On the Complete Convergence for 
END Random Variables

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Abstract

In this paper, the complete convergence for the maximum partial sums of extended negatively dependent random variables are established. In particular, the Marcinkiewicz-Zygmund type strong law of large numbers for extended negatively dependent random variables are obtained.

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1 Introduction

A sequence of random variables \( \{U_n, n \geq 1\} \) is said to converge completely to a constant \( C \) if

\[
\sum_{n=1}^{\infty} \mathbb{P}(|U_n - C| > \varepsilon) < \infty, \quad \text{for all} \quad \varepsilon > 0. \tag{1}
\]

The concept of complete convergence was introduced firstly by Hsu and Robbins [1]. In view of the Borel-Cantelli lemma, complete convergence implies that \( U_n \to C \) almost surely.

Let us recall the notations of slowly varying function and extended negatively dependent random variables. 

**Definition 1.1** ([2]) A real valued function \( l(x) \), positive and measurable on \((0, \infty)\), is said to be slowly varying if

\[
\lim_{x \to \infty} \frac{l(x\lambda)}{l(x)} = 1, \tag{2}
\]

for all \( \lambda \geq 1 \).

**Definition 1.2** ([3]) A sequence of random variables \( \{X_n, n \geq 1\} \) is said to be extended negatively dependent (END, in short) if there exists a constant \( 0 < M < \infty \) such that

\[
\mathbb{P}(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq M \prod_{i=1}^{n} \mathbb{P}(X_i > x_i), \tag{3}
\]

and

\[
\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq M \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i). \tag{4}
\]

hold for all \( n \geq 1 \) and all real numbers \( x_1, x_2, \ldots, x_n \).

When \( M = 1 \), the notion of extended negatively dependent random variables reduces to the well known notion of so-called negatively dependent (ND, in short) random variables. Liu (2009) investigated the asymptotic behavior of precise large deviations of partial sums of END random variables with heavy tails, and presented an example of a sequence of random variables which is END but is not ND. Joag-Dev and Proschan (1983) pointed out that negatively associated (NA, in short) random variables must be ND but ND random variables are not necessarily NA. Since END random variables are a much weaker concept than that of independent random variables, NA random variables, and ND random variables. Consequently, the study of the limit properties for END random variables is of much interest. For more details about the probability inequality, moment inequality, probability limit theorems and application for
END random variables, one can refer to Liu[3, 4], Zhang[5], Wang [6], Qiu [7], Wu [8], and so on.

The main purposes of the paper are give several generalized complete convergence for the extended negatively dependent random variables. Our main results are stated in Section 2 and all proofs are given in Section 3.

Throughout the paper, $C$ denote positive constants which may be different in variables places. $I(A)$ be the indicator functions of the set $A$.

## 2 Main Results

In the section, we state our main results and remarks.

**Theorem 2.1** Let $\{X_n, n \geq 1\}$ be a sequence of END random variables, $1 \leq p \leq 2, \alpha > 1/2$. If

$$
\lim_{x \to \infty} \sup_{n \geq 1} n^{-\alpha} \sum_{k=1}^{n} \mathbb{E}|X_k|I(|X_k| > x) = 0,
$$

and

$$
\sum_{n=1}^{\infty} n^{-\alpha p} \mathbb{E}|X_n|^p < \infty,
$$

then for any $\varepsilon > 0$,

$$
\sum_{n=1}^{\infty} n^{-1} \mathbb{P}\left( \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} (X_k - \mathbb{E}X_k) \right| > n^\alpha \varepsilon \right) < \infty.
$$

Consequently,

$$
n^{-\alpha} \sum_{k=1}^{n} (X_k - \mathbb{E}X_k) \to 0, \text{ a.s.}
$$

If $\alpha = 1$, then by Theorem 2.1, we can get the following corollary.

