Applied Mathematical Sciences, Vol. 12, 2018, no. 1, 47 - 55
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2018.710315

Numerical Analysis of the Torsion Problem Using Nonconforming Finite Elements

A. Garanza, M. Gawlik and F.T. Suttmeier[1]

Department of Mathematics
University of Siegen
D-57258 Siegen, Germany

Abstract

In this note we investigate a non-conforming finite element discretisation of the torsion problem. It turns out to be convenient to treat this problem by a Lagrangian formalism. The approach offers several alternatives for the numerical analysis of variational inequalities. Employing a conforming FE-method to the underlying saddle point problem yields instable behaviour of the dual variable, whereas the non-conforming Crouzeix-Raviart element produces stable results. We derive *a posteriori* error bounds for the corresponding discretisation error measured in the energy norm. Eventually numerical tests demonstrate the performance of the adaptive grid refinement based on the proposed reliable error estimate.

Mathematics Subject Classification: 65N30

Keywords: torsion problem, finite element method

1 Introduction

The work at hand continues our research, partly documented in Hage e.a. [8, 5], on variational inequalities with gradient constraints. The treatment of this subject is partly motivated by our membership in research groups supported

[1]Corresponding author

by the Deutsche Forschungsgemeinschaft. For instance, in collaboration with scientists from mechanical engineering, we provide the numerical analysis for problems arising in the field of highspeed machining.
A typical situation is sketched in Figure 1, where a miller machines a given workpiece. From mathematical point of view two problems have to be treated. First, we are faced problems of Signorini-type in the contact zone between miller and workpiece. Second, due to rotation, inelastic behaviour of parts of the miller has to be taken into account. In mathematical models this can be done by imposing restrictions on certain norms of derivatives of the displacement.
In this note, we investigate a non-conforming finite element discretisation of the torsion problem, a fundamental situation in the regime of variational inequalities with gradient constraints (c.f. Glowinski [6]).

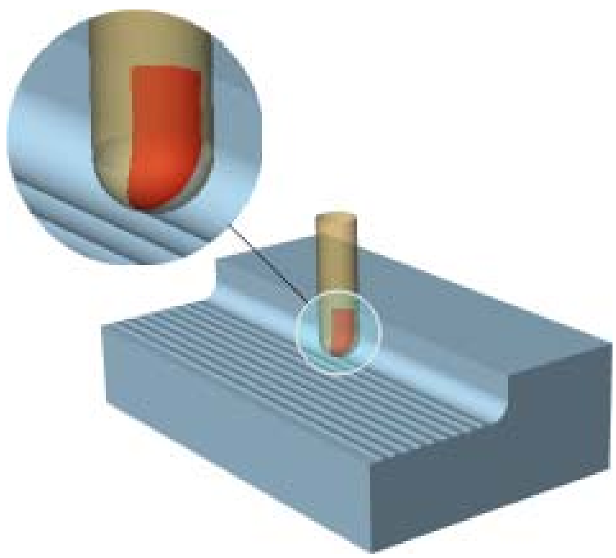

Figure 1: Sketch of model situation of a milling process.

It turns out to be convenient to treat this problem by a Lagrangian formalism. The approach offers several alternatives for the numerical analysis of variational inequalities. Employing a conforming FE-method to the underlying saddle point problem yields instable behaviour of the dual variable, whereas the non-conforming Crouzeix-Raviart element produces stable results. We derive an *a posteriori* error bounds for the corresponding discretisation error measured in the energy norm. Eventually numerical tests demonstrate the performance of the adaptive grid refinement based on the proposed reliable error estimate.

