Applied Mathematical Sciences, Vol. 12, 2018, no. 1, 27 - 36
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2018.712357

The Painlevé Test for a System of Coupled Equations from Nonlinear Birefringence

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Abstract

The Painlevé analysis has been performed for integrability of an example of two coupled equations taken from the field of nonlinear fiber optics. This is a generic system of highly nonlinear differential equations which describes the phenomena in nonlinear birefringence. In particular, the Painlevé test uncovers the generic non-integrable nature of the partial differential system such as the computed Fuchs indices. Last but not the least, the analysis performed so far will lay a solid foundation for further investigation of the closed form solutions.

Mathematics Subject Classification: 30E15, 35Q60, 37K10

Keywords: Painlevé analysis; differential equations; birefringence

1. Introduction

The field of nonlinear fiber optics plays an increasingly important role in science, engineering, and mathematics [5, 6, 10]. The reason behind such a tremendous growth was the advent of fiber amplifiers. Such amplifiers changed the design of fiber-optic communication systems, including those making use of optical solitons. The existence of the optical solitons stems from the presence of nonlinear effects in optical fibers. Optical amplifiers eventually permit propagation of lightwave signals over thousands of kilometers. Nonlinear fiber optics has the contributions in the design of of such high-capacity lightwave systems. In terms of mathematics, a major role is the nonlinear Schrödinger

(NLS) equation which consists of assuming that the incident light is preserved during its propagation inside an optical fiber [2, 8]. Practically speaking, all fibers exhibit some birefringence due to unintentional variations in the shape and stresses along the fiber. When the nonlinear effects in optical fibers tend to become dominant, a very intense optical field will induce the nonlinear birefringence whose magnitude is intensity dependent [9].

In this paper, we will explore the governing equations which were deduced from the nonlinear birefringence. In fact, the governing equations are coupled partial differential equations which are highly nonlinear and are complex-valued. Technically, we are unable to solve the general solutions of such complicated coupled system of partial differential equations. However, our focus is purely mathematical and we intend to perform the very famous Painlevé analysis [3] which is supposed to be the best tool to explore the integrability of a single differential equation [11] or a system of coupled differential equations [7]. The purpose is to investigate whether this can be achieved without any a priori knowledge of the solutions, with a powerful algorithm called the Painlevé test. If the system of the equations passes the Painlevé test, it is presumed integrable in some sense, and one can try to build the explicit information displaying the integrability, e.g., to build soliton solutions. If on the contrary the test fails, the system is non-integrable or even chaotic, but it may still be possible to find some exact solutions [4]. The example taken from nonlinear birefringence in this paper is targeted at illustrating the Painlevé test as well as the related analysis.

In Sect. 2 we shall first present the mathematical formulation of the problem from nonlinear birefringence. In Sect. 3 we shall perform the Painlevé test in order to explore the governing equations. Several things will be discussed such as the dominant behaviours of the solutions, the travelling wave reduction as well as the determination of the Fuchs indices of the indicial equation.

2. Mathematical formulation

We skip all the unnecessary details and immediately start to consider the governing equations such as the coupled equations from nonlinear birefringence [1].

$$\begin{cases} \mathrm{i}\Big(\dfrac{\partial u}{\partial \xi} + \delta \dfrac{\partial u}{\partial \tau}\Big) + bu + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \tau^2} + \Big(|u|^2 + \dfrac{2}{3}|v|^2\Big)u + \dfrac{1}{3}v^2 u^* \;=\; 0, \\ \mathrm{i}\Big(\dfrac{\partial v}{\partial \xi} - \delta \dfrac{\partial v}{\partial \tau}\Big) - bv + \dfrac{1}{2}\dfrac{\partial^2 v}{\partial \tau^2} + \Big(|v|^2 + \dfrac{2}{3}|u|^2\Big)v + \dfrac{1}{3}u^2 v^* \;=\; 0, \end{cases} \tag{1}$$

where the real parameters δ and b vary randomly along the fiber because of random birefringence fluctuations.

We attempt to study the governing equations by performing the Painlevé test. The analysis will include the assumption of general solitary wave solution and try to look for the dominant behaviours of the solutions. Next we would like to hunt for the Fuchs indices of the equations. Besides the nature of the solutions, we shall employ the travelling wave reduction to explore the Laurent series solutions of the equations. We will deduce the dominant behaviours as well and we should emphasize that with the final result we still need the help of algebra software for computations.

