_\epsilon||_{q+1}^p = \frac{K_1}{\epsilon^{\frac{N-p}{p-1}}} + O(\epsilon^{\frac{-N-p}{N}}) \tag{2.23}$$

$$||u_\epsilon||_p^p = \omega \int_0^1 r^\alpha |\varphi(r)|^p dr + O(\epsilon^{\frac{N-p}{p}}) \tag{2.24}$$

where $K_1$ and $K_2$ are positive constants such that $\frac{K_1}{K_2} = S$ , $\alpha = \frac{p^2-N-p+1}{p-1}$ and $\omega$ is the area of $S^{N1}$ . Proof of (2.22). We have

$$\nabla u_\epsilon = \frac{\varphi^{'}(r)}{(\epsilon^{\frac{p}{p-1}} + |r|^{\frac{p}{p-1}})^{\frac{(N-p)}{p}}} - \frac{\varphi(r) r^{\frac{1}{p-1}}}{(\epsilon^{\frac{p}{p-1}} + |r|^{\frac{p}{p-1}})^{\frac{N}{p}}}$$

And thus

$$||\nabla u_\epsilon[[_p^p = \omega \int_0^1 |\frac{\varphi^{'}(r) r^{\frac{N-1}{p}}}{(\epsilon^{\frac{p}{p-1}} + |r|^{\frac{p}{p-1}})^{\frac{(N-p)}{p}}} - (\frac{N-p}{p-1}) \frac{\varphi(r) r^{\frac{1}{p-1}+\frac{N-1}{p}}}{(\epsilon^{\frac{p}{p-1}} + |r|^{\frac{p}{p-1}})^{\frac{N}{p}}}|^p dr$$

Using the fact that $\varphi(0) = 1$ and $\varphi^{'}(0) = 0$ we obtain

$$\frac{\varphi^{'}(r) r^{\frac{N-1}{p}}}{(\epsilon^{\frac{p}{p-1}} + |r|^{\frac{p}{p-1}})^{\frac{(N-p)}{p}}} = r^{\frac{\alpha}{p}} \varphi^{'}(r) + O(\epsilon), \tag{2.25}$$

with $\alpha = \frac{p^2-N-p+1}{p-1}$

$$\frac{\varphi(r) r^{\frac{N(p-1)+1}{p(p-1)}}}{(\epsilon^{\frac{p}{p-1}} + |r|^{\frac{p}{p-1}})^{\frac{(N)}{p}}} = \frac{r^{\frac{N(p-1)+1}{p(p-1)}}}{(\epsilon^{\frac{p}{p-1}} + |r|^{\frac{p}{p-1}})^{\frac{(N)}{p}}} + O(1), , \tag{2.26}$$

and therefore

$$\frac{1}{\epsilon^{\frac{N-p}{p-1}}} ||\nabla u_\epsilon||_p^p = \omega |\epsilon^{\frac{N-p}{p-1}} \int_0^1 r^\alpha |\varphi^{'}(r)|^p dr + (\frac{N-p}{p-1})^p \epsilon^{\frac{N-p}{p-1}} \int_0^1 \frac{r^{\frac{p}{p-1}}}{(\epsilon^{\frac{p}{p-1}} + |r|^{\frac{p}{p-1}})^N} r^{N-1} dr| + O(\epsilon^{\frac{N-p}{p-1}}). \tag{2.27}$$

Also, we have

$$\int_0^1 \frac{r^{N-1+\frac{p}{p-1}}}{(\epsilon^{\frac{p}{p-1}} + |r|^{\frac{p}{p-1}})^N} dr = \frac{1}{\epsilon^{\frac{N-p}{p-1}}} \int_0^\infty \frac{s^{N-1+\frac{p}{p-1}}}{(1 + s^{\frac{p}{p-1}})^N} ds + O(1) \tag{2.28}$$

Combining (2.26)-(2.28) we obtain (2.23) with

$$K_1 = (\frac{N-p}{p-1})^p \omega \int_0^\infty \frac{s^{N-1+\frac{p}{p-1}}}{(1+s^{\frac{p}{p-1}})^N} ds.$$

