Finite Volume Discretization of a Class of Parabolic Equations with Discontinuous and Oscillating Coefficients

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Abstract

In this paper we consider the numerical approximation of a class of parabolic equations with highly oscillating and discontinuous coefficients. We use the method of lines with a finite volumes approach to discretize this problem. This discretization leads to an ordinary differential equation (ODE). We then analyze two families of numerical schemes corresponding to explicit and implicit discretization of the obtained ODE. Numerical results comparing the approximation obtained by this method and the homogenized problem’s solution are presented. They show that this method is accurate and robust for the class of problems studied.

Mathematics Subject Classification: 65M08, 65M20

Keywords: Parabolic equation, Finite volumes method, Method of lines, Homogenization

1 Introduction

We consider a parabolic equation modeling a diffusion process in a periodic medium (for example, an heterogeneous domain obtained by mixing period-
ically two different phases, one being the matrix and the other the inclu-
sions). To fix ideas, the periodic domain is called $\Omega$, a bounded open set in $\mathbb{R}^n$ ($n = 1, 2, 3$), with a smooth boundary $\Gamma$. Its period $\varepsilon$ (a positive number which is assumed to be very small in comparison with size of the domain), and the rescaled unit periodic cell $Y = (0, 1)^n$. For a final time $T > 0$, a source term $s(x,t)$, and an initial condition $f(x)$, we consider the Cauchy problem:

$$
(P_\varepsilon) \quad \left\{ \begin{array}{l}
c \left( \frac{x}{\varepsilon} \right) \frac{\partial u_\varepsilon}{\partial t} - \text{div} \left( k \left( \frac{x}{\varepsilon} \right) \nabla u_\varepsilon \right) = s(x,t) \quad \text{in } \Omega, \\
u_\varepsilon = 0 \quad \text{on } \Gamma \times (0,T), \\
u_\varepsilon(x,0) = f(x) \quad \text{in } \Omega.
\end{array} \right.
$$

$k$ is a symmetric and uniformly positive definite matrix in $\Omega$ which has jumps discontinuities across a given interface. $c$ is a positive and uniformly bounded function which has jumps discontinuities across a given interface. The case of piecewise constant coefficients matrix $k$ and piecewise constant function $c$ is very important for the applications.

Study of the problem can be done by different methods. When $\varepsilon$ is small enough, from homogenization tools (see, e.g. [2], [4], [5], [6], [7]), the original problem $(P_\varepsilon)$ can be replaced by the homogenized problem, modeling some average quantity without the oscillations.

Whenever homogenized equations are applicable they are very useful for computational purposes. When $\varepsilon$ is not small, the original equation have to be approximated directly.

The numerical approximation of partial differential equations with highly oscillating coefficients has been a problem of interest for many years and many methods have been developped. The case of elliptic equations where the matrix $k$ has continuous coefficients is the most studied (see, e.g. [3], [15],[16] and the bibliographies therein). The case of discontinuous coefficients has been addressed recently in [14] and a long before in some works as [3], [6], [7] and [16].

In this paper we are going to addres $(P_\varepsilon)$ by the method of lines by using a finite volumes approach. For a more advanced presentation of the method of lines and finite volume methods, the reader is referred to [10] and [18] and to [8] and [9], respectively.

Our study is limited to one-dimensional problem (1-D problem), and the obtained results can be generalized in the two-dimensional problem.