We rename ξ as t, τ as x and relax the coefficients of the equations by introducing the real numbers p and q, the system (1) becomes

$$E \equiv \begin{cases} \mathrm{i}\Big(u_t + \delta u_x\Big) + bu + pu_{xx} + q\Big(u\bar{u} + \dfrac{2}{3}v\bar{v}\Big)u + \dfrac{1}{3}v^2\bar{u} = 0, \\ \mathrm{i}\Big(v_t - \delta v_x\Big) - bv + pv_{xx} + q\Big(v\bar{v} + \dfrac{2}{3}u\bar{u}\Big)v + \dfrac{1}{3}u^2\bar{v} = 0, \end{cases} \tag{2}$$

where $(u, v) \in \mathbb{C}$, $(b, \delta, p, q) \in \mathbb{R}$, and $\bar{z}$ the complex conjugate of $z \in \mathbb{C}$.

3. Painlevé analysis of the system

The first analysis we want to perform is to look for any general solitary wave solution of the system of coupled equations from nonlinear birefringence. We restrict to the search for the most general solitary wave solution

$$u(x,t) = A(\xi)\, e^{\phi_1(\xi)}, \qquad v(x,t) = B(\xi)\, e^{\phi_2(\xi)}, \qquad \xi = x - ct, \tag{3}$$

in which (A, B, ϕ_1, ϕ_2) are complex-valued functions of the reduced independent variable ξ. We are ready to look for the Laurent series of the general solution of (2) which are the local representation of the solutions:

$$u = a_0\chi^{p_1}\Big(1 + a_1\chi + O(\chi^2)\Big), \quad v = b_0\chi^{p_2}\Big(1 + b_1\chi + O(\chi^2)\Big), \tag{4}$$

where $\chi = \xi - \xi_0$. In (4), $\{p_1, p_2, a_0, a_1, \cdots, b_0, b_1, \cdots\}$ are complex constants to be determined. Denote that $p_1 = p_{1r} + \mathrm{i}p_{1i}$ and $p_2 = p_{2r} + \mathrm{i}p_{2i}$. In order to compute the Laurent series of the form (4), we first examine the dominant behaviour of the four fields $(u, \bar{u}, v, \bar{v})$. Since the terms $\mathrm{i}\,(u_t + \delta u_x)$, bu, $\mathrm{i}\,(v_t - \delta v_x)$, $-bv$ are less singular, the dominant terms are

$$\hat{E} \equiv \begin{cases} pu_{xx} + q\left(u^2\bar{u} + \dfrac{2}{3}uv\bar{v}\right) + \dfrac{1}{3}v^2\bar{u}, \\ pv_{xx} + q\left(v^2\bar{v} + \dfrac{2}{3}uv\bar{u}\right) + \dfrac{1}{3}u^2\bar{v}. \end{cases} \tag{5}$$

Substituting the dominant terms of the forms $u \sim O(\chi^{p_{1r}})$, $\bar{u} \sim O(\chi^{p_{1r}})$, $v \sim O(\chi^{p_{2r}})$, $\bar{v} \sim O(\chi^{p_{2r}})$ into (5) and comparing the powers gives $u_{xx} \sim O(\chi^{p_{1r}-2})$, $u^2\bar{u} \sim O(\chi^{3p_{1r}})$, $uv\bar{v} \sim O(\chi^{p_{1r}+2p_{2r}})$, $v^2\bar{u} \sim O(\chi^{2p_{2r}+p_{1r}})$, $v_{xx} \sim O(\chi^{p_{2r}-2})$, $v^2\bar{v} \sim O(\chi^{3p_{2r}})$, $uv\bar{u} \sim O(\chi^{p_{2r}+2p_{1r}})$, $u^2\bar{v} \sim O(\chi^{2p_{1r}+p_{2r}})$. Carefully checking the dominant balance between the powers, we obtain

$$p_{1r} = -1, \qquad p_{2r} = -1.$$

Let us then denote the dominant behaviour as (rename p_{1i} as α and p_{2i} as β, both are real)

$$u \sim a_0\chi^{-1+\mathrm{i}\alpha}, \quad \bar{u} \sim \bar{a}_0\chi^{-1-\mathrm{i}\alpha}, \quad v \sim b_0\chi^{-1+\mathrm{i}\beta}, \quad \bar{v} \sim \bar{b}_0\chi^{-1-\mathrm{i}\beta}, \tag{6}$$

in which $(a_0, b_0) \in \mathbb{C}$, $(\alpha, \beta) \in \mathbb{R}$ are constants to be determined. Substituting (6) into (5) gives the dominant orders

$$\begin{cases} -3+\mathrm{i}\alpha &= -3+\mathrm{i}(2\beta-\alpha), \\ -3+\mathrm{i}\beta &= -3+\mathrm{i}(2\alpha-\beta). \end{cases} \tag{7}$$