**Corollary 2.1** Let $1 \leq p \leq 2$, $\{X_n, n \geq 1\}$ be a sequence of END random variables. If

$$
\lim_{x \to \infty} \sup_{n \geq 1} n^{-1} \sum_{k=1}^{n} \mathbb{E}|X_k|I(|X_k| > x) = 0,
$$

and

$$
\sum_{n=1}^{\infty} n^{-p} \mathbb{E}|X_n|^p < \infty,
$$

then for any $\varepsilon > 0$,

$$
\sum_{n=1}^{\infty} n^{-1} \mathbb{P}\left( \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} (X_k - \mathbb{E}X_k) \right| > n \varepsilon \right) < \infty.
$$
Remark 2.1 The Theorem 2.1 improve and extend the results of Bai[9], from pairwise i.i.d random variables to END random variables.

The following theorems present the complete convergence for the partial sums of the moving average processes of END random variables.

**Theorem 2.2** Let \( 1 < p < 2, \alpha p \geq 1, \{X, X_n, n \geq 1\} \) be a sequence of identically distributed END random variables with \( \mathbb{E}X = 0 \). Let \( l(x) > 0 \) be a slowly varying function, and \( \{a_n, n \geq 1\} \) be a sequence of real constants with \( \sum_{n=1}^{\infty} |a_n| < \infty \). If \( \mathbb{E}|X|^p l(|X|^{1/\alpha}) < \infty \), then for any \( \varepsilon > 0 \),
\[
\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \mathbb{P} \left( \left| \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i X_{i+k} \right| > \varepsilon n^\alpha \right) < \infty. \tag{12}
\]

If \( l(x) \equiv 1 \) in Theorem 2.2, we can get the following corollary.

**Corollary 2.2** Let \( 1 < p < 2, \alpha p \geq 1, \{X, X_n, n \geq 1\} \) be a sequence of identically distributed END random variables with \( \mathbb{E}X = 0 \). Let \( \{a_n, n \geq 1\} \) be a sequence of real constants with \( \sum_{n=1}^{\infty} |a_n| < \infty \). If \( \mathbb{E}|X|^p < \infty \), then for any \( \varepsilon > 0 \),
\[
\sum_{n=1}^{\infty} n^{\alpha p - 2} \mathbb{P} \left( \left| \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i X_{i+k} \right| > \varepsilon n^\alpha \right) < \infty. \tag{13}
\]

**Theorem 2.3** Let \( \alpha > 1/2, 1 < p < 2, \alpha p > 1, \{X, X_n, n \geq 1\} \) be a sequence of identically distributed END random variables with \( \mathbb{E}X = 0 \). Let \( l(x) > 0 \) be a slowly varying function, and \( \{a_n, n \geq 1\} \) be a sequence of real constants with \( \sum_{n=1}^{\infty} |a_n| < \infty \), and \( \lim_{x \to \infty} \sup_{n \geq 1} n^{1-\alpha} \mathbb{E}|X|^p l(|X| > x) = 0 \), when \( 1/2 < \alpha < 1 \). If \( \mathbb{E}|X|^p l(|X|^{1/\alpha}) < \infty \), then for any \( \varepsilon > 0 \),
\[
\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \mathbb{P} \left( \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} \sum_{i=-\infty}^{\infty} a_i X_{i+k} \right| > \varepsilon n^\alpha \right) < \infty. \tag{14}
\]

### 3 Proofs of Main results

#### 3.1 Some lemmas

To prove our results, we first give some lemmas as follows.

**Lemma 3.1** (Zhou [10]) Let \( l(x) \) is slowly varying on \((0,\infty)\), then
\[
(i) \sum_{k=1}^{n} k^p l(k) \leq C n^{p+1} l(n), \text{ for } p > -1 \text{ and positive integer } n;
\]
\[
(ii) \sum_{k=n}^{\infty} k^p l(k) \leq C n^{p+1} l(n), \text{ for } p < -1 \text{ and positive integer } n.
\]
Lemma 3.2 ([3]) Let \( \{X_n, n \geq 1\} \) be a sequence of END random variables. If \( \{f_n, n \geq 1\} \) is a sequence of all nonincreasing (or nondecreasing) functions, then \( \{f_n(X_n), n \geq 1\} \) are END random variables.