The paper is organized as follows. In section 2, a description of used methods is presented. Section 3 is devoted to numerical simulations. Lastly, some concluding remarks are presented in section 4.
2 Methods

Our study will focus on the 1-D problem and we assume (without loss of generality) that \( \Omega = [0, 1] \). In this case the problem \((P_\varepsilon)\) is written simply.

\[
\begin{aligned}
Q_T := & \{ 0 < x < 1, \; 0 < t \leq T \}, \\
\epsilon^u u_\varepsilon^t(x, t) - (k^\varepsilon(x) u_\varepsilon^x) &= s(x, t) \quad \text{in} \; Q_T, \\
u_\varepsilon(0, t) = u_\varepsilon(1, t) = 0, \; t \in ]0, T], \\
u_\varepsilon(x, 0) = f(x), \; x \in ]0, 1[, \\
\end{aligned}
\]

where \( k^\varepsilon(x) = k \left( \frac{x}{\varepsilon} \right) = k(y), \; c^\varepsilon(x) = c \left( \frac{x}{\varepsilon} \right) = c(y), \) with \( y = \frac{x}{\varepsilon} \). \( k \) and \( c \) are discontinuous and periodic functions of period 1 on \([0, 1] \). In all of this paper we make the following assumptions:

(A1) \( \alpha \leq k(y) \leq \beta, \; \text{a.e. in } ]0, 1[; \; \alpha, \beta \in \mathbb{R}_+^* \),

(A2) \( s \in L^2(Q_T), \)

(A3) \( f \in L^2([0, 1]), \)

(A4) \( c^- < c(y) \leq c^+ < \infty, \; \text{a.e. in } ]0, 1[, \) with some \( c^-, c^+ \in \mathbb{R}_+^* \).

Under the assumptions \((A1) - (A4)\), it is a well-know result that there exists a unique solution \( u_\varepsilon \) of (1). Furthermore, from the homogenization theory, when \( \varepsilon \) tends to zero, \( u_\varepsilon \) tends to \( u^0(x, t) \equiv u(x, t) \) the solution of the following homogenized problem.

\[
\begin{aligned}
c^* u_1(x, t) - (k^* u_2)_x &= s(x, t) \quad \text{in} \; Q_T, \\
u(0, t) = u(1, t) = 0, \; t \in ]0, T], \\
u(x, 0) = f(x), \; x \in ]0, 1[, \\
\end{aligned}
\]

where \( k^* \) is the mean harmonic value of \( k(y) \) on \( Y = (0, 1) \) and \( c^* \) is the mean arithmetic value of \( c(y) \) on \( Y = (0, 1) \).

In order to compute a numerical approximation of \( u_\varepsilon \), we are going to begin by a discretization of the problem (1) only in space, which leads us to a ODE in time. This approach is also called method of lines. The obtained ODE will be analysed by two families of numerical schemes corresponding to explicit and implicit discretization.

2.1 Semi-discrete approximation

By semi-discretization we mean discretization only in space, not in time. We discretize \((0, 1)\) into \( N \) equal size grid cells of size \( h = 1/N \), and de-
one introduces as in [8], [9] and [14], the auxiliary unknown

\[
\text{In a finite volume method the unknowns approximate the average of the solution over a grid cell. More precisely, we let } c_i^\varepsilon u_i^\varepsilon(t) \text{ be the approximation }
\]

\[
c_i^\varepsilon u_i^\varepsilon(t) := \frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} c_i^\varepsilon u^\varepsilon(x, t) \, dx; \text{ where } u_i^\varepsilon(t) \approx \frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} u^\varepsilon(x) \, dx, \text{ and } c_i^\varepsilon := c_i(x_i).
\]

Integrating (1) over the cell \( I_i \) and dividing by \( h \) we get

\[
\frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} c_i^\varepsilon(x) u_i^\varepsilon(x, t) \, dx = \frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} (k_i^\varepsilon(x) u_i^\varepsilon(x, t)) \, dx + \frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} s(x, t) \, dx. \tag{3}
\]

Let \( T = (I_i)_{i=0, \ldots, N-1} \) be an admissible uniform mesh of \( (0, 1) \) in the sense of Definition (2.1) such that the discontinuities of \( k^\varepsilon \) and \( c^\varepsilon \) coincide with the interfaces of the mesh. First integral in the right-hand term of (3) becomes:

\[
\int_{x_{i-1/2}}^{x_{i+1/2}} (k_i^\varepsilon(x) u_i^\varepsilon(x, t)) \, dx = (k_i^\varepsilon(x) u_i^\varepsilon(x)) (x_{i+1/2}, t) - (k_i^\varepsilon(x) u_i^\varepsilon(x)) (x_{i-1/2}, t). \tag{4}
\]

