On Optional Deterministic Server Vacations in a Single Server Queue Providing Two Types of First Essential Service Followed by Two Types of Additional Optional Service

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Abstract
We study a single server queue providing one of the two types of first essential service followed by one of the two types of optional additional service. On completion of services chosen by the customer, the server has the option to take a deterministic vacation of fixed duration ‘d’ with probability $\alpha$ or else with probability $1 - \alpha$, the server may continue staying in the system. We assume that the service times $S_{11}$ and $S_{12}$ of the two types of first essential services and the service times $S_{21}$ and $S_{22}$ of the two types of additional optional services follow heterogeneous general distributions. After completion of a vacation, the server instantly joins the system and starts providing service. We determine steady state solution in terms of the steady state probability generating functions for all states of the system.

Keywords: First essential service, Additional optional service, Poisson arrivals, Steady state, Deterministic vacations
1. Introduction

Service interruptions in queueing systems are quite common. A large number of researchers have studied queues with different types of service interruptions including random system failures, periodic overhauling of the system or server vacations. In recent times, queues on vacations have assumed a greater significance. Many researchers including Madan and Choudhury [7], Gaver [1], Kielson and Servi [2], Tegham [12] , Krishnamoorthy and Srinivasan [3], Shanthikumar [10], Takagi [11], Madan & Abu-Rub [5, 6] , have studied queues assuming a variety of vacation policies including Bernoulli schedule vacations or modified Bernoulli schedule vacations, working vacations or vacations based on exhaustive vacations. Further, majority of authors who studied many real life situations, the server may take a break or a vacation of fixed length as it happens in offices or factories, where the workers get a lunch break of fixed length. Madan [4, 8] studied quite a few queueing systems with deterministic vacations or deterministic vacations or both.

Recently Madan [9] has studied a queueing system with Bernoulli schedule vacations in a single server queue with two phase first essential service followed by optional two phase additional service. In this paper, we study a single server queue in which the server provides one of the two types of first essential service followed by one of the two types of additional optional service. On completion of service or services required by a customer, the server may take a deterministic vacation of fixed length ‘d’.

2. Description of the Mathematical Model

- Customers arrive at the system in batches of variable size in accordance with a compound Poisson process. Let $\lambda c_i dt (i = 1, 2, 3, \ldots)$ be the first order probability that a batch of $i$ customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq c_i \leq 1$, $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches. The arriving batches wait in the queue in the order of their arrival. We further assume that customers within each batch are pre-ordered for the purpose of service.

- The system has a single server who provides one of the two types of first essential service to all customers one by one on a first-come, first-served basis.
A customer may choose type j first essential service with probability $\alpha_j$, $j = 1, 2$, $\alpha_1 + \alpha_2 = 1$. Let $B_{ij}(x)$ and $b_{ij}(x)$ respectively be the distribution function and the density function of the service time $S_{ij}$ of type j first essential service and let $\mu_{ij}(x)dx$ be the conditional probability of completion of type j first essential service, given that the elapsed time is $x$, so that

$$\mu_{ij}(x) = \frac{b_{ij}(x)}{1 - B_{ij}(x)}$$

and, therefore,

$$b_{ij}(x) = \mu_{ij}(x)e^{-\int_0^x \mu_{ij}(t)dt}, \quad j = 1, 2. \quad (2.1)$$

After completion of the first essential service chosen by a customer, the server provides two types of additional optional service. A customer may choose one of type j additional optional service with probability $\theta$ or else with probability $1 - \theta$, the customer may leave the system. Each customer opting for the additional optional service may choose type j additional service with probability $\beta_j$, $j = 1, 2$, $\beta_1 + \beta_2 = 1$. Let $A_{2j}(v)$ and $a_{2j}(v)$ respectively be the distribution function and the density function of time $S_{2j}$ of type j additional optional service and let $\mu_{2j}(x)dx$ be the conditional probability of completion of phase j of additional optional service, given that the elapsed time is $x$, so that

$$\mu_{2j}(x) = \frac{a_{2j}(x)}{1 - A_{2j}(x)}$$

and, therefore,

$$a_{2j}(x) = \mu_{2j}(x)e^{-\int_0^x \mu_{2j}(t)dt}, \quad j = 1, 2. \quad (2.2)$$

As soon as the services required by a customer are completed, the server may opt to take a vacation with probability $\delta$, or else with probability $1 - \delta$, he may continue staying in the system.

