**Applied Mathematical Sciences, Vol. 12, 2018, no. 16, 783 - 795**
**HIKARI Ltd www.m-hikari.com**
**https://doi.org/10.12988/ams.2018.8345**

# Effect of Rotatory Inertia and Load Natural Frequency on the Response of Uniform Rayleigh Beam Resting on Pasternak Foundation Subjected to a Harmonic Magnitude Moving Load

**A. Jimoh[1] and E. O. Ajoge[2]**

[1] Department of Mathematical Sciences
Kogi State University, Anyigba, Nigeria

[2] Centre for Energy Research and Development
Obafemi Awolowo University, Ile-Ife, Nigeria



### Abstract

In this study, the effect of rotatory inertia on the transverse motion of uniform Rayleigh beam resting on Pasternak foundation subjected to harmonic magnitude moving load is investigated. We employed Fourier sine transform, Laplace integral transformation and convolution theorem as solution technique. It was observed from the results that, the amplitude of the deflection profile of the beam decreases with increase in the value of rotatory inertia and load natural frequency. Also increases in the values of the other structural parameters like shear modulus, foundation modulus, axial force, and damping coefficient lead to decreases in the deflection profile of the beam. It was also observed that the effect of the load natural frequency is more noticeable than that of the rotatory inertia.

**Keywords:** Rotatory inertia, load natural frequency, harmonic load, moving load, Pasternak foundation, damping coefficient, axial force, shear modulus, foundation

# 1 Introduction

The calculation of the dynamic response of elastic structures (beams, plates and shells) carrying one or more traveling loads (moving trains, tracks, cars, bicycles, cranes, etc.) is very important in Engineering, Physics and Applied Mathematics as applications relate, for example, to the analysis and design of highway, railway bridges, cable-railroads and the like. Generally, emphasis is placed on the dynamics of the elastic structural members rather on that of the moving loads.

Among the earliest researchers on the dynamic analysis of an elastic beam was Ayre et al [1] who studied the effect of the ratio of the weight of the load to the weight of a simply supported beam of a constantly moving mass load. Kenny [2] similarly investigated the dynamic response of infinite elastic beams on elastic foundation under the influence of load moving at constant speeds. He included the effects of viscous damping in the governing differential equation. Steel [3] also investigated the response of a finite simplify supported Bernouli-Euler beam to a unit force moving at a uniform velocity. He analysed the effects of this moving force on beams with and without an elastic foundation. Oni [4] considered the problem of a harmonic time-variable concentrated force moving at uniform velocity over a finite deep beam.

In the recent years, several other researchers that made tremendous feat in the study of dynamic of structures under moving loads include Chang and Liu [5], Oni and Omolofe [6], Oni and Awodola [7], Misra [8], Oni and Ogunyebi [9], Hsu [10], Achawakorn and Jearsiri Pongkul [11].

However, all the aforementioned researchers considered only the winkler approximation model which has been criticised variously by authors [12, 13, 14] because it predict discontinuities in the deflection of the surface of the foundation at the ends of a finite beam, which is in contradiction to observation made in practice.

To this end, in a more recent time, researchers who considered the dynamic response of elastic beam resting on Pasternak foundation are Coskum [15], Oni and Jimoh [16], Guter [17], Oni and Ahmed [18], Oni and Ahmed [19], Ma et al [20] and Ahmed [21].

In all those researches, rotatory inertia and load natural frequency were neglected. Thus, in this paper, we investigate the effect of rotatory intertia and load natural frequency on the response of uniform Rayleigh beam resting on Pasternak foundation and subjected to harmonic magnitude moving loads.

# 2. Mathematical Model

This paper considered the dynamic response of a uniform Rayleigh beam resting on Pasternak foundation when it is under the action of a harmonic magnitude moving load. The governing partial differential equation that described the motion of the dynamical system is given by [22]

$$EI\frac{\partial^4 V(x,t)}{\partial x^4}+\mu\frac{\partial^2 V(x,t)}{\partial x^2}-N\frac{\partial^2 V(x,t)}{\partial x^2}-\mu R_0^2\frac{\partial^4 V(x,t)}{\partial x^2\partial t^2}+\Sigma\frac{\partial v(x,t)}{\partial t}+PG(x,t)=P(x,t) \quad (1)$$

*EI* is the flexural rigidity of the beam, $\mu$ is the mass of the beam per unit length *L*, *N* is the axial force, *Σ* is the damping coefficient, PG is the foundation reaction, *E* is the young's modulus of the beam, *I* is the moment of inertia of the beam, *K* is the foundation modulus, P(x, t) is the transverse moving load, *x* is the spatial coordinate and *t* is the time

The boundary condition at the end $x = 0$ and $x = L$ are given by

$$V(0,t)=0$$

$$\frac{\partial V(L,t)}{\partial x}=0 \quad (2)$$

And the initial conditions

$$V(x,0)=0$$

$$\frac{\partial V(x,0)}{\partial t}=0 \quad (3)$$

The foundation reaction PG is given by

$$PG(x,t)=F\frac{\partial^2 V(x,t)}{\partial x^2}+KV(x,t) \quad (4)$$

Where *F* is the shear modulus and *K* is the foundation stiffness.