In order that the scheme be conservative, the discretization of the flux \( -k_i^\varepsilon(x) u_i^\varepsilon(x) \) at the point \( x_{i+\frac{1}{2}} \) should have the same value on \( I_i \) and \( I_{i+1} \). To this purpose, one introduces as in [8], [9] and [14], the auxiliary unknown \( u_{i+\frac{1}{2}}^\varepsilon \) (approximation of \( u^\varepsilon \) at \( x_{i+\frac{1}{2}} \)). Since on \( I_i \) and \( I_{i+1} \), \( k_i^\varepsilon \) is continuous, the approximation of \( k_i^\varepsilon u_i^\varepsilon \) may be performed on each side of \( x_{i+\frac{1}{2}} \) by using the finite difference principle:

\[
F_{1+\frac{1}{2}}^\varepsilon(t) = (k_i^\varepsilon(x) u_{i+\frac{1}{2}}^\varepsilon(x)) (x_{1+\frac{1}{2}}) \approx \frac{u_{i+1}^\varepsilon(t) - u_{i}^\varepsilon(t)}{h} \text{ on } I_i,
\]

\[
F_{1-\frac{1}{2}}^\varepsilon(t) = (k_i^\varepsilon(x) u_{i-\frac{1}{2}}^\varepsilon(x)) (x_{1-\frac{1}{2}}) \approx \frac{u_{i}^\varepsilon(t) - u_{i-1}^\varepsilon(t)}{h} \text{ on } I_{i+1},
\]

where \( k_i^\varepsilon \) (respectively \( k_{i+1}^\varepsilon \)) is the value of \( k_i^\varepsilon(x) \) on \( I_i \) (respectively on \( I_{i+1} \)). Requiring the two above approximation of \( (k_i^\varepsilon u_i^\varepsilon(x)) (x_{i+\frac{1}{2}}, t) \), one deduces:

\[
u_{i+\frac{1}{2}}^\varepsilon(t) = \frac{k_{i+1}^\varepsilon u_{i+1}^\varepsilon(t) - k_i^\varepsilon u_i^\varepsilon(t)}{k_{i+1}^\varepsilon + k_i^\varepsilon}.
\]
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So we get:

\[ F_{i+\frac{1}{2}}^\varepsilon(t) = \frac{\tau_{i+\frac{1}{2}}^\varepsilon}{h} \left( u_{i+1}^\varepsilon(t) - u_i^\varepsilon(t) \right), \]  

(6)

where

\[ \tau_{i+\frac{1}{2}}^\varepsilon = \frac{2k_i^\varepsilon h_{i+1}^\varepsilon}{k_i^\varepsilon h_{i+1}^\varepsilon}. \]  

(7)

From (3), (4), (6) and (7) we get the following scheme for the inner points \( x_i, 1 \leq i \leq N - 2 \),

\[ c_i^\varepsilon \frac{du_i^\varepsilon(t)}{dt} = \frac{1}{h} \left( F_{i+\frac{1}{2}}^\varepsilon(t) - F_{i-\frac{1}{2}}^\varepsilon(t) \right) + s_i(t), \]

\[ c_i^\varepsilon \frac{d\bar{u}_i^\varepsilon(t)}{dt} = \frac{1}{h^2} \left[ \tau_{i+\frac{1}{2}}^\varepsilon \left( u_{i+1}^\varepsilon(t) - u_i^\varepsilon(t) \right) - \tau_{i-\frac{1}{2}}^\varepsilon \left( u_i^\varepsilon(t) - u_{i-1}^\varepsilon(t) \right) \right] + s_i(t). \]