Whenever the server decides to take a vacation, his vacation period is of constant length ‘d’.

We assume that on completion of a vacation, the server instantly takes up a customer (at the head of the queue) for service if there is a customer waiting in the queue. However, if on returning to the system the server finds the queue empty, the server still joins the system and remains idle until a new batch of customers arrives in the system.

Various stochastic processes involved in the system are independent of each other.
3. Definitions and Notations

Let $W^{(1,j)}_n(x), \ j = 1, 2$ be the steady state probability that there are $n \geq 0$ customers in the queue excluding one customer in type $j$ first essential service with elapsed service time $x$. Accordingly, $W^{(1,j)}_n = \int_{x=0}^{\infty} W^{(1,j)}_n(x) \, dx$ denotes the steady state probability that there are $n$ customers in the queue excluding one customer in type $j$ first essential service irrespective of the value of $x$.

Next, let $W^{(2,j)}_n(x), \ j = 1, 2$ be the steady state probability that there are $n \geq 0$ customers in the queue excluding one customer in type $j$ additional optional service with elapsed service time $x$. Accordingly, $W^{(2,j)}_n = \int_{x=0}^{\infty} W^{(2,j)}_n(x) \, dx$ denotes the steady state probability that there are $n$ customers in the queue excluding one customer in type $j$ additional optional service irrespective of the value of $x$. Let $V_n$ denote the steady state probability that there are $n$ customers in the queue and the server is on vacation. Further, let $P_n = \sum_{j=1}^{2} W^{(1,j)}_n + \sum_{j=1}^{2} W^{(2,j)}_n + V_n$ denote the steady state probability that there are $n \geq 0$ customers in the queue irrespective of whether the server is providing any type of the first essential service or any type of additional optional service or is on vacation. Finally, let $Q$ be the steady state probability that there is no customer in the system and the server is idle. We further assume that $K_r$ is the probability of $r$ arrivals during the vacation period and therefore,

$$K_r = \frac{\exp(\lambda d)(\lambda d)^r}{r!}, \quad r = 0,1,2,\ldots$$

Next, we define the following Probability Generating Functions (PGFs):

In addition, we define the following probability generating functions (PGFs):

$$W^{(1,j)}(x,z) = \sum_{n=0}^{\infty} W^{(1,j)}_n(x) z^n, W^{(1,j)}(z) = \sum_{n=0}^{\infty} W^{(1,j)}_n z^n, \quad j = 1,2, \quad (3.2)$$

$$W^{(2,j)}(x,z) = \sum_{n=0}^{\infty} W^{(2,j)}_n(x) z^n, W^{(2,j)}(z) = \sum_{n=0}^{\infty} W^{(2,j)}_n z^n, \quad j = 1,2, \quad (3.3)$$
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\[ V(z) = \sum_{n=1}^{\infty} V_n z^n, \]  
\[ K(z) = \sum_{i=1}^{\infty} K_n z^n = \sum_{n=0}^{\infty} \frac{\exp(-\lambda d)(\lambda d)^n}{n!} z^n = \exp \left[ -\lambda d(1-z) \right], \quad |z| \leq 1. \]  
\[ C(z) = \sum_{i=1}^{\infty} c_i z^i, \quad |z| \leq 1. \]

Following the usual probability arguments, we obtain the following steady equations for our model:

### 4. Steady State Equations

Applying the usual probability arguments based on the underlying assumptions, we obtain the following steady state equations:

\[ \frac{d}{dx} W_{n}^{(1,1)}(x) + (\lambda + \mu_{11}(x))W_{n}^{(1,1)}(x) = \lambda \sum_{i=1}^{n} c_i W_{n-i}^{(1,1)}(x), \quad n \geq 1, \]  
\[ \frac{d}{dx} W_{0}^{(1,1)}(x) + (\lambda + \mu_{11}(x))W_{0}^{(1,1)}(x) = 0 \]  
\[ \frac{d}{dx} W_{n}^{(1,2)}(x) + (\lambda + \mu_{12}(x))W_{n}^{(1,2)}(x) = \lambda \sum_{i=1}^{n} c_i W_{n-i}^{(1,2)}(x), \quad n \geq 1, \]  
\[ \frac{d}{dx} W_{0}^{(1,2)}(x) + (\lambda + \mu_{12}(x))W_{0}^{(1,2)}(x) = 0 \]  
\[ \frac{d}{dx} W_{n}^{(2,1)}(x) + (\lambda + \mu_{21}(x))W_{n}^{(2,1)}(x) = \lambda \sum_{i=1}^{n} c_i W_{n-i}^{(2,1)}(x), \quad n \geq 1, \]  
\[ \frac{d}{dx} W_{0}^{(2,1)}(x) + (\lambda + \mu_{21}(x))W_{0}^{(2,1)}(x) = 0 \]  
\[ \frac{d}{dx} W_{n}^{(2,2)}(x) + (\lambda + \mu_{22}(x))W_{n}^{(2,2)}(x) = \lambda \sum_{i=1}^{n} c_i W_{n-i}^{(2,2)}(x), \quad n \geq 1, \]
\[
\frac{d}{dx} W_{0}^{(2,2)}(x) + (\lambda + \mu_{22}(x))W_{0}^{(2,2)}(x) = 0 ,
\] (4.8)

\[
V_{n} = \delta(1 - \theta)\sum_{j=1}^{\infty} \int_{0}^{\infty} W_{n}^{(1,j)}(x)\mu_{1j}(x)dx + \delta \sum_{j=1}^{\infty} \int_{0}^{\infty} W_{n}^{(2,j)}(x)\mu_{2j}(x)dx, \quad n \geq 0 ,
\] (4.9)

\[
\lambda Q = (1 - \delta)(1 - \theta)\int_{0}^{\infty} W_{0}^{(1,j)}(x)\mu_{1j}(x)dx + (1 - \delta)\sum_{j=1}^{\infty} \int_{0}^{\infty} W_{0}^{(2,j)}(x)\mu_{2j}(x)dx + V_{0}K_{0} .
\] (4.10)

We will solve the above equations subject to the following boundary conditions:

\[
W_{n}^{(1,1)}(0) = (1 - \theta)(1 - \delta)\alpha\sum_{j=1}^{\infty} \int_{0}^{\infty} W_{n+1}^{(1,j)}(x)\mu_{1j}(x)dx + (1 - \delta)\alpha\sum_{j=1}^{\infty} \int_{0}^{\infty} W_{n+1}^{(2,j)}(x)\mu_{2j}(x)dx,
\]

\[
\alpha_{1}(V_{0}K_{n+1} + V_{1}K_{n} + \ldots + V_{n+1}K_{0}) + \lambda \alpha_{1}c_{n+1}Q , \quad n \geq 1
\] (4.11)

\[
W_{n}^{(1,2)}(0) = (1 - \theta)(1 - \delta)\alpha\sum_{j=1}^{\infty} \int_{0}^{\infty} W_{n+1}^{(1,j)}(x)\mu_{1j}(x)dx + (1 - \delta)\alpha\sum_{j=1}^{\infty} \int_{0}^{\infty} W_{n+1}^{(2,j)}(x)\mu_{2j}(x)dx,
\]

\[
\alpha_{2}(V_{0}K_{n+1} + V_{1}K_{n} + \ldots + V_{n+1}K_{0}) + \lambda \alpha_{2}c_{n+1}Q , \quad n \geq 1
\] (4.12)

\[
W_{n}^{(2,1)}(0) = \theta\beta\sum_{j=1}^{\infty} \int_{0}^{\infty} W_{n}^{(1,j)}(x)\mu_{1j}(x)dx , \quad n \geq 0 ,
\] (4.13)

\[
W_{n}^{(2,2)}(0) = \theta\beta\sum_{j=1}^{\infty} \int_{0}^{\infty} W_{n}^{(1,j)}(x)\mu_{1j}(x)dx , \quad n \geq 0 ,
\] (4.14)