The choosen prototype problem can be formulated as follows. On a bounded domain $\Omega \subset \mathbb{R}^2$ a scalar potential u is sought, which minimizes the functional

$$J(\varphi) = \frac{1}{2}a(\varphi,\varphi) - (f,\varphi) \tag{1.1}$$

over the convex set

$$K = \{\varphi \in H_0^1 \mid |\nabla\varphi| \leq 1\}.$$

where we define

$$a(v,\varphi) = (\nabla v, \nabla\varphi),$$

and f is a given scalar function. Here, and in what follows, $W^{m,p} = W^{m,p}(\Omega)$ denotes the standard Sobolev space of L^p-functions with derivatives in $L^p(\Omega)$ up to the order m. The subscript in $W_0^{m,p}$ indicates zero boundary conditions and we write $H^m = W^{m,p}$ for $p = 2$. Furthermore, $(.,.)$ represents the L^2 inner product and $\|.\|$ the corresponding norm.
The variational setting in form of an inequality is given by

$$a(u, \varphi - u) \geq (f, \varphi - u) \qquad \forall\varphi \in K. \tag{1.2}$$

Problem (1.2) is uniquely solvable (c.f. Glowinski [6]) and, under appropriate smoothness conditions on the boundary and data, the solution is known to satisfy the regularity result $u \in W^{2,p}(\Omega)\,, 1 < p < \infty$ (see Brézis and Stampacchia [3]).
Remark: A significant number of results are obtained by rewriting the original torsion problem as an unilateral one. But following Kunze & Rodriguez [9], we note that this equivalence cannot be exploited in the case of non-constant gradient constraints.
Below, we will apply the finite element method on decompositions $\mathbb{T}_h = \{T_i \mid 1 \leq i \leq N_h\}$ of Ω consisting of N_h triangular elements T_i, satisfying the usual condition of shape regularity. The width of the mesh $\mathbb{T}_h$ is characterised in terms of a piecewise constant mesh size function $h = h(x) > 0$, where $h_T := h_{|T} = \operatorname{diam}(T)$.
The solution $u \in K$ is approximated by $u_h \in K_h \subset K$ with

$$a(u_h, \varphi - u_h) \geq (f, \varphi - u_h) \qquad \forall\varphi \in K_h \tag{1.3}$$

where

$$K_h = \{\varphi \in V_h \mid |\nabla\varphi| \leq 1\}, \quad V_h = \{\varphi \in H_0^1(\Omega) \mid \varphi \text{ linear on } T \in \mathbb{T}_h\}.$$

A first suboptimal error estimate may be found in Glowinski [6].

2 Saddle point problem

In this section we focus on so called Lagrange-settings, which can be employed to handle situations with given constraints as described above. In this way additional auxiliary variables are introduced which are determined simultaneously to the original primal solution within a so-called mixed system.
This approach turns out to offer some advantages: On the one hand direct access is established to terms hidden in the original setting, e.g. for obstacle problems additional Lagrange parameters contain information about contact forces (see e.g. Biermann e.a. [1]). On the other hand more convenient solution processes on the basis of mixed systems can be constructed as indicated now. Numerical algorithms for solving discrete problems of contact type are straightforward, since the restrictions only apply to the solutions itself. For example one can employ point projection schemes as described for the applications shown below. (c.f. Glowinski e.a. [7])
In contrast, in the present case, we have to deal with pointwise restrictions for the gradient of the solution. To handle this situation, we introduce the Lagrange functional

$$\mathcal{L}(\varphi, w) = \frac{1}{2}a(\varphi,\varphi) - (f,\varphi) + \int_\Omega w((\nabla\varphi)^2 - 1)\,dx \tag{2.4}$$

for $\varphi \in V := H_0^1(\Omega)$ and $w \in \Lambda = \{q \in L^\infty \mid q \geq 0 \text{ a.e.}\}$.
If $\mathcal{L}(.,.)$ has a saddle point $(u,\lambda) \in V \times \Lambda$ – the existence is for example proven for constant f by Brézis [2] – it can be shown that the first component u minimises the functional (1.1). In what follows, we assume f to be chosen in such a way, that $\mathcal{L}(.,.)$ has a solution in $V \times \Lambda$. Using this approach one gets the relation

$$((1+\lambda)\nabla u, \nabla\varphi) = (f,\varphi) \qquad \forall \varphi \in V\,. \tag{2.5}$$