We note that $K_1 = \int_{\mathbb{R}^N} |\nabla U|^p dx$, where $U(x) = \frac{1}{(1+|x|^{\frac{p}{p-1}})^{\frac{(N-p)}{p}}}$ ∎

PROOF OF (2.22): We have

$$\begin{aligned} ||u_\varepsilon||_{q+1}^{q+1} &= \omega \int_0^1 \frac{\varphi(r)^{q+1} r^{N-1}}{(\varepsilon^{\frac{p}{p-1}} + r^{\frac{p}{p-1}})^N} dr \\ &= \omega \int_0^1 \frac{[\varphi(r)^{q+1} - 1] r^{N-1}}{(\varepsilon^{\frac{p}{p-1}} + r^{\frac{p}{p-1}})^N} dr + \omega \int_0^1 \frac{r^{N-1}}{(\varepsilon^{\frac{p}{p-1}} + r^{\frac{p}{p-1}})^N} dr \\ &= I_1 + I_2. \end{aligned}$$

Since $\varphi(0) = 1$ and $\varphi'(0) = 0$ we get

$$|I_1| \le C \int_0^1 \frac{r^{N+1}}{(\varepsilon^{\frac{p}{p-1}} + r^{\frac{p}{p-1}})^N} dr = O(\varepsilon^{1-\frac{Np}{p-1}})$$

$$|I_2| = \frac{\omega}{\varepsilon^{\frac{N}{p-1}}} \int_0^\infty \frac{r^{N-1}}{(\varepsilon^{\frac{p}{p-1}} + r^{\frac{p}{p-1}})^N} dr + O(1).$$

Therefore we find

$$||u_\varepsilon||_{q+1}^{q+1} = \frac{1}{\varepsilon^{\frac{N}{p-1}}} [\omega \int_0^\infty \frac{r^{N-1}}{(\varepsilon^{\frac{p}{p-1}} + r^{\frac{p}{p-1}})^N} dr + O(\varepsilon^{\frac{N-p}{p-1}})]$$

and (2.22) follows with

$$K_2 = [\omega \int_0^\infty \frac{r^{N-1}}{(\varepsilon^{\frac{p}{p-1}} + r^{\frac{p}{p-1}})^N} dr]^{\frac{p}{q+1}} = ||U||_{q+1}^p.$$

PROOF OF (2.23): We have

$$\begin{aligned} ||u_\varepsilon||_p^p &= \omega \int_0^1 \frac{|\varphi(r)|^p r^{N-1}}{(\varepsilon^{\frac{p}{p-1}} + r^{\frac{p}{p-1}})^{N-p}} dr \\ &= \omega \int_0^1 r^\alpha |\varphi(r)|^p dr + O(\varepsilon). \end{aligned}$$

Combining (2.21), (2.22) and (2.23) we obtain

$$\mathcal{Q}_\lambda(u_\varepsilon) = S + \varepsilon^{\frac{N-p}{p-1}} \frac{\omega}{K_2} \int_0^1 r^\alpha [|\varphi(r)|^p - \lambda |\varphi'(r)|^p] dr + O(1).$$

To choose a function $\varphi$ such that $\int_0^1 r^\alpha[|\varphi'(r)|^p - \frac{1}{2^p}\lambda_1|\varphi(r)|^p]dr = 0$ we consider the following problem with Neuman conditions:

$$\begin{cases} -(r^\alpha|u'|^{p-2}u')' = \lambda_1 r^\alpha |u|^{p-2}u \quad \text{in} \quad ]0,1[ \\ u'(0) = u'(1) = 0. \end{cases}$$