From (7), we deduce that $\alpha = \beta$, and hence

$$\begin{cases} \Big(p(-1+\mathrm{i}\alpha)(-2+\mathrm{i}\alpha) + q(|a_0|^2 + \frac{2}{3}\,|b_0|^2)\Big)\,a_0 + \frac{1}{3}\,b_0^2\bar{a}_0 &= 0, \\ \Big(p(-1+\mathrm{i}\beta)(-2+\mathrm{i}\beta) + q(|b_0|^2 + \frac{2}{3}\,|a_0|^2)\Big)\,b_0 + \frac{1}{3}\,a_0^2\bar{b}_0 &= 0. \end{cases} \tag{8}$$

We need to find the two complex constants a_0 and b_0 but unfortunately we cannot find any non-trivial solution. The following is a simple proof. We first divide the first equation of (8) by a_0 and the second equation by b_0. Next, since $\alpha = \beta$, we subtract one by the other ($q = 1$),

$$\frac{1}{3}|a_0|^2 - \frac{1}{3}|b_0|^2 = \frac{1}{3}\left(\frac{a_0^2\bar{b}_0}{b_0} - \frac{b_0^2\bar{a}_0}{a_0}\right).$$

Take $a_0 = ae^{\mathrm{i}\theta_a}$ and $b_0 = be^{\mathrm{i}\theta_b}$, we obtain

$$\frac{a^2}{b^2} = \frac{1-e^{2\mathrm{i}(\theta_b-\theta_a)}}{1-e^{2\mathrm{i}(\theta_a-\theta_b)}} = -e^{2\mathrm{i}(\theta_b-\theta_a)}.$$

The above further implies that

$$a = b \qquad \text{and} \qquad 2(\theta_b-\theta_a) = \pm\pi,\ \pm3\pi,\ \pm5\pi,\ \cdots,$$

and then for $k = 0, \pm1, \pm2, \cdots$, we have

$$b_0 = be^{\mathrm{i}(\theta_a+\frac{2k+1}{2}\pi)} = a_0\cdot e^{\mathrm{i}(k\pi)}\cdot e^{\mathrm{i}(\pi/2)} = \pm\mathrm{i}\,a_0.$$

Substituting the above into the first equation of (8) gives

$$\alpha = \beta = 0 \qquad \text{and} \qquad |a_0|^2 + \frac{d^2}{2}.$$

The above determines the leading order behaviour of the solutions u and v. However, it should be noted that (6) may not be the best assumption. With the assumption we in prior set the conjugate of power and coefficients. However we emphasize that in general we should write

$$u \sim a_0\chi^{p_1}, \qquad \bar{u} \sim b_0\chi^{p_2}, \qquad v \sim c_0\chi^{p_3}, \qquad \bar{v} \sim d_0\chi^{p_4},$$

in which $(a_0, b_0, c_0, d_0, p_1, p_2, p_3, p_4)$ are all complex constants to be determined.

Another analysis we want to perform is to look for the Fuchs indices of the system. The Fuchs indices are important to tell the integrability of the system of nonlinearly coupled equations. By investigating the dominant behaviours of the solutions, we are able to deduce the indicial equation of the target equation. The roots of the indicial equation are called Fuchs indices of the linear system. The number of Fuchs indices is at most equal to the differential order of the equation. In fact, recall (2) and its complex conjugates,