Lemma 3.3 ([11, 12]) Let \( r > 1, \{X_n, n \geq 1\} \) be a sequence of END random variables with \( \mathbb{E}X_n = 0 \) and \( \mathbb{E}|X_n|^r < \infty \). Then there exists positive constant \( c_r \) depending only on \( r \), for all \( n \geq 1 \),

\[
\mathbb{E} \left| \sum_{k=1}^{n} X_k \right|^r \leq c_r \sum_{k=1}^{n} \mathbb{E}|X_k|^r, \tag{15}
\]

holds when \( 1 < r < 2 \), and

\[
\mathbb{E} \left| \sum_{k=1}^{n} X_k \right|^r \leq c_r \left\{ \sum_{k=1}^{n} \mathbb{E}|X_k|^r + \left( \sum_{k=1}^{n} \mathbb{E}|X_k|^2 \right)^{r/2} \right\} \tag{16}
\]

holds when \( r \geq 2 \).

Lemma 3.4 ([5]) Let \( r > 1, \{X_n, n \geq 1\} \) be a sequence of END random variables with \( \mathbb{E}X_n = 0 \) and \( \mathbb{E}|X_n|^r < \infty \). Then there exists positive constant \( c_r \) depending only on \( r \), for all \( n \geq 1 \),

\[
\mathbb{E} \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} X_k \right|^r \leq c_r (\log n)^r \sum_{k=1}^{n} \mathbb{E}|X_k|^r, \tag{17}
\]

holds when \( 1 < r < 2 \), and

\[
\mathbb{E} \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} X_k \right|^r \leq c_r (\log n)^r \left\{ \sum_{k=1}^{n} \mathbb{E}|X_k|^r + \left( \sum_{k=1}^{n} \mathbb{E}|X_k|^2 \right)^{r/2} \right\} \tag{18}
\]

holds when \( r \geq 2 \).

Lemma 3.5 Let \( \alpha > 0, 1 \leq p \leq 2, \{X_n, n \geq 1\} \) be a sequence of END random variables. If

\[
\sum_{n=1}^{\infty} n^{-\alpha p} \mathbb{E}|X_n|^p < \infty, \tag{19}
\]

then for any \( \varepsilon > 0 \),

\[
\sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \left| \sum_{k=1}^{n} (X_k - \mathbb{E}X_k) \right| > n^\alpha \varepsilon \right) < \infty. \tag{20}
\]
**proof** By $\sum_{n=1}^{\infty} n^{-\alpha} E|X_n|^p < \infty$ and Kronecker lemma, we can get that

$$n^{-\alpha} \sum_{k=1}^{n} E|X_k|^p \to 0, \text{ as } n \to \infty. \quad (21)$$

Hence for any $\varepsilon > 0$, we have that for $n$ large enough

$$n^{-\alpha} \left| \sum_{k=1}^{n} E X_k I(|X_k| > n^\alpha) \right| \leq n^{-\alpha} \sum_{k=1}^{n} E|X_k|I(|X_k| > n^\alpha) \leq n^{-\alpha} \sum_{k=1}^{n} n^{\alpha(1-p)} E|X_k|^p I(|X_k| > n^\alpha) \leq n^{-\alpha} \sum_{k=1}^{n} E|X_k|^p \leq \frac{\varepsilon}{2}. \quad (22)$$

By Lemma 3.2, $\{X_k I(|X_k| \leq n^\alpha) - E X_k I(|X_k| \leq n^\alpha), n \geq 1, 1 \leq k \leq n\}$ is a sequence of END random variables with zero mean. From (22), Lemma 3.3 and Markov's inequality, we can get that

$$\sum_{n=1}^{\infty} n^{-1} P \left( \left| \sum_{k=1}^{n} (X_k - E X_k) \right| > \varepsilon n^\alpha \right) \leq \sum_{n=1}^{\infty} n^{-1} \sum_{k=1}^{n} P(|X_k| > n^\alpha) + \sum_{n=1}^{\infty} n^{-1} P \left( \left| \sum_{k=1}^{n} (X_k I(X_k \leq n^\alpha) - E X_k I(X_k \leq n^\alpha)) \right| > \frac{\varepsilon n^\alpha}{2} \right) \leq \sum_{n=1}^{\infty} n^{-\alpha p} \sum_{k=1}^{n} E|X_k|^p + C \sum_{n=1}^{\infty} n^{-2\alpha - 1} \sum_{k=1}^{n} E|X_k|^2 I(|X_k| \leq n^\alpha) \quad (23)$$