It is well known that the first eigenfunction associated to the first eigenvalue $\lambda_1$ change the sign and is antisymmetric (odd) i.e. $u(r) = -u(1-r)$. Now take $\varphi(r) = \frac{1}{k}u(\frac{1}{2}r)$, $k$ is a constant such that $u(0) = k$. Since the above problem is $p$-homogeneous, the function $\frac{1}{k}u(r)$ is a solution of this problem, still denoted $u(r)$. Using $u'(0) = 0$ and $u(\frac{1}{2}) = 0$ we obtain

$$\begin{aligned} \int_0^1 r^\alpha|\varphi'(r)|^p dr &= \frac{1}{2^p}\int_0^1 r^\alpha|u'(\frac{1}{2}r)|^p dr \\ &= \frac{2^\alpha}{2^{p-1}}\int_0^{\frac{1}{2}} t^\alpha|u'(t)|^p dt \\ &= \frac{2^\alpha}{2^{p-1}}\lambda_1\int_0^{\frac{1}{2}} t^\alpha|u(t)|^p dt \\ &= \frac{1}{2^p}\lambda_1\int_0^1 r^\alpha|\varphi(r)|^p dr \end{aligned}$$

and thus

$$\mathcal{Q}_\lambda(u_\varepsilon) = S + (\frac{\lambda_1}{2^p} - \lambda)C\varepsilon^{\frac{N-p}{p-1}}$$

for some positive constant C.

## 3 Existence of Positive Solution for $-\Delta_p u = u^q + f(x,u)$ in $\Omega$, $u = 0$ on $\partial\Omega$ where $q+1 = \frac{Np}{N-p}$ and $f(x,u)$ is a Lower-Order Perturbation.

Let $\Omega \subset \mathbb{R}^N$, $N > p$, be a bounded domain and the function $f(x,u) : \Omega \times [0,+\infty[\to \mathbb{R}$ is measurable in $x$, continuous in $u$ and that $\sup_{\{x\in\Omega, 0\le u\le M\}} |f(x,u)| < \infty$ for every $M > 0$.

Let $q+1 = \frac{Np}{N-p}$. We assume that $f(x,0) = 0$ and that $f$ is a lower-order perturbation of $u^q$, that is,

$$\lim_{u\to+\infty}\frac{f(x,u)}{u^q} = 0.$$

Our main result is the existence of a function $u$ satisfying:

$$\begin{cases} -\Delta_p u = u^q + f(x,u) \quad \text{in} \quad \Omega \\ u > 0 \quad \text{in} \quad \Omega \qquad (3.1) \\ u = 0 \quad \text{in} \quad \partial\Omega \end{cases}$$

We assume that $f(x,u)$ is as follows

$$f(x,u) = a(x)|u|^{p-2}u + g(x,u), \qquad (3.2)$$

with

$$a(x) \in L^\infty(\Omega), \qquad (3.3)$$

$$g(x,u) = o(u^{p-1}) \quad u \to 0^+, \text{ uniformly in x}, \qquad (3.4)$$

$$g(x,u) = o(u^q) \quad u \to +\infty, \text{ uniformly in x}, \qquad (3.5)$$

Moreover we assume that the operator $T(u) = -\Delta_p u - a(x)|u|^{p-2}u$ has its least eigenvalue positive that is

$$\int |\nabla\Phi|^p - a\Phi^p \geq \alpha \int |\Phi|^p \text{ for all } \Phi \in W_0^{1,p} \qquad \alpha > 0 \quad (3.6)$$

or equivalently

$$\int |\nabla\Phi|^p - a\Phi^p \geq \alpha' \int |\nabla\Phi|^p \text{ for all } \Phi \in W_0^{1,p} \qquad \alpha' > 0. \quad (3.6')$$

Now since we are looking for a positive solution $u$, the values of $f(x,u)$ for $u < 0$ are irrelevant so we take

$$f(x,u) = 0 \text{ for } x \in \Omega, \quad u \leq 0.$$

Set

$$F(x,u) = \int_0^u f(x,t)dt \text{ for } x \in \Omega, \quad u \in \mathbb{R}$$

and

$$\Psi(u) = \int [\frac{1}{p}|\nabla u|^p - \frac{1}{q+1}|u|^{q+1} - F(x,u)] \text{ for } u \in W_0^{1,p}.$$