Choosing K_h and V_h as above, we introduce $\Lambda_h \subset \Lambda$ by

$$\Lambda_h = \{w \in \Lambda \mid w \text{ constant over each } T \in \mathbb{T}_h\}\,.$$

Analogously there holds

$$((1+\lambda_h)\nabla u_h, \nabla\varphi) = (f,\varphi) \qquad \forall \varphi \in V_h\,. \tag{2.6}$$

Now, standard Uzawa-type schemes can be employed, to solve the discrete torsion problem (c.f. Glowinski [6]).
Uzawa's scheme: The discrete problems are solved by the following iterative algorithm.

1. Choose an initial iterate λ_h^0 and $\rho > 0$

2. Solve the linear problem $\int_\Omega (1+\lambda_h^\nu)\nabla u_h \nabla\varphi dx = (f,\varphi)\ \forall\varphi \in V_h$
3. Update: $\lambda_h^{\nu+1} = \max(0, \lambda_h^\nu + \rho(|\nabla u_h^\nu|^2 - 1))$ on each cell.
4. Set $\nu = \nu + 1$ and go back to 2.

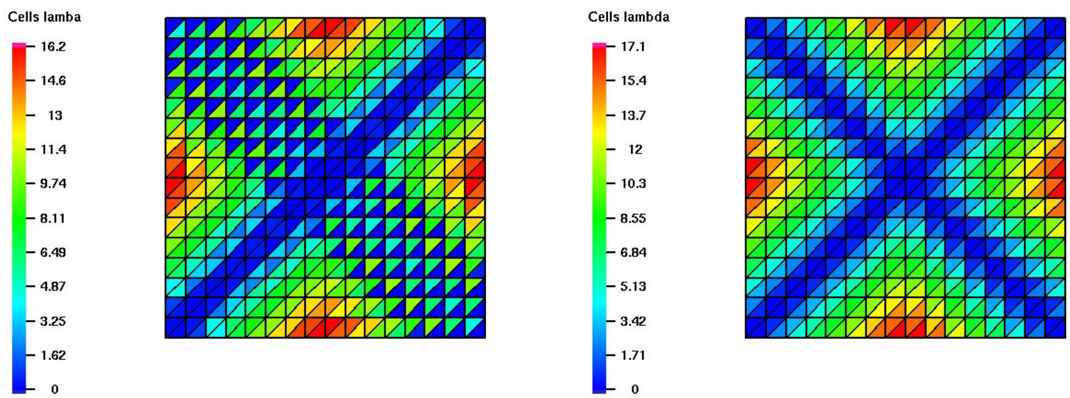

Figure 2: Conforming discretisation, unstable checker-board modes for λ_h (left). Non-conforming discretisation, stable $\bar{\lambda}_h$ (right)

Proposition 3.5 in Glowinski [6] guarantees the existence of a saddle point $(u_h, \lambda_h) \in V_h \times \Lambda_h$ of the discrete analogue of (2.4). The first component u_h is then the solution of the original problem (1.3). But in numerical tests – details described in sections below – one observes instable "checker-board"-modes for the dual variable λ_h as indicated in Figure 2 (left).
Similar to techniques initiated by Crouzeix & Raviart [4] for the treatment of Stokes-problem, we propose to utilize non-conforming approximation spaces for the discretisation of the primal variable.
More precisely we choose the discrete space

$$W_h = \{\varphi \text{ linear on } T \in \mathbb{T}_h \mid \varphi \text{ continous in the midpoints of element edges}\}\,.$$

and consider

$$((1+\bar{\lambda}_h)\nabla v_h, \nabla\varphi)_h = (f,\varphi) \qquad \forall\varphi \in W_h\,. \tag{2.7}$$

where, due to the discontinuities of functions in W_h along element edges, the bilinear form $(.,.)_h$ is defined as the sum over all triangles, i.e. we set

$$(.,.)_h := \sum_{T\in\mathbb{T}_h} (.,.)_T$$

Now replacing V_h with W_h in Uzawa's scheme described above, one obtains stable results even for the corresponding $\bar{\lambda}_h$ as depicted in Figure 2(right).