$$\leq \sum_{n=1}^{\infty} n^{-\alpha p - 1} \sum_{k=1}^{n} E|X_k|^p + C \sum_{n=1}^{\infty} \sum_{k=1}^{n} E|X_k|^p \leq C \sum_{k=1}^{\infty} E|X_k|^p \sum_{n=k}^{\infty} n^{-\alpha p - 1} \leq C \sum_{n=1}^{\infty} n^{-\alpha p} E|X_n|^p < \infty.$$
3.2 Proof of Theorem 2.1

Note that

$$\lim_{x \to \infty} \sup_{n \geq 1} n^{-\alpha} \sum_{k=1}^{n} \mathbb{E}[|X_k| I(|X_k| > x)] = 0,$$

then for every $\varepsilon > 0$, there is a constant $x = x(\varepsilon) > 0$ such that for all $n \geq 1$,

$$n^{-\alpha} \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} \mathbb{E}X_k I(|X_k| > x) \right| \leq n^{-\alpha} \sum_{k=1}^{n} \mathbb{E}|X_k| I(|X_k| > x) \leq \frac{\varepsilon}{4}. \quad (24)$$

Since

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} (X_k - \mathbb{E}X_k) \right| > n^\alpha \varepsilon \right)$$

$$\leq \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} (X_k I(|X_k| \leq x) - \mathbb{E}X_k I(|X_k| \leq x)) \right| > \frac{n^\alpha \varepsilon}{4} \right) \quad (25)$$

$$+ \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} (X_k I(|X_k| > x) - \mathbb{E}X_k I(|X_k| > x)) \right| > \frac{3n^\alpha \varepsilon}{4} \right)$$

$$\triangleq I_1 + I_2.$$

To prove (7), it is enough to show that $I_1 < \infty$ and $I_2 < \infty$. Note that $\alpha > 1/2$, by the condition (5), Lemma 3.4 and Markov’s inequality,

$$I_1 = \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} (X_k I(|X_k| \leq x) - \mathbb{E}X_k I(|X_k| \leq x)) \right| > \frac{n^\alpha \varepsilon}{4} \right)$$

$$\leq C \sum_{n=1}^{\infty} n^{-1-2\alpha} \mathbb{E} \left[ \left( \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} (X_k I(|X_k| \leq x) - \mathbb{E}X_k I(|X_k| \leq x)) \right| \right)^2 \right]$$

$$\leq C \sum_{n=1}^{\infty} n^{-1-2\alpha} (\log n)^2 \sum_{k=1}^{n} \mathbb{E}X_k^2 I(|X_k| \leq x)$$

$$\leq C \sum_{n=1}^{\infty} n^{-2\alpha} (\log n)^2 < \infty. \quad (26)$$
For $I_2$, we have by (24) and Lemma 3.5 that
\[
I_2 = \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} (X_k I(|X_k| > x) - \mathbb{E} X_k I(|X_k| > x)) \right| > \frac{3n^\alpha \varepsilon}{4} \right)
\]
\[
\leq C \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} X_k I(|X_k| > x) \right| > \frac{n^\alpha \varepsilon}{2} \right)
\]
\[
\leq C \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \sum_{k=1}^{n} |X_k| I(|X_k| > x) > \frac{n^\alpha \varepsilon}{2} \right)
\]
\[
\leq C \sum_{n=1}^{\infty} n^{-1} \mathbb{P} \left( \sum_{k=1}^{n} (|X_k| I(|X_k| > x) - \mathbb{E} X_k I(|X_k| > x)) > \frac{n^\alpha \varepsilon}{4} \right)
\]
\[
< \infty.
\]  
(27)

### 3.3 Proof of Theorem 2.2

By $\mathbb{E} |X|^p l(|X|^{1/\alpha}) < \infty$, we can get that
\[
\mathbb{E} |X|^{p-\delta} < \infty, \quad \text{for any } 0 < \delta < p.
\]  
(28)