**Theorem 3.1.** *Assume (3.2)-(3.6) hold and, moreover, suppose that there exists some $v_0 \in W_0^{1,p}$, $v_0 \geq 0$ on $\Omega$, $v_0$ not identically null such that*

$$\sup_{t \geq 0} \Psi(tv_0) < \frac{1}{N} S^{\frac{N}{p}}. \qquad (3.7)$$

*Then, the problem (3.1) has a solution.*

**Remark 3.1.** *In the case $f(x,u) = \lambda|u|^{p-2}u$, assumption (3.6) corresponds to $\lambda < \lambda_1$ while the assumption (3.7) is equivalent to $S_\lambda < S$. Indeed*

$$\sup_{t \geq 0} \Psi(tv_0) = \frac{1}{N}[\frac{A}{B^{\frac{p}{q+1}}}]^{\frac{N}{p}} \qquad (3.8)$$

*with $A = ||\nabla v_0||_p^p - \lambda||v_0||_p^p$ and $B = ||v_0||_{q+1}^{q+1}$.*

As in [5], the proof of Theorem 3.1 is based on a variant of the mountain pass Theorem of Ambrosetti and Rabinowitz without (PS) condition (see [1]):

**Theorem 3.2.** *Let $\Phi$ be a $C^1$ function on a Banach space $E$. Suppose there exists a neighborhood $U$ of $0$ in $E$ and a constant $\rho$ such that $\Phi(u) \geq \rho$ for every $u$ in the boundary of $U$,*

$$\Phi(0) < \rho \quad \text{and} \quad \Phi(v) < \rho \quad \text{for some} \quad v \notin U. \tag{3.9}$$

*Set*

$$c = \inf_{P \in \mathcal{P}} \max_{w \in P} \Phi(w) \geq \rho, \tag{3.10}$$

*where $\mathcal{P}$ denotes the class of continuous paths joining $0$ to $v$. Then there is a sequence $u_j$ in $E$ such that*

$$\Phi(u_j) \to c \quad \text{and} \quad \Phi'(u_j) \to 0 \quad \text{in} \quad E^* \tag{3.11}$$

*where $E^*$ is the dual of the space $E$.*

**Proof:**
Proof of Theorem 3.1
We use the argument of H. Brezis and L. Nirenberg (see Theorem 2.1 in [5]) but here we define on $E = W_0^{1,p}$ the following function:

$$\Phi(u) = \int [\frac{1}{p}|\nabla u|^p + \frac{1}{p}\mu u^p - \frac{1}{q+1}(u^+)^{q+1} - F(x, u^+) - \frac{1}{p}\mu (u^+)^p].$$

From (3.4) and (3.5) we find for all $u \in W_0^{1,p}$, and for $\varepsilon > 0$ small enough

$$\Phi(u) = \int [\frac{1}{p}|\nabla u|^p - \frac{1}{p}a(x)(u^+)^p - \frac{1}{p}\varepsilon (u^+)^p - \frac{C+1}{q+1}(u^+)^{q+1}],$$

where $C$ is some constant depending on $\varepsilon$.
Hence there exists constants $k > 0$ and $C'$ such that

$$\Phi(u) \geq k||u||^p_{W_0^{1,p}} - C'||u||^{q+1}_{W_0^{1,p}} \quad \text{for all} \quad u \in W_0^{1,p},$$

which implies (3.9) with some $\rho > 0$ (and $U$ a small ball in $W_0^{1,p}$).
Also, by (3.5) we deduce that for any $u \in W_0^{1,p}$, $u \geq 0$, $u \not\equiv 0$,

$$\lim_{t \to +\infty} \Phi(tu) = -\infty.$$

Thus, there are many $v$ satisfying (3.9). We take the special $v = t_0 v_0$ where $v_0$ is given by (3.8) and $t_0 > 0$ is chosen large enough so that $v \notin U$ and $\Phi(v) \leq 0$. It follows from (3.8) that

$$\sup_{t \geq 0} \Phi(tv) < \frac{1}{N} S^{\frac{N}{p}}$$

and therefore we have

$$c < \frac{1}{N} S^{\frac{N}{p}}. \qquad (3.12)$$

Theorem 2.2 guarantees that there exists a sequence $(u_j)$ in $W_0^{1,p}$ such that $\Phi(u_j) \to c$ and $\Phi'(u_j) \to 0$ in $W_0^{1,p}$, that is,