Furthermore, the harmonic magnitude moving force *P*(x,t) acting on the beam is given by

$$P(x,t)=P\cos wt\, f(x-Vt) \quad (5)$$

Where *w* is the load natural frequency and f(*) is the dirac-delta function.

When equations (4) and (5) are substituted into (1), the result is a non-homogeneous system of partial differential equations given by

$$EI\frac{\partial^4 V(x,t)}{\partial x^4}+\mu\frac{\partial^2 V(x,t)}{\partial x^2}-N\frac{\partial^2 V(x,t)}{\partial x^2}-\mu R_0^2\frac{\partial^4 V(x,t)}{\partial x^2\partial t^2}+\Sigma\frac{\partial v(x,t)}{\partial t}+F\frac{\partial^2 V(x,t)}{\partial x^2}+KV(x,t)=P\cos wt\, f(x-Vt) \quad (6)$$

## 3. Approximate Analytical Solution

The effective applicable method of handling 6 is the integral transform technique, specifically, the Fourier transformation for the length coordinates and the Laplace transformation for the time coordinate with boundary and initial conditions are used in this work. The finite Fourier sine integral transformation for the length coordinate is defined as

$$V(m,t) = \int_0^L V(x,t)\sin\frac{mux}{L} \tag{7}$$

With the inverse transform defined as

$$V(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} V(m,t)\sin\frac{mux}{L} \tag{8}$$

Thus, by invoking equation (7) on equation (6), we have

$$\left(1 + R_0\frac{m^2\pi^2}{L^2}\right)V_{tt}(m,t) + \frac{\sum V_t(m,t)}{\mu} + \left(\frac{EI}{\mu}\left(\frac{m\pi}{L}\right)^4 + \frac{N}{\mu}\left(\frac{m\pi}{L}\right)^2 - \frac{F}{\mu}\left(\frac{m\pi}{L}\right)^2 + \frac{K}{\mu}\right)V(m,t) = \frac{P}{\mu}\cos wt\sin\frac{m\pi vt}{L} \tag{9}$$

Equation (9) can be conveniently be written as:

$$V_{tt}(m,t) + \Delta_{11}V_t(m,t) + \Delta_{12}V(m,t) = \Delta_{13}\cos wt\sin\Delta_0 t \tag{10}$$

Where

$$b_0 = 1 + R_0\left(\frac{m\pi}{L}\right)^2,\ b_1 = \frac{\Sigma}{\mu},\quad b_2 = \frac{EI}{\mu}\left(\frac{m\pi}{L}\right)^4,\quad b_3 = \frac{1}{\mu}\left(\frac{m\pi}{L}\right)^2(N-F),\ b_4 = \frac{K}{\mu},\ b_5 = \frac{P}{\mu},\ P = mg \tag{11}$$

$$\Delta_{11} = \frac{b_1}{b_0},\qquad \Delta_{12} = \frac{b_2 + b_3 + b_4}{b_0},\qquad \Delta_{13} = \frac{b_5}{b_0},\qquad \Delta_0 = \frac{m\pi v}{L}$$

Equation (10) represent the first Fourier transformed governing equations of the uniform Rayleigh beam subjected to harmonic magnitude moving load moving with constant velocity

### 3.1 Laplace Transformed Solutioins

To solve equation (10) above, we apply the method of the Laplace integral transformation for the time coordinate between 0 and ∞. The operation of the Laplace transform is indicated by the notation.

$$L\big(f(t)\big) = \int_0^{\infty} f(t)\,e^{-st}dt \tag{12}$$

Here

*'L'* and *'s'* are the Laplace transform operator and Laplace transform variable respectively.