So

\[ c_i^\varepsilon \frac{d\bar{u}_i^\varepsilon}{dt} = \frac{1}{h^2} \left[ \tau_{i+\frac{1}{2}}^\varepsilon \bar{u}_{i+1}^\varepsilon - \left( \tau_{i-\frac{1}{2}}^\varepsilon + \tau_{i+\frac{1}{2}}^\varepsilon \right) \bar{u}_i^\varepsilon + \tau_{i+\frac{1}{2}}^\varepsilon \bar{u}_{i+1}^\varepsilon \right] + s_i, \quad i = 1, \ldots, N-2. \]  

(8)

To complete the scheme (8) we need update formula also for the boundary points \( i = 0 \) and \( i = N - 1 \). These must be derived by taking the boundary conditions into account. We introduce the ghost cells \( I_{-1} \) and \( I_{N} \) which located just outside the domain.

The boundary conditions are used to fill these cells with values \( u_{-1}^\varepsilon \) and \( u_N^\varepsilon \), based on the values \( u_i^\varepsilon \) in the interior cells. The same update formula (8) as before can then be used also for \( i = 0 \) and \( i = N - 1 \).

Let us consider our boundary conditions \( u^\varepsilon(0, t) = 0 \) and \( u^\varepsilon(1, t) = 0 \).

Since the center of cells \( I_0 \) and \( I_{N-1} \) are not on the boundary, we take the average of two cells to approximate the value in between,

\[ 0 = u^\varepsilon(0, t) \approx \frac{u^\varepsilon(x_0, t) + u^\varepsilon(x_{-1}, t)}{2} \approx \frac{u_0^\varepsilon(t) + u_{-1}^\varepsilon(t)}{2} \]  

(9)

and

\[ 0 = u^\varepsilon(1, t) \approx \frac{u^\varepsilon(x_{N-1}, t) + u^\varepsilon(x_{N}, t)}{2} \approx \frac{u_{N-1}^\varepsilon(t) + u_N^\varepsilon(t)}{2} \]  

(10)

for \( i = 0 \) and \( i = N - 1 \), (8) becomes:

\[ c_0^\varepsilon \frac{du_0^\varepsilon}{dt} = \frac{1}{h^2} \left[ \tau_{-\frac{1}{2}}^\varepsilon u_{-1}^\varepsilon - \left( \tau_{-\frac{1}{2}}^\varepsilon + \tau_{\frac{1}{2}}^\varepsilon \right) u_0^\varepsilon + \tau_{\frac{1}{2}}^\varepsilon u_1^\varepsilon \right] + s_0, \]  

(11)

\[ c_{N-1}^\varepsilon \frac{du_{N-1}^\varepsilon}{dt} = \frac{1}{h^2} \left[ \tau_{N-\frac{1}{2}}^\varepsilon u_{N-2}^\varepsilon - \left( \tau_{N-\frac{1}{2}}^\varepsilon + \tau_{N-\frac{1}{2}}^\varepsilon \right) u_{N-1}^\varepsilon + \tau_{N-\frac{1}{2}}^\varepsilon u_N^\varepsilon \right] + s_{N-1} \]  

(12)

from (9) and (10) we get:

\[ u_{-1}^\varepsilon(t) = -u_0^\varepsilon(t) \quad \text{et} \quad u_N^\varepsilon(t) = -u_{N-1}^\varepsilon(t) \]  