$$\int [\frac{1}{p}|\nabla u_j|^p + \frac{1}{p}\mu u_j^p - \frac{1}{q+1}(u_j^+)^{q+1} - F(x,u_j^+) - \frac{1}{p}\mu(u_j^+)^p] = c + o(1), \qquad (3.13)$$

and

$$-\Delta_p u_j + \mu|u_j|^{p-2}u_j - (u_j^+)^q - f(x,u_j^+) - \mu|u_j^+|^{p-1} = \xi_j \qquad (3.14)$$

with $\xi_j \to 0$ in $W^{-1,p}$. We claim that

$$||u_j||_{W_0^{1,p}} \le C. \qquad (3.15)$$

Indeed, multiplying (3.14) by $u_j$ we have

$$\int [|\nabla u_j|^p + \mu u_j^p - (u_j^+)^{q+1} - f(x,u_j^+)u_j^+ - \mu(u_j^+)^p] =< \xi_j, u_j > . \qquad (3.16)$$

Taking (3.13) $- \frac{1}{p}$(3.16) we obtain

$$\frac{1}{N}\int (u_j)^{q+1} \le \int [F(x,u_j^+) - \frac{1}{p} f(x,u_j^+)u_j^+] + c + o(1) + ||\xi_j||_{W^{-1,p}}||u_j||_{W_0^{1,p}}. \qquad (3.17)$$

On the other hand, from (3.5) we have for all $\varepsilon > 0$ that there exists a $C$ such that

$$|f(x,u)| \le \varepsilon u^q + C \quad \text{for a.e. } x \in \Omega, \quad \text{and for all } u \ge 0, \qquad (3.18)$$

and so

$$|F(x,u)| \le \frac{\varepsilon}{q+1} u^{q+1} + C \quad \text{for a.e. } x \in \Omega, \quad \text{and for all } u \ge 0. \qquad (3.18')$$

We conclude with $\varepsilon$ small enough that

$$\int (u_j^+)^{q+1} \le C + C||u_j||_{W_0^{1,p}}. \qquad (3.19)$$

Combining (3.13) and (3.19) we obtain (3.15).
Since the sequence $(u_j)_j$ is bounded in $W_0^{1,p}$, we can extract a subsequence denoted also by $(u_j)_j$, such that

$$u_j \rightharpoonup u \quad \text{weakly in } W_0^{1,p},$$

$$u_j \to u \quad \text{strongly in} \quad L^r \quad \text{for all} \quad r < q+1,$$
$$u_j \to u \quad \text{a.e. on} \quad \Omega,$$
$$(u_j^+)^q \rightharpoonup (u^+)^q \quad \text{weakly in} \quad (L^{q+1})^*,$$
$$f(x, u_j^+) \rightharpoonup f(x, u^+) \quad \text{weakly in} \quad (L^{q+1})^*.$$

By passing to the limit in (3.14) we obtain

$$-\Delta_p u + \mu |u|^{p-2} u = (u^+)^q + f(x, u^+) + \mu |u^+|^{p-1} \quad \text{in} \quad W^{-1,p}. \tag{3.20}$$

We deduce that the right-hand side is positive, and by the maximum principle see (Damascelli [7]), that $u \geq 0$ in $\Omega$ and therefore $u$ satisfies

$$-\Delta_p u = u^q + f(x, u). \tag{3.21}$$

We shall now verify that $u \not\equiv 0$ which would then imply $u > 0$ in $\Omega$ by the strong maximum principle.
Suppose that $u \equiv 0$. Then by (3.18), (3.18'), the embedding $W_0^{1,p} \subset L^{q+1}$ and $u_j \to 0$ strongly in $L^p$ we obtain

$$\int f(x, u_j^+) u_j^+ \to 0,$$

$$\int F(x, u_j^+) \to 0,$$

so we may extract another subsequence still denoted $u_j$ such that

$$\int |\nabla u_j|^p \to l$$

for some constant $l \geq 0$. Passing to the limit in (3.16) and then in (3.13) we get

$$\int (u^+)^{q+1} \to l \tag{3.22}$$

$$\frac{1}{N} l = c. \tag{3.23}$$

On the other hand, we have

$$||\nabla u_j||_p^p \geq S ||u_j||_{q+1}^p \geq S ||u_j^+||_{q+1}^p$$

and using (3.21) and (3.22) we find in the limit

$$l \geq S l^{\frac{p}{q+1}}. \tag{3.24}$$

Finally, by (3.23) and (3.24) we conclude that

$$c \geq \frac{1}{N} S^{\frac{N}{p}}, \tag{3.25}$$

which contradicts (3.12). Thus $u \not\equiv 0$.