In particular, we use

$$L\big(V(m,t)\big) = V(m,s) = \int_0^\infty V(m,t)\, e^{-st} dt \tag{13}$$

Using the transformation in equation (13) on equation (10) in conjunction with the set of the initial conditions in equation (3) and upon simplification we obtain

$$V(m,s) = \frac{\Delta_{13}}{2(c_1-c_2)}\left(\left(\frac{1}{s-c_1}\right)\left(\frac{d_1}{s^2-d_1^2}\right) - \left(\frac{1}{s-c_2}\right)\left(\frac{d_1}{s^2-d_1^2}\right) - \left(\frac{1}{s-c_1}\right)\left(\frac{d_2}{s^2-d_2^2}\right) + \left(\frac{1}{s-c_2}\right)\left(\frac{d_2}{s^2-d_2^2}\right)\right) \tag{14}$$

Where

$$c_1 = -\frac{\Delta_{11}}{2} + \frac{\sqrt{\Delta_{11}^2 - 4\Delta_{12}}}{2} \tag{15}$$

$$c_2 = -\frac{\Delta_{11}}{2} - \frac{\sqrt{\Delta_{11}^2 - 4\Delta_{12}}}{2}$$

$$d_1 = w + \Delta_0$$

$$d_2 = w - \Delta_0 \tag{16}$$

In order to obtain the Laplace inversion of equation (14), we shall adopt the following representations

$$F_1(s) = \left(\frac{1}{s-c_1}\right)$$

$$F_2(s) = \left(\frac{1}{s-c_2}\right) \tag{17}$$

$$G_1(s) = \left(\frac{d_1}{s^2-d_1^2}\right)$$

$$G_2(s) = \left(\frac{d_1}{s^2-d_2^2}\right)$$

So that the Laplace inversion of equation (14) is the convolution of $f_i$ $and$ $g_j$ defined as

$$f_i \cdot g_j = \int_0^t f_i(t-u)\, g_j(u)du, \quad i = 1,2,;j = 1,2 \tag{18}$$

Thus the Laplace inversion of (14) is given by

$$z(m,t) = \frac{\Delta_{13}}{2(c_1 - c_2)}(F_1(s)G_1(s) - F_2(s)G_1(s) - F_1(s)G_2(s) + F_2(s)G_2(s) \tag{19}$$

Where

$$F_1G_1 = e^{c_1 t}\int_0^t e^{-c_1 u} \sin d_1 u\, du \tag{20}$$

$$F_1G_2 = e^{c_1 t}\int_0^t e^{-c_1 u} \sin d_2 u\, du \tag{21}$$

$$F_2G_1 = e^{c_2 t}\int_0^t e^{-c_2 u} \sin d_1 u\, du \tag{22}$$

$$F_2G_2 = e^{c_2 t}\int_0^t e^{-c_2 u} \sin d_2 u\, du \tag{23}$$

Evaluating the integrals (20 – 23) to obtain

$$F_1(t)G_1(t) = \frac{d_1}{d_1^2 + c_1^2}\left(\left((e^{c_1 t} - \cos d_1 t) - \frac{c_1}{d_1}\sin d_1 t\right)\right) \tag{24}$$

(25)

$$F_1(t)G_2(t) = \frac{d_2}{d_2^2 + c_1^2}\left(\left((e^{c_1 t} - \cos d_2 t) - \frac{c_1}{d_2}\sin d_2 t\right)\right)$$

(26)

$$F_2(t)G_1(t) = \frac{d_1}{d_1^2 + c_2^2}\left(\left((e^{c_2 t} - \cos d_1 t) - \frac{c_2}{d_1}\sin d_1 t\right)\right) \tag{27}$$

$$F_2(t)G_2(t) = \frac{d_2}{d_2^2 + c_2^2}\left(\left((e^{c_2 t} - \cos d_2 t) - \frac{c_2}{d_2}\sin d_2 t\right)\right)$$

Putting equations (24- 27) to obtain

$$V(m,t) = \frac{\Delta_{13}}{2(c_1 - c_2)}\left(\frac{d_1}{d_1^2 + c_1^2}\left(\left((e^{c_1 t} - \cos d_1 t) - \frac{c_1}{d_1}\sin d_1 t\right)\right)\right)$$
$$- \frac{d_2}{d_2^2 + c_1^2}\left(\left((e^{c_1 t} - \cos d_2 t) - \frac{c_1}{d_2}\sin d_2 t\right)\right)$$
$$- \frac{d_1}{d_1^2 + c_2^2}\left(\left((e^{c_2 t} - \cos d_1 t) - \frac{c_2}{d_1}\sin d_1 t\right)\right)$$
$$+ \frac{d_2}{d_2^2 + c_2^2}\left(\left((e^{c_2 t} - \cos d_2 t) - \frac{c_2}{d_2}\sin d_2 t\right)\right) \quad (28)$$