(13)
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from (13) one deduces:

\[
c_0^\varepsilon \frac{d u_0^\varepsilon}{dt} = \frac{1}{h^2} \left[ -\left(2\tau_{i-\frac{1}{2}}^\varepsilon + \tau_i^\varepsilon \right) u_0^\varepsilon + \tau_{i+\frac{1}{2}}^\varepsilon u_1^\varepsilon \right] + s_0
\] (14)

\[
c_{N-1}^\varepsilon \frac{d u_{N-1}^\varepsilon}{dt} = \frac{1}{h^2} \left[ \tau_{N-\frac{3}{2}}^\varepsilon u_{N-2} - \left( \tau_{N-\frac{3}{2}}^\varepsilon + 2\tau_{N-\frac{1}{2}}^\varepsilon \right) u_{N-1}^\varepsilon \right] + s_{N-1}
\] (15)

from (8), (14) and (15) we get the ODE

\[
\frac{d u^\varepsilon(t)}{dt} = A^\varepsilon u^\varepsilon(t) + S(t),
\] (16)

where \( A^\varepsilon \) is given by:

\[
\begin{align*}
A_{00}^\varepsilon &= -\frac{1}{c_0^\varepsilon h^2} \left(2\tau_{-\frac{1}{2}}^\varepsilon + \tau_0^\varepsilon \right), \quad A_{01}^\varepsilon = \frac{\tau_0^\varepsilon}{c_0^\varepsilon h^2}, \\
A_{ii-1}^\varepsilon &= \frac{\tau_{i-\frac{1}{2}}^\varepsilon}{c_i^\varepsilon h^2}, \quad 1 \leq i \leq N-2, \\
A_{ii+1}^\varepsilon &= \frac{\tau_{i+\frac{1}{2}}^\varepsilon}{c_i^\varepsilon h^2}, \quad 1 \leq i \leq N-2, \\
A_{ii}^\varepsilon &= -\frac{1}{c_i^\varepsilon h^2} \left(\tau_{i-\frac{1}{2}}^\varepsilon + \tau_{i+\frac{1}{2}}^\varepsilon \right), \quad 1 \leq i \leq N-2, \\
A_{N-1,N-2}^\varepsilon &= \frac{\tau_{N-\frac{3}{2}}^\varepsilon}{c_{N-1}^\varepsilon h^2}, \quad A_{N-1,N-1}^\varepsilon = -\frac{1}{c_{N-1}^\varepsilon h^2} \left(\tau_{N-\frac{3}{2}}^\varepsilon + 2\tau_{N-\frac{1}{2}}^\varepsilon \right),
\end{align*}
\] (17)

and

\[
S(t) = \begin{bmatrix}
\frac{s_0(t)}{c_0^\varepsilon} \\
\frac{s_1(t)}{c_1^\varepsilon} \\
\vdots \\
\frac{s_{N-2}(t)}{c_{N-2}^\varepsilon} \\
\frac{s_{N-1}(t)}{c_{N-1}^\varepsilon}
\end{bmatrix}.
\] (18)
2.2 Fully discrete approximation

The system (16) can be solved by different methods. In this paper we examine two numerical schemes corresponding to explicit and implicit discretization. We note $\Delta t = \frac{T}{M}$ ($M > 0$), the time step.

2.2.1 Explicit scheme

The forward Euler method to solve the ODE (16) is given by:

$$U_{n+1}^\varepsilon = U_n^\varepsilon + \Delta t \left[ A^\varepsilon U_n^\varepsilon + S^n \right], \quad U_n^\varepsilon \approx U^\varepsilon(t_n), \quad S^n \approx S(t_n), \quad t_n = n\Delta t, \quad n = 0, ..., M,$$

(19)

where

$$U^\varepsilon(t) = \begin{bmatrix} u_0^\varepsilon(t) \\ u_1^\varepsilon(t) \\ \cdot \\ \cdot \\ u_{N-2}^\varepsilon(t) \\ u_{N-1}^\varepsilon(t) \end{bmatrix}$$

As for any ODE method, there arises the question of stability whose the answer is given by the following proposition.