$$\left\{\begin{array}{rcl} \mathrm{i}\,(u_t+\delta u_x)+bu+\dfrac{d^2}{2}u_{xx}+\left(u\bar{u}+\dfrac{2}{3}v\bar{v}\right)u+\dfrac{1}{3}v^2\bar{u} &=& 0,\\ -\mathrm{i}\,(\bar{u}_t+\delta \bar{u}_x)+b\bar{u}+\dfrac{d^2}{2}\bar{u}_{xx}+\left(\bar{u}u+\dfrac{2}{3}\bar{v}v\right)\bar{u}+\dfrac{1}{3}\bar{v}^2u &=& 0,\\ \mathrm{i}\,(v_t-\delta v_x)-bv+\dfrac{d^2}{2}v_{xx}+\left(v\bar{v}+\dfrac{2}{3}u\bar{u}\right)v+\dfrac{1}{3}u^2\bar{v} &=& 0,\\ -\mathrm{i}\,(\bar{v}_t-\delta \bar{v}_x)-b\bar{v}+\dfrac{d^2}{2}\bar{v}_{xx}+\left(\bar{v}v+\dfrac{2}{3}\bar{u}u\right)\bar{v}+\dfrac{1}{3}\bar{u}^2v &=& 0. \end{array}\right. \tag{9}$$

The indicial equation is the determinant of the fourth order matrix

$$\begin{pmatrix} 2a_0b_0 & a_0^2+\dfrac{1}{3}c_0^2 & \dfrac{2}{3}a_0d_0+\dfrac{2}{3}b_0c_0 & \dfrac{2}{3}a_0c_0\\ b_0^2+\dfrac{1}{3}d_0^2 & 2a_0b_0+\dfrac{2}{3}c_0d_0 & \dfrac{2}{3}b_0d_0 & \dfrac{2}{3}b_0c_0+\dfrac{2}{3}a_0d_0\\ \dfrac{2}{3}b_0c_0+\dfrac{2}{3}a_0d_0 & \dfrac{2}{3}a_0c_0 & 2c_0d_0+\dfrac{2}{3}a_0b_0 & c_0^2+\dfrac{1}{3}a_0^2\\ \dfrac{2}{3}b_0d_0 & \dfrac{2}{3}a_0d_0+\dfrac{2}{3}b_0c_0 & d_0^2+\dfrac{1}{3}b_0^2 & 2c_0d_0+\dfrac{2}{3}a_0b_0 \end{pmatrix}+D(j), \tag{10}$$

where

$$\begin{aligned} D(j) \;:=\; \mathrm{diag}\Big(&p(j-1+\mathrm{i}\alpha)(j-2+\mathrm{i}\alpha),\ p(j-1-\mathrm{i}\alpha)(j-2-\mathrm{i}\alpha),\\ &p(j-1+\mathrm{i}\beta)(j-2+\mathrm{i}\beta),\ p(j-1-\mathrm{i}\beta)(j-2-\mathrm{i}\beta)\Big). \end{aligned} \tag{11}$$

We need to substitute the values of $(a_0, b_0, c_0, d_0, \alpha, \beta)$ into (10) in order to obtain a polynomial of degree eight in j. We construct explicitly the polynomial for determining the Fuchs indices. The final determination of the Fuchs indices will need the involvement of algebra software for the computations.

Next we study the travelling wave reduction

$$\left\{\begin{array}{rcl} u(x,t) &=& M_1(\xi)\,e^{\mathrm{i}(\varphi_1(\xi)-\omega t)},\\ v(x,t) &=& M_2(\xi)\,e^{\mathrm{i}(\varphi_2(\xi)-\omega t)},\quad \xi \;=\; x-ct, \end{array}\right. \tag{12}$$

in which $(M_1, M_2, \varphi_1, \varphi_2)$ are real-valued functions of the reduced independent variable ξ, and $c\in\mathbb{R}$, $\omega\in\mathbb{R}$. Substituting (12) into (2) and multiplying both sides of the first equation by M_1/u, the second equation by M_2/v, we obtain

$$\begin{cases} pM_1'' + \mathrm{i}\left(\delta - c + 2p\psi_1\right) M_1' + \left(\omega + b - \delta\psi_1 + c\psi_1 - p\psi_1^2 + \mathrm{i}p\psi_1'\right) M_1 \\ \qquad + \dfrac{2}{3}\, qM_1M_2^2 + \dfrac{1}{3}\, M_1M_2^2\, e^{-2\mathrm{i}\,(\varphi_1(\xi)-\varphi_2(\xi))} + qM_1^3 \;=\; 0, \\ pM_2'' + \mathrm{i},\left(-\delta - c + 2p\psi_2\right) M_2' + \left(\omega - b + \delta\psi_2 + c\psi_2 - p\psi_2^2 + \mathrm{i}p\psi_2'\right) M_2 \\ \qquad + \dfrac{2}{3}\, qM_1^2M_2 + \dfrac{1}{3}\, M_1^2M_2\, e^{2\mathrm{i}\,(\varphi_1(\xi)-\varphi_2(\xi))} + qM_2^3 \;=\; 0, \end{cases} \tag{13}$$

where $\psi_j \equiv \varphi_j' \equiv \dfrac{d\varphi_j}{d\xi}, \quad M_j' \equiv \dfrac{dM_j}{d\xi}, \quad j = 1, 2.$