Take $0 < \delta < \min\{\frac{\alpha p - 1}{\alpha}, p - 1\}$, which implies $1 - \alpha p + \alpha \delta < 0$ and $1 - p + \delta < 0$. Note that $\sum_{i=-\infty}^{\infty} |a_i| < \infty$ and (28), then
\[
n^{-\alpha} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \mathbb{E} X_j I(|X_j| > n^\alpha) \right|
\]
\[
\leq n^{-\alpha} \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} \mathbb{E} |X_j| I(|X_j| > n^\alpha)
\]
\[
\leq C n^{1-\alpha} \mathbb{E} |X| I(|X| > n^\alpha)
\]
\[
\leq C n^{1-\alpha p + \alpha \delta} \mathbb{E} |X|^{p-\delta} I(|X| > n^\alpha) \to 0.
\]  
(29)

Hence for any $\varepsilon > 0$, we have that for $n$ large number
\[
n^{-\alpha} \left| \mathbb{E} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} X_j I(|X_j| \leq n^\alpha) \right| < \frac{\varepsilon}{4}.
\]  
(30)

Denote
\[
X_{i}^{(n)}(n) = X_i I(|X_i| \leq n^\alpha) - \mathbb{E} X_i I(|X_i| \leq n^\alpha), \quad i \geq 1, \ n \geq 1.
\]
By \( \mathbb{E}X = 0 \) and (30), then

\[
\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \mathbb{P} \left( \left| \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i X_{i+k} \right| > n^{\alpha} \varepsilon \right) = \sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \mathbb{P} \left( \left| \sum_{i=-\infty}^{\infty} a_i \sum_{k=i+1}^{i+n} X_k \right| > n^{\alpha} \varepsilon \right)
\leq \sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \mathbb{P} \left( \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} X_j I(|X_j| > n^{\alpha}) \right| > \frac{n^{\alpha} \varepsilon}{2} \right) + \sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \mathbb{P} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} X_j^{(n)} \right| > \frac{n^{\alpha} \varepsilon}{4} \right) \]
\[= J_1 + J_2. \]

By Lemma 3.1 and Markov’s inequality

\[
J_1 \leq C \sum_{n=1}^{\infty} n^{\alpha p - \alpha - 2} l(n) \mathbb{E} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} X_j I(|X_j| > n^{\alpha}) \right|
\leq C \sum_{n=1}^{\infty} n^{\alpha p - \alpha - 2} l(n) \mathbb{E} \left| \sum_{i=-\infty}^{\infty} a_i n X I(|X| > n^{\alpha}) \right|
\leq C \sum_{n=1}^{\infty} n^{\alpha p - \alpha - 1} l(n) \mathbb{E} \left| X I(|X| > n^{\alpha}) \right|
\leq C \sum_{n=1}^{\infty} n^{\alpha p - \alpha - 1} l(n) \sum_{k=n}^{\infty} \mathbb{E} |X| I(k^{\alpha} < |X| \leq (k + 1)^{\alpha})
\leq C \sum_{k=1}^{\infty} \mathbb{E} |X| I(k^{\alpha} < |X| \leq (k + 1)^{\alpha}) \sum_{n=1}^{k} n^{\alpha p - \alpha - 1} l(n)
\leq C \sum_{k=1}^{\infty} l(k) k^{\alpha p - \alpha} \mathbb{E} |X| I(k^{\alpha} < |X| \leq (k + 1)^{\alpha})
\leq C \mathbb{E} |X|^{p l(|X|^{1/\alpha})} < \infty.
\]
For $J_2$, by Lemma 3.1, Lemma 3.3, Markov’s inequality and Hölder inequality,

$$J_2 = \sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \mathbb{P} \left( \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} X_j^{(n)} \right| > \frac{n^\alpha \varepsilon}{4} \right)$$

$$\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 2} l(n) \mathbb{P} \left( \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} X_j^{(n)} \right|^2 \right)$$

$$\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 2} l(n) \sum_{i=-\infty}^{\infty} |a_i| \sum_{j=i+1}^{i+n} \mathbb{E} |X^2 I(\{|X| \leq n^\alpha\})$$