**Proposition 2.2** Let $C = \max_{0 \leq i \leq N-1} \left( \frac{\tau_{i-\frac{1}{2}}^\varepsilon + \tau_{i+\frac{1}{2}}^\varepsilon}{c_i^\varepsilon} \right)$ and $\mathcal{T} = ( I_i )_{i=0,...,N-1}$ be an admissible uniform mesh of $(0,1)$ in the sense of Definition 2.1, such as the discontinuities of $k^\varepsilon$ and $c^\varepsilon$ coincide with the interfaces of the mesh. Then under the CFL condition:

$$CFL := \Delta t \leq \frac{h^2}{C},$$

the explicit scheme (19), where the matrix $A^\varepsilon$ is given by (17) and $S(t)$ is given by (18) is $L^\infty$ stable.

**Proof.** The proof is identical to that given in [13]. We write the scheme (19) in the following form for the inner points:

$$u_{n+1,i}^\varepsilon = \frac{\Delta t}{c_i^\varepsilon h^2} \tau_{i-\frac{1}{2}}^\varepsilon u_{n,i-1}^\varepsilon + \left[ 1 - \frac{\Delta t}{c_i^\varepsilon h^2} \left( \frac{\tau_{i-\frac{1}{2}}^\varepsilon + \tau_{i+\frac{1}{2}}^\varepsilon}{c_i^\varepsilon} \right) \right] u_{n,i}^\varepsilon + \frac{\Delta t}{c_i^\varepsilon h^2} \tau_{i+\frac{1}{2}}^\varepsilon u_{n,i+1}^\varepsilon + \frac{\Delta t}{c_i^\varepsilon} s_n^i, \quad 1 \leq i \leq N - 2.$$

(20)
The key point of the proof is that, under the CFL condition, the right-hand side of the equation (20) is a convex combination of $u_{n,i-1}^\varepsilon$, $u_{n,i}^\varepsilon$ and $u_{n,i+1}^\varepsilon$, plus the term $\frac{\Delta t}{c_i^\varepsilon} s_i^n$. From (20) on can deduce:

$$|u_{n+1,i}^\varepsilon| \leq \left| \frac{\Delta t}{c_i^\varepsilon} \frac{\varepsilon}{h^2} \left( \frac{\varepsilon}{i-\frac{1}{2}} u_{n,i-1}^\varepsilon + \frac{\varepsilon}{i+\frac{1}{2}} u_{n,i+1}^\varepsilon \right) \right| u_{n,i}^\varepsilon | + \left| \frac{\Delta t}{c_i^\varepsilon} \frac{\varepsilon}{h^2} \left( \frac{\varepsilon}{i-\frac{1}{2}} u_{n,i}^\varepsilon + \frac{\varepsilon}{i+\frac{1}{2}} u_{n,i+1}^\varepsilon \right) \right| u_{n,i}^\varepsilon | + \left| \frac{\Delta t}{c_i^\varepsilon} s_i^n \right| ,$$

$1 \leq i \leq N - 2$. Using (11), (12) and (13) we get the previous inequality for $i = 0$ and $i = N - 1$. One deduces

$$\|u_{n+1}^\varepsilon\|_\infty \leq \|u_n^\varepsilon\|_\infty + \frac{\Delta t}{c^-} \max_{0 \leq k \leq M} \|S^k\|.$$

By obvious recurrence, we get

$$\|U_{n+1}^\varepsilon\|_\infty \leq \|U_0^\varepsilon\|_\infty + \frac{\Delta t}{c^-} \max_{0 \leq k \leq M} \|S^k\| \leq \|U_0^\varepsilon\|_\infty + \frac{T}{c^-} \max_{0 \leq k \leq M} \|S^k\|, \forall n \in \{0, 1, ..., M\} .$$

Hence the $L^\infty$ stability is proven.

**Remark.** Proposition 2.2 gives a time step restriction of the type

$$\Delta t \leq Ch^2.$$

This is a severe restriction, which is seldom warranted from an accuracy point of view. It leads to unnecessarily expensive methods.