The inversion of equation (28) yields

$$V(m,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\frac{\Delta_{13}}{2(c_1 - c_2)}\left(\frac{d_1}{d_1^2 + c_1^2}\left(\left((e^{c_1 t} - \cos d_1 t) - \frac{c_1}{d_1}\sin d_1 t\right)\right)\right)\right.$$
$$- \frac{d_2}{d_2^2 + c_1^2}\left(\left((e^{c_1 t} - \cos d_2 t) - \frac{c_1}{d_2}\sin d_2 t\right)\right)$$
$$- \frac{d_1}{d_1^2 + c_2^2}\left(\left((e^{c_2 t} - \cos d_1 t) - \frac{c_2}{d_1}\sin d_1 t\right)\right) \quad (29)$$
$$\left. + \frac{d_2}{d_2^2 + c_2^2}\left(\left((e^{c_2 t} - \cos d_2 t) - \frac{c_2}{d_2}\sin d_2 t\right)\right)\right]\sin\frac{mux}{L}$$

Equation (29) represents the transverse displacement response to a harmonic magnitude moving load with constant velocity of uniform Rayleigh beam resting on Pasternak foundation.

## 4. Numerical Calculations and Discussion of Results

In this section, numerical results for the uniform Rayleigh beam are presented in plotted curves. An elastic beam of length 12.9 m is considered. Other values used are modulus of elasticity $E$ =2.10924 x $10^{10}$ $N/m^2$, the moment of inertial $I$= 2.87698 x $10^{-3}$ $m$ and mass per unit length of the beam $\mu = 3401.563\ kg/m$. The value of the foundation stiffness (k) is varied between ( 0 & 400000) $N/m^3$, the value of the axial force $N$ is varied between (0 & 2.0 x $10^8$)$N$, the values of the shear modules ($G$) varied between (0 and 1300000) $N/m^3$ , damping coefficient ($\Sigma$) varied between (1.8 & 3.0), rotatory inertia $R_0$ varied between (0 & 12), and the value of Natural Frequency of the moving load is between (0 & 12), the results are shown in the graphs below, and from the graphs it was observed that an increase in all the structural parameters lead to decreases in the deflection profile of the uniform Rayleigh beam subjected to harmonically varying magnitude moving load. It was also observed that the effect of load natural frequency is more pronounced than that of rotatory inertial.

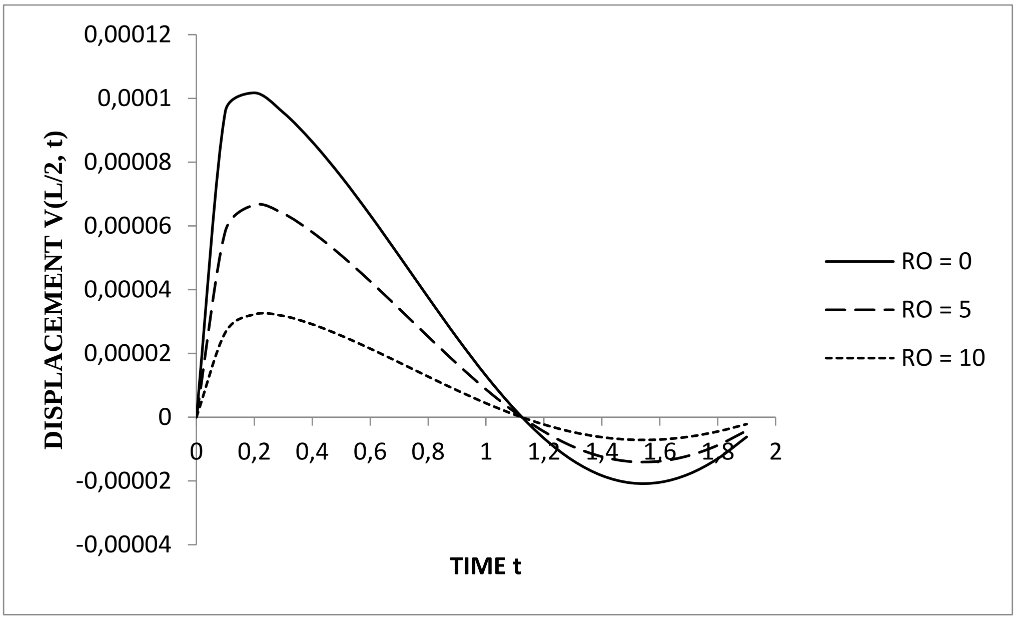


Fig 1: Deflection profile of uniform Rayleigh Beam subjected to harmonically varying moving loads for various values of Rotatory Inertial ($R_0$) and fixed values of Shear Modulus (G), Foundation Modulus (K), Damping Coefficient (ʢ), Axial Force (N), and Load Natural Frequency (w).