3 Error estimates

In this section we present the error analysis for the nonconforming discretisation using W_h. As indicated above, we have to work with the bilinear form $(.,.)_h$. For notational simplicity, we ommit the subscript h below.
In what follows the discretisation error is denoted by $e = u - u_h$. Below the subscription w_i denotes the application of a standard interpolation operator to a function $w \in H^1$ into V_h.
We start with

$$\begin{aligned}(\nabla e, \nabla e) &= (\nabla e, \nabla(u - u_h^c)) + (\nabla e, \nabla(u_h^c - u_h)) \\ &\le (\nabla e, \nabla(u - u_h^c)) + \frac{1}{2}\|\nabla e\|^2 + \frac{1}{2}\|\nabla(u_h^c - u_h)\|^2 \end{aligned} \tag{3.8}$$

where $u_h^c \in V_h$, $|\nabla u_h^c| \le 1$ is obtained within a postprocess as the results of

$$\|\nabla(u_h^c - u_h)\|^2 \le \|\nabla(v_h - u_h)\|^2 \qquad \forall v_h \in V_h \text{ with } |\nabla v_h| \le 1\,,$$

and focus now on the first term on the right-hand-side in (3.8). For abbreviation we set $e^c = u - u_h^c$ and exploit $|\nabla u| = 1$ in regions with $\lambda > 0$, $|\nabla u_h| \le 1$ and $|\nabla u_h^c| \le 1$. This allows us to employ the estimate

$$\lambda\left(|\nabla u||\nabla u_h^c| - |\nabla u|^2\right) \le 0\,.$$

Eventually we proceed by

$$\begin{aligned}&(\nabla e, \nabla e^c) \\ &= ((1+\lambda)\nabla u, \nabla e^c) - ((1+\lambda_h)\nabla u_h, \nabla e^c) + (\lambda_h \nabla u_h - \lambda \nabla u, \nabla e^c) \\ &= (f, e^c) - ((1+\lambda_h)\nabla u_h, \nabla e^c) + (\lambda_h \nabla u_h - \lambda \nabla u, \nabla(u - u_h^c)) \\ &\le (f, e^c) - ((1+\lambda_h)\nabla u_h, \nabla e^c) + \int_\Omega \lambda_h\left(|\nabla u||\nabla u_h| - \nabla u_h \nabla u_h^c\right) dx \\ &\le (f, e^c) - ((1+\lambda_h)\nabla u_h, \nabla e^c) + \int_\Omega \lambda_h\left(1 - \nabla u_h \nabla u_h^c\right) dx\,.\end{aligned}$$

Using (2.7) we may insert e_i^c yielding

$$\begin{aligned}(\nabla e, \nabla e^c) \le &\int_\Omega \lambda_h\left(1 - \nabla u_h \nabla u_h^c\right) dx \\ &+ (f, e^c - e_i^c) - ((1+\lambda_h)\nabla u_h, \nabla(e^c - e_i^c))\,.\end{aligned} \tag{3.9}$$