### 2.2.2 Implicit scheme

The implicit Euler scheme to solve the system (16) is given by:

$$U_{n+1}^\varepsilon = U_n^\varepsilon + \Delta t \left[ A^\varepsilon U_{n+1}^\varepsilon + S^{n+1} \right] , \quad U_n^\varepsilon \approx U^\varepsilon(t_n) , \quad S^n \approx S(t_n) , \quad t_n = n\Delta t , \quad n = 0, ..., M .$$

So

$$(I - \Delta t A^\varepsilon) U_{n+1}^\varepsilon = U_n^\varepsilon + \Delta t S^{n+1} , \quad \text{so} \quad U_{n+1}^\varepsilon = (I - \Delta t A^\varepsilon)^{-1} \left[ U_n^\varepsilon + \Delta t S^{n+1} \right] .$$

**Proposition 2.3** Let $T = (I_i)_{i=0, ..., N-1}$ be an admissible mesh of $(0,1)$ in the sense of Definition 2.1, such as the discontinuities of $k^\varepsilon$ and $c^\varepsilon$ coincide with the interfaces of the mesh. Then the implicit scheme (21), where the matrix $A^\varepsilon$ is given by (17) is $L^\infty$ stable.
Proof. We write the scheme (21) in the following form for the inner points:

\[
\left[ 1 + \frac{\Delta t}{c^2_i h^2} \left( \tau^\varepsilon_{i-\frac{1}{2}} + \tau^\varepsilon_{i+\frac{1}{2}} \right) \right] u^\varepsilon_{n+1,i} - \frac{\Delta t}{c^2_i h^2} \tau^\varepsilon_{i-\frac{1}{2}} u^\varepsilon_{n+1,i-1} - \frac{\Delta t}{c^2_i h^2} \tau^\varepsilon_{i+\frac{1}{2}} u^\varepsilon_{n+1,i+1} = u^\varepsilon_{n,i} + \frac{\Delta t}{c^2_i} s^s_{n+1} ,
\]

\[ 1 \leq i \leq N - 2. \]  

(23)

We are going to prove that:

\[
\max_{0 \leq i \leq N-1} u^\varepsilon_{n,i} \leq \max_{0 \leq i \leq N-1} u^\varepsilon_{n+1,i} + \Delta t \sum_{p=1}^{N-1} \max_{0 \leq i \leq N-1} \left( \frac{s^p_i}{c^2_i} \right) ,
\]

(24)

and

\[
\min_{0 \leq i \leq N-1} u^\varepsilon_{n,i} \geq \min_{0 \leq i \leq N-1} u^\varepsilon_{n+1,i} + \Delta t \sum_{p=1}^{N-2} \min_{0 \leq i \leq N-1} \left( \frac{s^p_i}{c^2_i} \right) .
\]

(25)

Let \( i_0 \) such as \( u^\varepsilon_{n+1,i_0} = \max_{1 \leq i \leq N-2} u^\varepsilon_{n+1,i} \).

Then \( u^\varepsilon_{n+1,i_0} \leq u^\varepsilon_{n+1,i} \geq 0 \), for \( j = i_0 \pm 1 \). From (23), we have

\[
u^\varepsilon_{n+1,i_0} = u^\varepsilon_{n+1,i_0} \leq u^\varepsilon_{n,i_0} + \frac{\Delta t}{c^2_{i_0}} s^s_{n+1} .
\]

In particular

\[
\max_{1 \leq i \leq N-2} u^\varepsilon_{n+1,i} \leq \max_{1 \leq i \leq N-2} u^\varepsilon_{n,i} + \Delta t \max_{1 \leq i \leq N-2} \left( \frac{s^s_{n+1}}{c^2_i} \right) .
\]

(26)

By recurrence and from (26) we get

\[
\max_{1 \leq i \leq N-2} u^\varepsilon_{n,i} \leq \max_{1 \leq i \leq N-2} u^\varepsilon_{n,i} + \Delta t \sum_{p=1}^{N-1} \max_{1 \leq i \leq N-2} \left( \frac{s^p_i}{c^2_i} \right) .
\]

(27)

Using (11), (12), (13) and (27), we get (24). The proof of (25) can be done by similar way. Hence the \( L^\infty \) stability of the scheme (21) is proven.