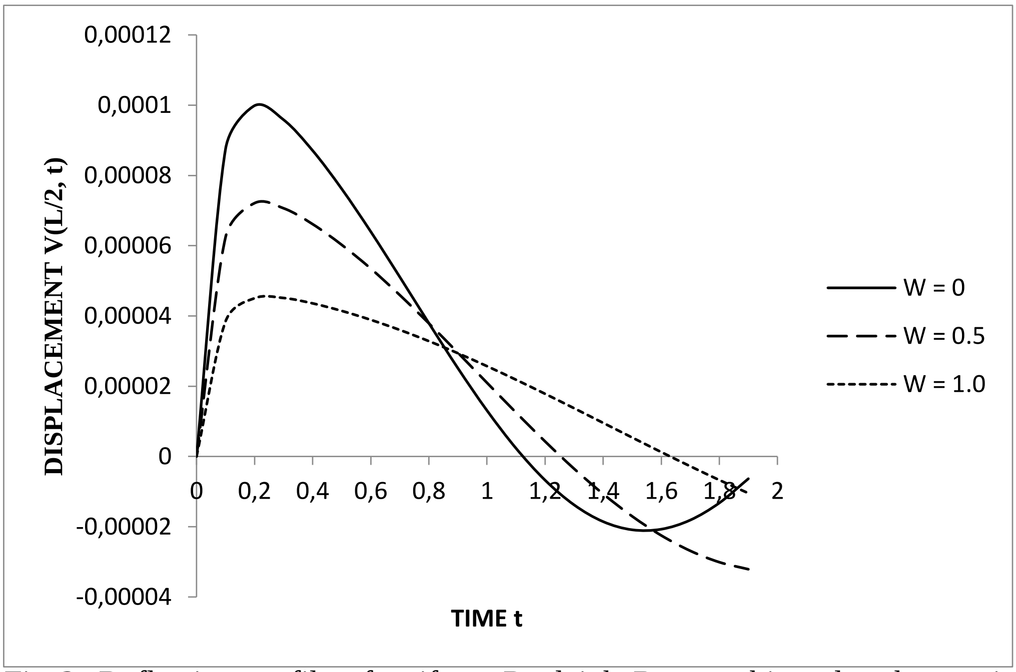


Fig 2: Deflection profile of uniform Rayleigh Beam subjected to harmonically varying moving loads for various values of Load Natural Frequency (w) and fixed values of Shear Modulus (G), Foundation Modulus (K), Damping Coefficient (ʢ), Axial Force (N), and Rotatory Inertial ($R_0$).

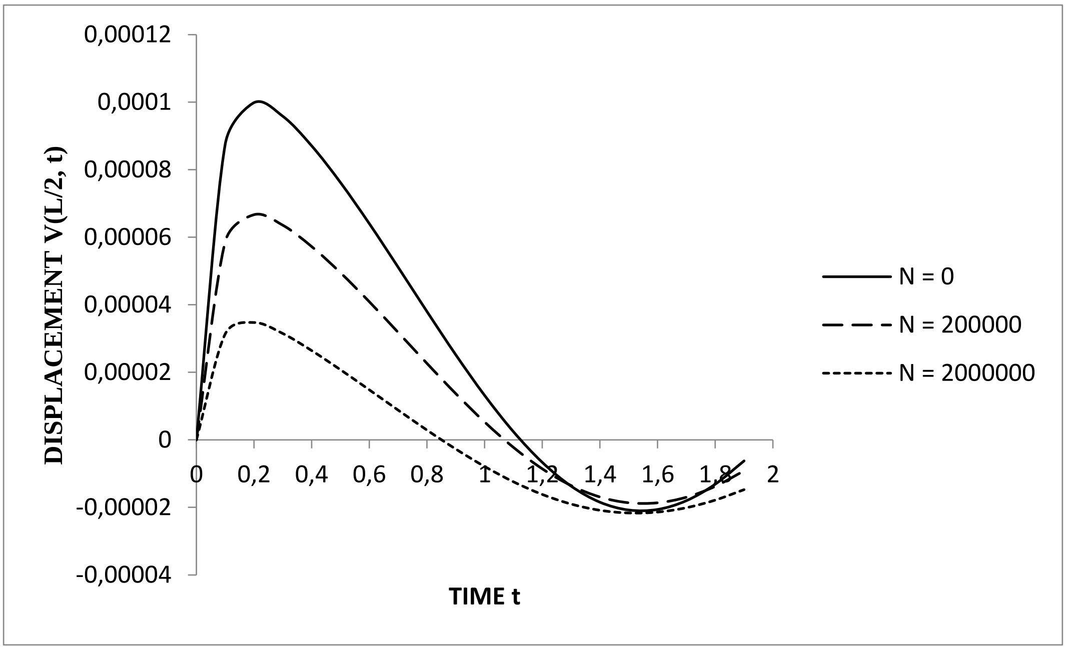


Fig 3: Deflection profile of uniform Rayleigh Beam subjected to harmonically varying moving loads for various values of Axial Force (N) and fixed values of Shear Modulus (G), Foundation Modulus (K), Damping Coefficient (Ʃ), Rotatory Inertial ($R_0$), and Load Natural Frequency (w).