Cell-wise integration by parts in (3.9) results in the a posteriori error bound

$$(\nabla e, \nabla e^c) \le \sum_{T \in \mathbb{T}_h} \omega_T \rho_T \;+\; \kappa\,, \tag{3.10}$$

where somehow the discrete complementarity condition is measured by

$$\kappa = \int_\Omega \lambda_h \left(1 - \nabla u_h \nabla u_h^c\right) dx$$

and local *residuals* ρ_T and *weights* ω_T defined by

$$\begin{aligned}
\rho_T &:= h_T \|f + (1+\lambda_h)\Delta u_h\|_T + \tfrac{1}{2} h_T^{1/2} \|n \cdot [(1+\lambda_h)\nabla u_h]\|_{\partial T} \,, \\
\omega_T &:= \max\left\{ h_T^{-1} \|e^c - e_i^c\|_T, h_T^{-1/2} \|e^c - e_i^c\|_{\partial T} \right\},
\end{aligned}$$

where for interior interelement boundaries $[(1+\lambda_h)\partial_n u_h]$ denotes the jump of the normal derivative $(1+\lambda_h)\partial_n u_h$.
Next, one uses the interpolation estimates

$$\omega_T \le C_{i,T} \|\nabla e^c\|_T \tag{3.11}$$

$$\le C_{i,T} \left(\|\nabla e\|_T + \|\nabla(u_h - u_h^c)\|_T\right) , \tag{3.12}$$

Summarising the above one obtains

$$\|e\|_1^2 \le C \sum_{T \in \mathbb{T}_h} \rho_T^2 + c \sum_{T \in \mathbb{T}_h} \|u_h - u_h^c\|_T^2 + \kappa \tag{3.13}$$

measuring the discretisation error in the energy norm.

4 Numerical Results

The numerical results presented throughout this work are obtained by FE-implementations based on the DEAL-library [11].
The *a posteriori* mesh design is organised as follows. Let an error tolerance TOL or a maximal number of cells N_{max} be given. Starting from some initial coarse mesh the refinement criteria are chosen in terms of the *local error indicators* η_T. Then for the mesh refinement, we use the following *fixed fraction* strategy: In each refinement cycle, the elements are ordered according to the size of η_T and then a fixed portion of about 30% of the elements with largest η_T is refined resulting in approximately a doubling of the number N of cells. This process is repeated until the stopping criterion $\eta \le TOL$ is fulfilled or N_{max} is exceeded. A more detailed discussion about mesh refinement strategies can be found, e.g., in Rannacher [10].