VERIFICATION OF (3.26) and (3.26'):
We consider again the sequence $u_j$ as in the proof of Theorem 3.1. So from the fact that $f(x,u)$ is lower order that $u^q$, that the sequence $(u_j)$ is bounded in $L^{q+1}$ with $u_j \to u$ a.e. on $\Omega$ and the Lebesgue Convergence we obtain:

$$\int f(x,u_j^+)u_j^+ \to \int f(x,u^+)u^+ \quad \text{and} \quad \int F(x,u_j^+) \to \int F(x,u). \tag{3.29}$$

We set $v_j = u_j - u$, so that

$$\int |\nabla u_j|^p = \int |\nabla u|^p + \int |\nabla v_j|^p + o(1), \tag{3.30}$$

and from [6] we deduce that

$$\int (u_j^+)^{q+1} = \int (u)^{q+1} + \int (v_j^+)^{q+1} + o(1). \tag{3.31}$$

Combining (3.13) and (3.16) with (3.29), (3.30) and (3.31) leads to

$$\Phi(u) + \int [\frac{1}{p}|\nabla v_j|^p - \frac{1}{q+1}(v_j^+)^{q+1}] = c + o(1),$$

$$\int [|\nabla u|^p - u^{q+1} - f(x,u)u] + \int [|\nabla v_j|^p - (v_j^+)^{q+1}] = o(1). \tag{3.32}$$

Which reduced by (3.28) to

$$\int |\nabla v_j|^p = \int (v_j^+)^{q+1} + o(1). \tag{3.33}$$

Combining the previous equalities we get

$$\Phi(u) + \frac{1}{N}\int [|\nabla v_j|^p = c + o(1). \tag{3.34}$$

In the end, we may assume for a subsequence of $v_j$ denoted also $v_j$ that

$$\int |\nabla v_j|^p \to k \geq 0 \quad \text{and} \quad \int (v_j^+)^{q+1} \to k. \tag{3.35}$$

Sobolev's inequality leads to $k \geq S k^{\frac{p}{q+1}}$. Thus we have, either $k = 0$, or $k \geq S^{\frac{N}{p}}$, which proves (3.26) and (3.26'). ∎

**Remark 3.2.** *As in* [5], we obtain an additional property that the solution $u$ satisfies

$$\Phi(u) = c \qquad (3.26)$$

or

$$\Phi(u) \leq c - \frac{1}{N} S^{\frac{N}{p}} < 0. \qquad (3.26')$$

In some cases, (3.26') is excluded if we have:

$$F(x,u) \leq \frac{1}{p} f(x,u)u + \frac{1}{N} u^{q+1} \quad \text{for a.e. } \Omega, \quad \text{and for all } u \geq 0. \qquad (3.27)$$

This assumption could been hold for example if $f(x,u) = a(x)|u|^{p-2}u + \mu u^r$ with $a \in L^\infty$, $\mu \geq 0$ and $1 \leq r \leq q$. Indeed if $u$ is a solution of (3.1) we have

$$\int |\nabla u|^p = \int [u^{q+1} + f(x,u)u] \qquad (3.28)$$

and thus

$$\begin{aligned} \Phi(u) &= \int [\frac{1}{p} |\nabla u|^p - \frac{1}{q+1} u^{q+1} - F(x,u)] \\ &= \int [\frac{1}{N} |u|^{q+1} + \frac{1}{p} f(x,u)u - F(x,u)]. \end{aligned}$$

By using (3.27), we have $\Phi(u) \geq 0$. In fact, when we assume (3.27) the argument below shows that $\Phi$ satisfies the condition $(PS)_c$ (introduced in [4]) for every $c < \frac{1}{N} S^{\frac{N}{p}}$.

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