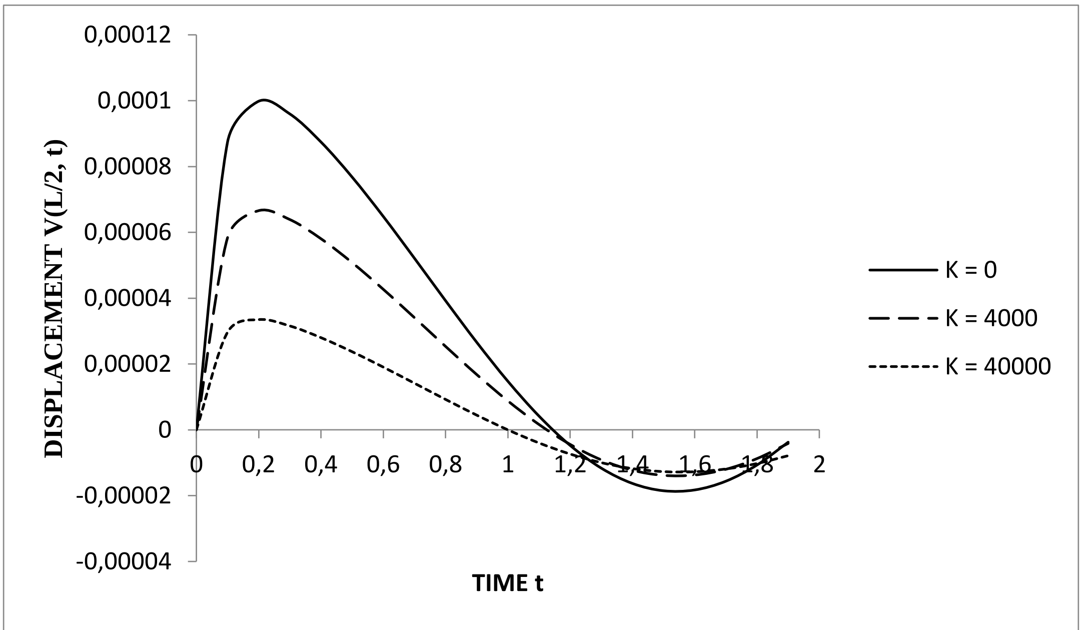


Fig 4: Deflection profile of uniform Rayleigh Beam subjected to harmonically varying moving loads for various values of Foundation Modulus (K) and fixed values of Shear Modulus (G), Damping Coefficient (Ʃ), Rotatory Inertial ($R_0$), Load Natural Frequency (w), and Axial Force (N).