3 Numerical simulations

In this section, we are going to present numerical results obtained by using the implicit scheme (22), and compare them with the homogenized solution (Homog). More especially, we shall present numerical results obtained with following data:

\[
k(y) = \begin{cases} 
  k_1 & \text{if } 0 < y < \frac{1}{2}, \\
  k_2 & \text{if } \frac{1}{2} < y < 1,
\end{cases}
\]

\[
k^\varepsilon(x) = \begin{cases} 
  k_1 & \text{if } p\varepsilon < x < (p + \frac{1}{2}) \varepsilon \\
  k_2 & \text{if } (p + \frac{1}{2}) \varepsilon < x < (p + 1) \varepsilon, 
\end{cases}
\]

\[ k_1, k_2 \in \mathbb{R}_+^\ast. \]
$$c(y) = \begin{cases} 
c_1 & \text{if } 0 < y < \frac{1}{2}, \\
c_2 & \text{if } \frac{1}{2} < y < 1,
\end{cases}$$

$$c^\varepsilon(x) = \begin{cases} 
c_1, & \text{if } p\varepsilon < x < (p + \frac{1}{2})\varepsilon \\
c_2, & \text{if } (p + \frac{1}{2})\varepsilon < x < (p + 1)\varepsilon,
\end{cases}$$

$$\varepsilon = \frac{1}{n_p}, \text{ where } n_p \text{ is a positive integer, } 0 < p < n_p - 1.$$ 

First test problem involved simulations with $k_1 = 10^4$, $k_2 = 1$, $c_1 = 1$, $c_2 = 0.5$, the source function is $s(x, t) = 1$, and the initial condition function is $f(x) = 0$.

The below graphics of Figure 1 confirm that when $\varepsilon$ is enough small the homogenized solution gives a good approximation of the original problem’s solution, otherwise a direct approximation of the original problem is required.

![Figure 1: Test problem 1.](image)

Graphics of Figure 2 give the temperature distribution after 15, 35 and 45 seconds (graphic on left) and the temperature distribution for $\varepsilon = \frac{1}{8}$, $\varepsilon = \frac{1}{16}$ and $\varepsilon = \frac{1}{32}$.
In the below second test problem, we changed the specific heat: $c_1 = 2060$ and $c_1 = 4182$; $\Delta t = 0.01$.
All of the others data remain the same as in the first test problem. The left one graphic gives the temperature distribution after 10 and 30 seconds and the right one gives the temperature distribution after 800 and 1000 seconds and shows that the numerical solution is stable from 800 seconds.
The last series of tests was done with following data: $k_1 = 1$, $k_1 = 10^3$, $c_1 = 1$, $c_2 = 4000$, the initial condition function $f(x) = 0$, and the source function $s(x, t) = \begin{cases} 100 & \text{if } 0 < y < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < y < 1. \end{cases}$

The below left one graphic gives the temperature distribution after 25, 50, 800 and 1000 seconds and shows again the stability of the numerical solution from 800 seconds, and the right one graphic gives the temperature distribution for $\varepsilon = 0.5$, $\varepsilon = 0.25$ and $\varepsilon = 0.1$. 

Figure 3: Test problem 2.
4 Concluding remarks

The purpose of this paper was to apply the method of lines with a finite volumes approach for a class of parabolic equations with discontinuous and oscillating coefficients.

We studied and analyzed numerical explicit and implicit schemes, for solving the ODE obtained. After proving that the explicit scheme is $L^\infty$ stable under CFL condition, and that the implicit scheme is $L^\infty$ stable without any CFL condition, we presented numerical results obtained by using the implicit scheme, showing that this approach is well adapted to the discretization of this class of problems. The extension of this technique to the two-dimensional problem with uniform rectangular grids is straightforward.

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References


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