$$\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 1} l(n) \mathbb{E} |X|^2 I(\{|X| \leq n^\alpha\})$$

$$\leq C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 1} l(n) \sum_{k=1}^{\infty} \mathbb{E} |X|^2 I((k-1)^\alpha < |X| \leq k^\alpha)$$

$$\leq C \sum_{k=1}^{\infty} \mathbb{E} |X|^2 I((k-1)^\alpha < |X| \leq k^\alpha) l(k) k^{\alpha p - 2}$$

$$\leq C \mathbb{E} |X|^p l(|X|^{1/\alpha}) < \infty.$$

### 3.4 Proof of Theorem 2.3

From the assumption $\lim_{x \to \infty} \sup_{n \geq 1} n^{1-\alpha} \mathbb{E} |X| I(|X| > x) = 0$, where $1/2 < \alpha < 1$, then for any every $\varepsilon > 0$, there is a $x = x(\varepsilon) > 0$ such that

$$n^{1-\alpha} \left( \sum_{i=-\infty}^{\infty} |a_i| \right) \mathbb{E} |X| I(|X| > x) \leq C n^{1-\alpha} \mathbb{E} |X| I(|X| > x) < \frac{\varepsilon}{4}, \ \forall n \geq 1. \quad (33)$$

Let

$$Y_i(x) = X_i I(|X_i| \leq x) - \mathbb{E} X_i I(|X_i| \leq x), \quad Z_i(x) = X_i I(|X_i| > x) - \mathbb{E} X_i I(|X_i| > x).$$

$$Y'_i(x) = |X_i| I(|X_i| > x) - \mathbb{E} |X_i| I(|X_i| > x).$$
Note that $X_i = Y_i + Z_i$ for all $i \geq 1$, then
\[
\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P} \left( \sum_{1 \leq m \leq n} \sum_{k=1}^{m} a_i X_{i+k} > n^\alpha \varepsilon \right) = \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P} \left( \sum_{1 \leq m \leq n} \sum_{k=1}^{m} a_i Y_{i+k} > \frac{n^\alpha \varepsilon}{4} \right) + \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P} \left( \sum_{1 \leq m \leq n} \sum_{k=1}^{m} a_i Z_{i+k} > \frac{3n^\alpha \varepsilon}{4} \right)
\]
$\triangleq H_1 + H_2$.

In order to prove (2.8), it is enough to prove that $H_1 < \infty$ and $H_2 < \infty$. By Markov’s inequality, Hölder inequality and Lemma 3.4
\[
H_1 = \sum_{n=1}^{\infty} n^{\alpha p-2\alpha-2} l(n) \mathbb{E} \left( \sum_{1 \leq m \leq n} \sum_{i=1}^{m} a_i \sum_{j=1}^{i+m} Y_j \right)^2 \\
\leq C \sum_{n=1}^{\infty} n^{\alpha p-2\alpha-2} l(n) \mathbb{E} \left( \sum_{i=1}^{\infty} |a_i| \sum_{j=1}^{i+m} Y_j \right)^2 \\
\leq C \sum_{n=1}^{\infty} n^{\alpha p-2\alpha-1} l(n) (\log n)^2 \mathbb{E} |X|^2 I(|X| \leq x) \\
\leq C \sum_{n=1}^{\infty} n^{\alpha p-2\alpha-1} l(n) (\log n)^2 < \infty.
\]

For $H_2$, by (33) and Theorem 2.2,
\[
H_2 = \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P} \left( \sum_{1 \leq m \leq n} \sum_{k=1}^{m} a_i Z_{i+k} > \frac{3n^\alpha \varepsilon}{4} \right) \\
\leq C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P} \left( \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} |a_i X_{i+k}| I(|X_{i+k}| > x) > \frac{n^\alpha \varepsilon}{2} \right) \\
\leq C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P} \left( \sum_{k=1}^{m} \sum_{i=-\infty}^{\infty} |a_i Y'_{i+k}| > \frac{n^\alpha \varepsilon}{4} \right) \\
< \infty.
\]
References


[12] A. Shen, Probability inequalities for END sequence and their applications, 

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