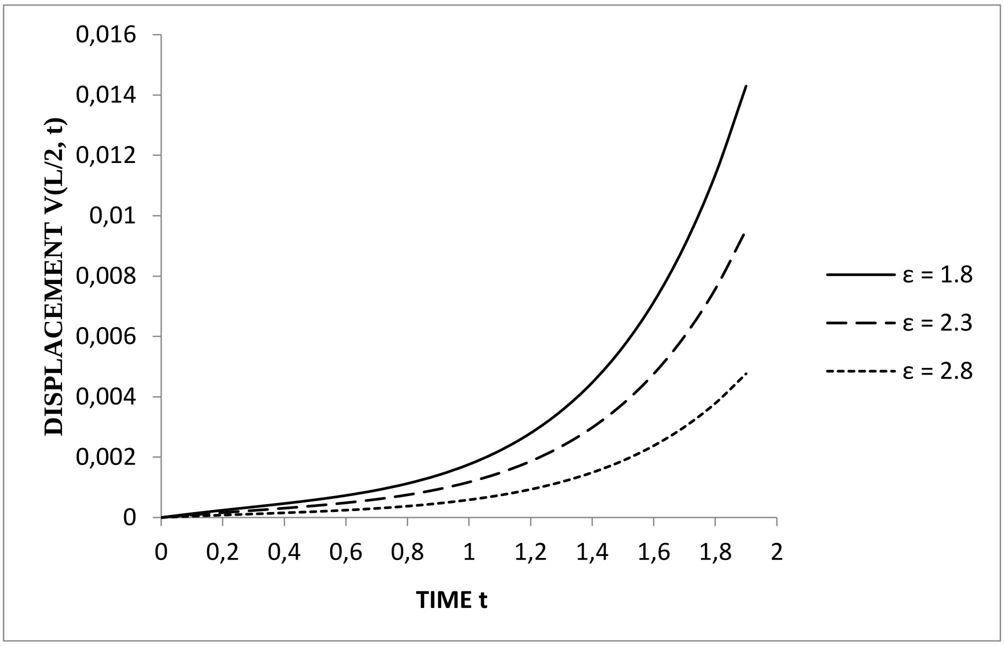


Fig 5: Deflection profile of uniform Rayleigh Beam subjected to harmonically varying moving loads for various values of Damping Coefficient ($\xi$) and fixed values of Shear Modulus (G), Rotatory Inertial ($R_0$), Load Natural Frequency (w), Axial Force (N), and Foundation Modulus (K).

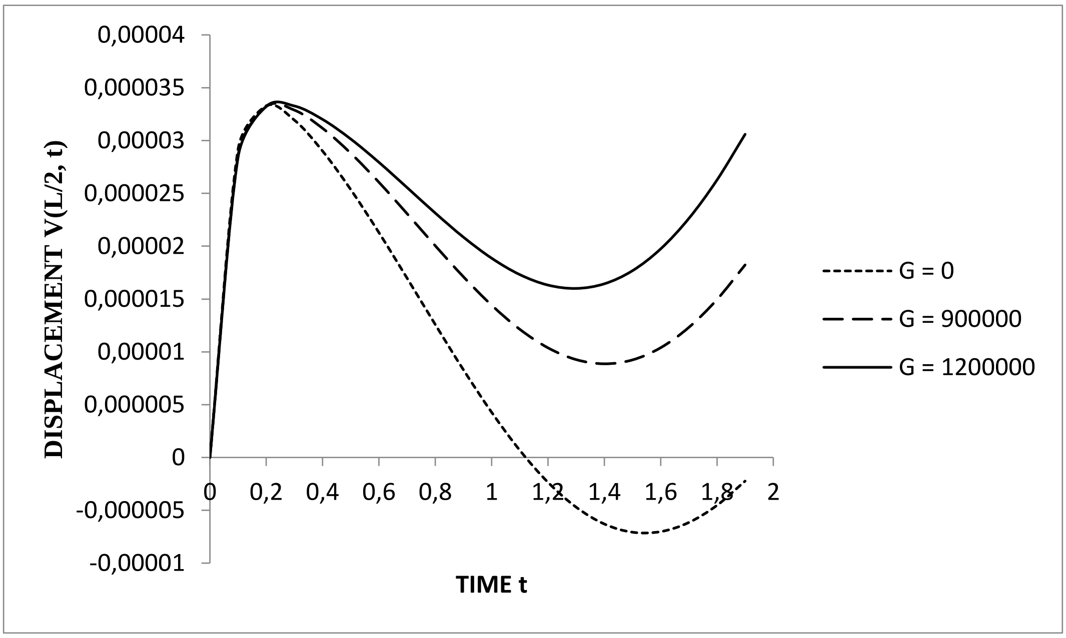


Fig 6: Deflection profile of uniform Rayleigh Beam subjected to harmonically varying moving loads for various values of Shear Modulus (G) and fixed values of Shear Modulus (G), Rotatory Inertial ($R_0$), Load Natural Frequency (w), Axial Force (N), Foundation Modulus (K), and Damping Coefficient ($\xi$).

## 5. Conclusion

In this paper, the problem of the dynamic response to harmonic magnitude moving load of uniform Rayleigh beam resting on Pasternak foundation is investigated. The approximation technique is based on the finite Fourier sine transform, Laplace transformation and convolution theorem. Analytical solutions and numerical analysis show that load natural frequency and rotatory inertia decreases the deflection profile of beam subjected to harmonic magnitude moving load. Furthermore, increases in other structural parameter such as shear modulus foundation stiffness, axial force and damping coefficient decreases the response amplitude of the beam. Finally, it was observed that rotatory inertia had more noticeable effect than that of the load natural frequency.

## References

[1] R. S. Ayre, L. S. Jacobsen and C. S. Hsu, Transverse vibration of one and of two-span beams under the action of a moving mass load, P*roceedings of the first U.S National Congress of Applied Mechanics*, (1951), 81-90.

[2] J. Kenny, Steady state vibrations of a beam on an elastic foundation for a moving load, *J. Appl. Mech.*, **76** (1984), 359-364.

[3] C. R. Steel, Beams and shells with moving loads, *Int. Journal of Solids and Structures*, **7** (1971), 1171-1198. https://doi.org/10.1016/0020-7683(71)90060-6

[4] S. T. Oni, On the thick beams under the action of a variable traveling transverse load. Abacus, *Journal of Mathematical Association of Nigeria*, **25** (1997), no. 2, 531 -546.