By assuming

$$e^{2\mathrm{i}(\varphi_1(\xi)-\varphi_2(\xi))} \;\equiv\; 1, \tag{14}$$

and separating real and imaginary terms, we obtain the following four equations

$$pM_1'' + \frac{1}{3}\,(2q+1)\, M_1M_2^2 + \left(\omega + b + c\psi_1 - \delta\psi_1 - p\psi_1^2\right) M_1 + qM_1^3 \;=\; 0, \tag{15}$$

$$pM_1\psi_1' + 2pM_1'\psi_1 - (c-\delta)\, M_1' \;=\; 0, \tag{16}$$

$$pM_2'' + \frac{1}{3}\,(2q+1)\, M_1^2M_2 + \left(\omega - b + c\psi_2 + \delta\psi_2 - p\psi_2^2\right) M_2 + qM_2^3 \;=\; 0, \tag{17}$$

$$pM_2\psi_2' + 2pM_2'\psi_2 - (c+\delta)\, M_2' \;=\; 0. \tag{18}$$

In order to find the Laurent series, we first determine the dominant behaviour of the four fields $(M_1, M_2, \psi_1, \psi_2)$. Let us denote the dominant behaviour as

$$\begin{cases} M_1 \;\sim\; M_{10}\chi^{-1}, & \psi_1 \;\sim\; \alpha\chi^{-1}, \\ M_2 \;\sim\; M_{20}\chi^{-1}, & \psi_2 \;\sim\; \beta\chi^{-1}, \qquad \varphi_1 - \varphi_2 \;\sim\; c_{12}, \end{cases} \tag{19}$$

in which $(M_{10}, M_{20}, \alpha, \beta) \in \mathbb{R}$ are constants to be determined. Recall (15)–(18) and the indicial equation is the determinant of the fourth order matrix

$$P(j) \;=\; \begin{pmatrix} M_{20}^2 - p\alpha^2 + 3M_{10}^2 & -2p\alpha M_{10} & 2M_{10}M_{20} & 0 \\ -\dfrac{1}{3}M_{20}^2 & 2p(-M_{10}) & -\dfrac{2}{3}M_{10}M_{20} & 0 \\ 2M_{10}M_{20} & 0 & M_{10}^2 - p\alpha^2 + 3M_{20}^2 & -2p\alpha M_{20} \\ 0 & 0 & 0 & 2p(-M_{20}) \end{pmatrix}$$
$$+ \operatorname{diag}\Big(p(j-1)(j-2),\; p(j-1),\; p(j-1)(j-2),\; p(j-1)\Big). \tag{20}$$

We need to substitute the values of (M_{10}, M_{20}) into (20) and obtain a polynomial in j. Recall the equations (15)–(18) (by setting $p = \frac{1}{2}$, $q = 1$, and multiplying the equations by 2)

$$M_1'' + 2M_1M_2^2 + 2(\omega + b)\, M_1 + 2(c - \delta)\, \psi_1 M_1 - \psi_1^2 M_1 + 2M_1^3 \;=\; 0, \tag{21}$$

$$M_1\psi_1' + 2M_1'\psi_1 - 2(c - \delta)\, M_1' \;=\; 0, \tag{22}$$

$$M_2'' + 2M_1^2M_2 + 2(\omega - b)\, M_2 + 2(c + \delta)\, \psi_2 M_2 - \psi_2^2 M_2 + 2M_2^3 \;=\; 0, \tag{23}$$

$$M_2\psi_2' + 2M_2'\psi_2 - 2(c + \delta)\, M_2' \;=\; 0. \tag{24}$$

We are going to eliminate unknowns M_2 and ψ_1 in the equations and expect to find a single sixth-order ODE in M_1 only. However, this is not as easy as we would expect to eliminate M_2 and ψ_1. We have to make use of computer software like Maple which enables us to simplify a lot of tedious computations and yet the commands are quite simple. We try to illustrate the idea below for reference. In fact, this can be done by finding