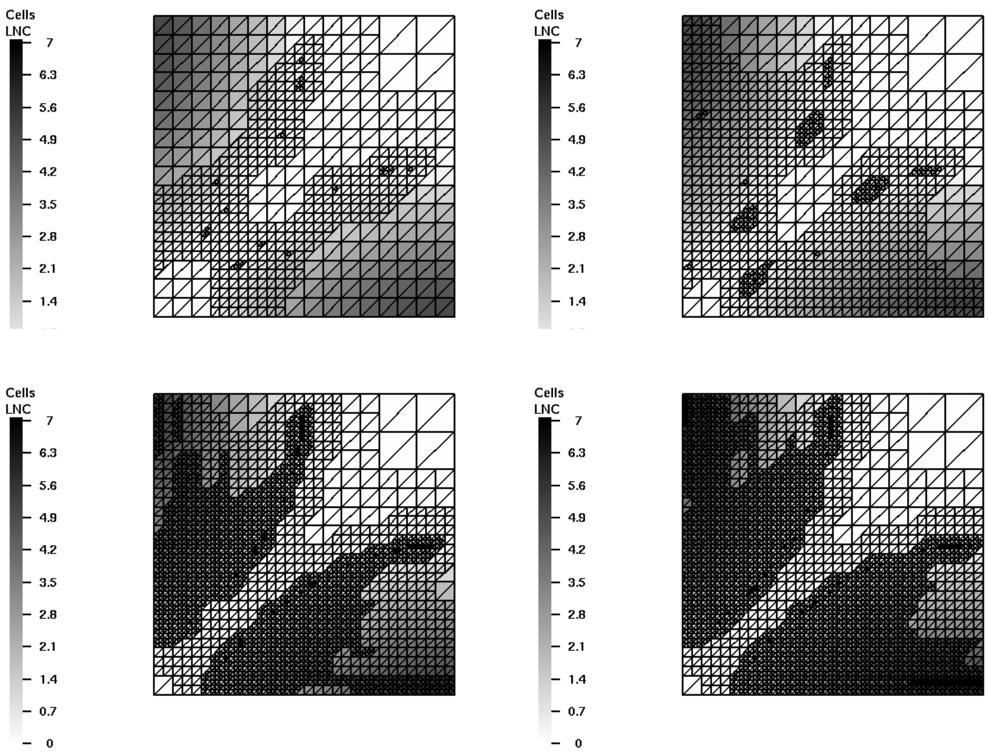

Figure 3: Sequence of adaptively refined grids, showing that especially the transition zone between areas with and without cavitation is well-resolved.

As a test we compute FE-soulutions of the torsion problem on the square $\Omega = (-1,1)^2$, where the right-hand-side is determined by

$$f(x,y) = 10 \cdot (1-x^2)(1-y^2)\,.$$

Exploiting symmetry, we reduce the computations to a quarter of the domain. In Figure 3 we show a sequence of locally refined grids produced on the basis of the *a posteriori* error estimate introduced in this paper. The background colour is determined by the Lagrange multiplyer $\bar{\lambda}_h$. In this way areas where plastification takes place are marked with “grey”. One observes that especially the transition zone between the elastic and plastic part of the solution is automatically well-resolved by our adaptive scheme.

References

[1] D. Biermann, H. Blum, I. Iovkov, N. Klein, A. Rademacher and F.T. Suttmeier, Stabilization techniques and a posteriori error estimates for the obstacle problem, *Appl. Math. Sciences*, **7** (2013), no. 127, 6329-6346. https://doi.org/10.12988/ams.2013.39504

[2] H. Brézis, Problémes Unilatéraux, (These), *J. Math. Pures Appl.*, 1972.

[3] H. Brézis and G. Stampacchia, Sur la régularité de la solution d'inéquations elliptiques, *Bull. Soc. Math. France*, **96** (1968), 153-180. https://doi.org/10.24033/bsmf.1663

[4] M. Crouzeix and P.-A. Raviart, Conforming and nonconforming finite element methods for solving the stationary stokes equations I, *Rev. Franaise d'Automat. Informat. Recherche Oprationnelle*, **7** (1973), no. R-3, 33-75. https://doi.org/10.1051/m2an/197307r300331

[5] J. Frohne, D. Hage and F.T. Suttmeier, Consistent FE-analysis of Strangs problem based on mixed formulations, *Int. Math. Forum*, **8** (2013), no. 1, 1-8. https://doi.org/10.12988/imf.2013.13001

[6] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer Series in Comp. Physics, Springer, 1984. https://doi.org/10.1007/978-3-662-12613-4

[7] R. Glowinski, J.L. Lions and R. Trémolières, *Numerical Analysis of Variational Inequalities*, North-Holland, 1981. https://doi.org/10.1016/s0168-2024(08)x7016-x

[8] D. Hage and F.T. Suttmeier, Finite elements for a certain class of variational problems with gradient constraints, *Applied Mathematical Sciences*, **6** (2012), no. 88, 4397-4404.

[9] M. Kunze and J.F. Rodrigues, An elliptic quasi-variational inequality with gradient constraint and some of its applications, *Math. Meth. Appl. Sciences*, **23** (2000), 897-908. https://doi.org/10.1002/1099-1476(20000710)23:10<897::aid-mma141>3.0.co;2-h

[10] R. Rannacher, Error control in finite element computations, Chapter in *Error Control and Adaptivity in Scientific Computing*, Kluwer-Verlag, 1999, 247-278. https://doi.org/10.1007/978-94-011-4647-0_12

[11] F.-T. Suttmeier, On concepts of PDE-software: The cellwise oriented approach in DEAL, *Int. Math. Forum*, **2** (2007), no. 1, 1-20. https://doi.org/10.12988/imf.2007.07001

Received: November 9, 2017; Published: January 10, 2018