[5] T. P. Chang and H. W. Liu, Vibration analysis of a uniform beam traversed by a moving vehicle with random mass and random velocity, *Structural Engineering & Mechanics, An International Journal*, **31** (2009), no. 6, 737-749. https://doi.org/10.12989/sem.2009.31.6.737

[6] S. T. Oni and B. Omolofe, Vibration analysis of non-prismatic beam resting on elastic subgrade and under the actions of accelerating masses, *Journal of the Nigerian Mathematical Society*, **30** (2011), 63-109.

[7] S. T. Oni and T. O. Awodola, Dynamic behaviour under moving concentrated masses of simply supported rectangular plates resting on variable winkler elastic foundation, *Latin American Journal of Solids and Structures*, **8** (2011), 373 – 392. https://doi.org/10.1590/s1679-78252011000400001

[8] R. K. Misra, Free Vibration analysis of isotropic plate using multiquadric radial basis function, *International Journal of Science, Environment and Technology*, **1** (2012), no. 2, 99 – 107.

[9] S. T. Oni and S.N. Ogunyebi, Dynamical analysis of a prestressed elastic beam with general boundary conditions under the action of uniform distributed masses, J*ournal of the Nigerian Association of Mathematical Physics*, **12** (2008), 87-102. https://doi.org/10.4314/jonamp.v12i1.45492

[10] M. H. Hsu, Vibration characteristics of rectangular plates resting on elastic foundations and carrying any number of spring masses, *International of Applied Sciences and Engineering*, **4** (2006), no. 1, 83-89.

[11] K. Achawakorn and T. Jearipongkul, Vibration analysis of Exponential cross-section beam of exponential cross-section beam using Galerkin's method, *International Journal of Applied Science and Technology*, **2** (2012), no. 6, 7-13.

[12] A. D. Kerr, Elastic and Viscoelastic foundation models, *Journal of Applied Mechanics*, **31** (1964), 491-499. https://doi.org/10.1115/1.3629667

[13] G. L. Anderson, The influence of a Weghardt type elastic foundation on the stability of some beams subjected to distributed tangential forces, *Journal of Sound and Vibration*, **44** (1976), no, 1, 103 -118. https://doi.org/10.1016/0022-460x(76)90710-0

[14] P. L. Pasternak, On a new method of analysis of an elastic foundation by means of two foundation constants in (Russian). Gosudertrennve itdate Ustre Literature, Postroetalstivvi Arkhiteektre, Moscow.

[15] I. Coskum, The response of a finite beam on a tensionless Pasternak foundation subjected to a harmonic load, *Eur. J. Mech. A- Solide*, **22** (2003), no. 1, 151 -161. https://doi.org/10.1016/s0997-7538(03)00011-1

[16] S. T. Oni and A. Jimoh, On the dynamic response to moving concentrated loads of non-uniform Bernoulli-Euler Beam resting on Bi-parametric subgrades with other boundary conditions, *A Journal of National Mathematical Sciences*, **3** (2014), no. 1, 515-538.

[17] K. Guter, Circular elastic plate resting on tensionless Pasternak foundation, *J. Eng. Mech.*, **130** (2003), no. 10, 1251-1254. https://doi.org/10.1061/(asce)0733-9399(2004)130:10(1251)

[18] S. T. Oni and A. Jimoh, Dynamic response to moving concentrated loads of simply supported pre-stressed Bernoulli-Euler beam resting on Bi- parametric

subgrades, I*nternational Journal of Scientific and Engineering Research*, **7** (2016), no. 9, 584-600.

[19] S. T. Oni and A. Jimoh, On the dynamic response to moving concentraed loads of pre-stressed Bernoulli-Euler subgrades with other boundary conditions, *A Journal of National Mathematical Centre*, *Abuja*, **4** no 1, 857-880.

[20] X. Ma, I. W. Butterworth and G. C. Cliffon, Static analysis of an infinite beam resting on a tensionless Pasternak foundation, *Eur. J. Mech. A. Solid*, **28** (2009), 697-703. https://doi.org/10.1016/j.euromechsol.2009.03.003

[21] S. T. Oni and A. Jimoh, Dynamic response to moving concentrated loads of non-uniform simply supported pre-stressed Bernoulli-Euler beam resting on bi-parametric subgrades, *International Journal of Scientific and Engineering Research*, **7** (2016), no. 3, 754-770.

[22] L. Fryba, *Vibration of Solids and Structures Under Moving Loads*, Springer, Gronigen, Noordhoff, 1972. https://doi.org/10.1007/978-94-011-9685-7

**Received: March 21, 2018; Published: July 6, 2018**