$$\begin{aligned}
\text{res}_1 \;&:=\; \text{resultant}\Big(\text{resultant}\big((21)',\ (22),\ \psi_1'\big),\ (21),\ \psi_1\Big)\\
&\equiv\; \text{expr}(M_1, M_1', M_1'', M_1''', M_2, M_2')\\
\text{res}_2 \;&:=\; \text{resultant}\Big(\text{resultant}\big((23)',\ (24),\ \psi_2'\big),\ (23),\ \psi_2\Big)\\
&\equiv\; \text{expr}(M_2, M_2', M_2'', M_2''', M_1, M_1')\\
\text{res}_3 \;&:=\; \text{resultant}\Big(\text{resultant}\big(\text{resultant}\big(\text{res}_2,\ \text{res}_1'',\ M_2'''\big),\ \text{res}_1',\ M_2''\big),\ \text{res}_1,\ M_2'\Big)\\
&\equiv\; \text{expr}(M_1, M_1', M_1'', M_1''', M_1^{(4)}, M_1^{(5)}, M_2)\\
\text{res}_4 \;&=\; \text{resultant}\Big(\text{resultant}\big(\text{res}_3',\ \text{res}_1,\ M_2'\big),\ \text{res}_3,\ M_2\Big)\\
&\equiv\; \text{expr}(M_1, M_1', M_1'', M_1''', M_1^{(4)}, M_1^{(5)}, M_1^{(6)})\\
\text{ODE} \;&:=\; \text{factor}(\text{res}_4)
\end{aligned}$$

Note that the expressions for res_1 and res_2 are short and are respectively given by

$$\begin{aligned}
\text{res}_1 \;&=\; M_1^2 \Big[(8(\omega + b) + 4(c - \delta)^2)\, M_1M_1' \\
&\qquad + 12M_1^3M_1' + 3M_1'M_1'' + M_1M_1''' + 8M_1M_1'M_2^2 + 4M_1^2M_2M_2' \Big]^2,\\
\text{res}_2 \;&=\; M_2^2 \Big[(8(\omega - b) + 4(c + \delta)^2)\, M_2M_2' \\
&\qquad + 12M_2^3M_2' + 3M_2'M_2'' + M_2M_2''' + 8M_2M_2'M_1^2 + 4M_2^2M_1M_1' \Big]^2.
\end{aligned} \tag{25}$$

The dominant behaviours of M_1, M_2, ψ_1 and ψ_2 will be determined below. Recall the equations (9) with dominant terms only (by adding the two real parameters p and q)

$$\begin{cases} pu_{xx} + q\left(u^2\bar{u} + \frac{2}{3}uv\bar{v} + \frac{1}{3}v^2\bar{u}\right) &= 0, \\ p\bar{u}_{xx} + q\left(\bar{u}^2u + \frac{2}{3}\bar{u}\bar{v}v + \frac{1}{3}\bar{v}^2u\right) &= 0, \\ pv_{xx} + q\left(v^2\bar{v} + \frac{2}{3}uv\bar{u} + \frac{1}{3}u^2\bar{v}\right) &= 0, \\ p\bar{v}_{xx} + q\left(\bar{v}^2v + \frac{2}{3}\bar{u}\bar{v}u + \frac{1}{3}\bar{u}^2v\right) &= 0. \end{cases} \tag{26}$$

Substituting (12) into (26) and multiplying both sides of the first equation by M_1/u, the second equation by M_2/v, we obtain

$$\begin{cases} pM_1'' + 2\mathrm{i}p\,\psi_1 M_1' + p\left(-\psi_1^2 + \mathrm{i}\psi_1'\right)M_1 & \\ +\frac{1}{3}q\left(2 + e^{-2\mathrm{i}(\varphi_1(\xi)-\varphi_2(\xi))}\right)M_1M_2^2 + qM_1^3 &= 0, \\ pM_1'' - 2\mathrm{i}p\,\psi_1 M_1' + p\left(-\psi_1^2 - \mathrm{i}\psi_1'\right)M_1 & \\ +\frac{1}{3}q\left(2 + e^{2\mathrm{i}(\varphi_1(\xi)-\varphi_2(\xi))}\right)M_1M_2^2 + qM_1^3 &= 0, \\ pM_2'' + 2\mathrm{i}p\,\psi_2 M_2' + p\left(-\psi_2^2 + \mathrm{i}\psi_2'\right)M_2 & \\ +\frac{1}{3}q\left(2 + e^{2\mathrm{i}(\varphi_1(\xi)-\varphi_2(\xi))}\right)M_2M_1^2 + qM_2^3 &= 0, \\ pM_2'' - 2\mathrm{i}p\,\psi_2 M_2' + p\left(-\psi_2^2 - \mathrm{i}\psi_2'\right)M_2 & \\ +\frac{1}{3}q\left(2 + e^{-2\mathrm{i}(\varphi_1(\xi)-\varphi_2(\xi))}\right)M_2M_1^2 + qM_2^3 &= 0, \end{cases} \tag{27}$$

where $\psi_j \equiv \varphi_j' \equiv \frac{d\varphi_j}{d\xi}$, $M_j' \equiv \frac{dM_j}{d\xi}$, $j = 1, 2$. Taking $u = Me^{\mathrm{i}\varphi}$ gives

$$\begin{cases} u_x &= u\left[\frac{M_x}{M} + \mathrm{i}\varphi\right], \quad \text{and} \\ u_{xx} &= u\left[\left(\frac{M_x}{M} + \mathrm{i}\varphi\right)^2 + \frac{M_{xx}}{M} - \frac{M_x^2}{M^2} + \mathrm{i}\varphi_x\right]. \end{cases}$$

Substituting the above into (26) and dividing by u gives

$$0 = p\left(\frac{M_{1,xx}}{M_1} + 2\mathrm{i}\varphi_{1,x}\frac{M_{1,x}}{M_1} - \varphi_{1,x}^2 + \mathrm{i}\varphi_{1,xx}\right) + q\left(M_1^2 + \frac{2}{3}M_2^2 + \frac{1}{3}M_2^2\,e^{-2\mathrm{i}(\varphi_1-\varphi_2)}\right),$$

and other three more equations. Consider the dominant behaviours

$$M_1 \sim M_{10}\chi^{-1}, \qquad M_2 \sim M_{20}\chi^{-1}, \qquad \varphi_1 \sim \alpha\log\chi, \qquad \varphi_2 \sim \beta\log\chi,$$

where $\varphi_1 - \varphi_2 \sim a_{12}\chi^0$, we obtain the nonlinear system of four equations

$$\begin{cases} p\left(2-\alpha^2-3\mathrm{i}\alpha\right)+q\left(M_{10}^2+\frac{2}{3}M_{20}^2+\frac{1}{3}M_{20}^2\,e^{-2\mathrm{i}a_{12}}\right) &= 0,\\ p\left(2-\alpha^2+3\mathrm{i}\alpha\right)+q\left(M_{10}^2+\frac{2}{3}M_{20}^2+\frac{1}{3}M_{20}^2\,e^{2\mathrm{i}a_{12}}\right) &= 0,\\ p\left(2-\beta^2-3\mathrm{i}\beta\right)+q\left(M_{20}^2+\frac{2}{3}M_{10}^2+\frac{1}{3}M_{10}^2\,e^{2\mathrm{i}a_{12}}\right) &= 0,\\ p\left(2-\beta^2+3\mathrm{i}\beta\right)+q\left(M_{20}^2+\frac{2}{3}M_{10}^2+\frac{1}{3}M_{10}^2\,e^{-2\mathrm{i}a_{12}}\right) &= 0,\end{cases} \tag{28}$$

in which $(M_{10}, M_{20}, \alpha, \beta, e^{2\mathrm{i}a_{12}})$ are the unknowns. We conclude that there are two possible families of solutions and the dominant behaviours of the solutions are governed by (28).

4. **Conclusion**

We present a local analysis, called the Painlevé test, in order to investigate the nature and the behaviours of the solutions of the system of coupled partial differential equations. The example was taken from nonlinear birefringence. We illustrate using this example the importance of taking account of the singularity structure in the complex plane to determine the general solution of nonlinear equations. Our analytical analysis provide a fundamental framework to possibly resolve the yet unsolved problems such as the construction of closed form particular solutions or first integrals.

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Received: December 21, 2017; Published: January 9, 2018