### 5. Steady State Solution in Terms of Probability Generating Functions

We multiply both sides of equation (4.1) by suitable powers of \( z \), add equation (4.2) in the result and use (3.2) and on simplifying we obtain,

\[
\frac{d}{dz} W^{(1,1)}(x, z) + (\lambda - \lambda C(z) + \mu_{11}(x))W^{(1,1)}(x, z) = 0 .
\] (5.1)
After performing similar operations on equations (4.3)-(4.9), we obtain

\[
\frac{d}{dz} W^{(1,2)}(x, z) + (\lambda - \lambda C(z) + \mu_{i_2}(x)) W^{(1,2)}(x, z) = 0, \tag{5.2}
\]

\[
\frac{d}{dz} W^{(2,1)}(x, z) + (\lambda - \lambda C(z) + \mu_{i_1}(x)) W^{(2,1)}(x, z) = 0, \tag{5.3}
\]

\[
\frac{d}{dz} W^{(2,2)}(x, z) + (\lambda - \lambda C(z) + \mu_{i_2}(x)) W^{(2,2)}(x, z) = 0, \tag{5.4}
\]

\[
V(z) = \delta(1-\theta) \sum_{j=1}^{\infty} \int_{0}^{\infty} W^{(1,j)}(x, z) \mu_{j}(x) dx + \delta \sum_{j=1}^{\infty} \int_{0}^{\infty} W^{(2,j)}(x, z) \mu_{2,j}(x) dx. \tag{5.5}
\]

Yet again, a similar operations on the boundary conditions (4.11) - (4.14), using (4.9) and simplifying yield

\[
z W^{(1,1)}(0, z) = \alpha_1 \left[ (1-\theta)(1-\delta) \sum_{j=1}^{\infty} \int_{0}^{\infty} W^{(1,j)}(x, z) \mu_{1,j}(x) dx + (1-\delta) \sum_{j=1}^{\infty} \int_{0}^{\infty} W^{(2,j)}(x, z) \mu_{2,j}(x) dx \right]
+ V(z) K(z) + \lambda (C(Z)-1) Q, \tag{5.6}
\]

\[
z W^{(1,2)}(0, z) = \alpha_2 \left[ (1-\theta)(1-\delta) \sum_{j=1}^{\infty} \int_{0}^{\infty} W^{(1,j)}(x, z) \mu_{1,j}(x) dx + (1-\delta) \sum_{j=1}^{\infty} \int_{0}^{\infty} W^{(2,j)}(x, z) \mu_{2,j}(x) dx \right]
+ V(z) K(z) + \lambda (C(Z)-1) Q, \tag{5.7}
\]

\[
W^{(2,1)}(0, z) = \theta \beta_1 \sum_{j=1}^{\infty} \int_{0}^{\infty} W^{(1,j)}(x) \mu_{1,j}(x) dx, \ n \geq 0, \tag{5.8}
\]

\[
W^{(2,2)}(0, z) = \theta \beta_2 \sum_{j=1}^{\infty} \int_{0}^{\infty} W^{(1,j)}(x) \mu_{1,j}(x) dx, \ n \geq 0, \tag{5.9}
\]

Next, we integrate equations (5.1), (5.2), (5.3) and (5.4) between the limits 0 and x and obtain

\[
W^{(1,1)}(x, z) = W^{(1,1)}(0, z) \exp \left[ - (\lambda - \lambda C(z)) x - \int_{0}^{z} \mu_{i_1}(t) dt \right], \tag{5.10}
\]

\[
W^{(1,2)}(x, z) = W^{(1,2)}(0, z) \exp \left[ - (\lambda - \lambda C(z)) x - \int_{0}^{z} \mu_{i_2}(t) dt \right], \tag{5.11}
\]

\[
W^{(2,1)}(x, z) = W^{(2,1)}(0, z) \exp \left[ - (\lambda - \lambda C(z)) x - \int_{0}^{z} \mu_{i_1}(t) dt \right], \tag{5.12}
\]

\[
W^{(2,2)}(x, z) = W^{(2,2)}(0, z) \exp \left[ - (\lambda - \lambda C(z)) x - \int_{0}^{z} \mu_{i_2}(t) dt \right], \tag{5.13}
\]
\[ W^{(1,2)}(x, z) = W^{(1,2)}(0, z) \exp \left[ -\left( \lambda - \lambda C(z) \right) x - \int_{0}^{\infty} \mu_{12}(t) dt \right], \quad (5.11) \]

\[ W^{(2,1)}(x, z) = W^{(2,1)}(0, z) \exp \left[ -\left( \lambda - \lambda C(z) \right) x - \int_{0}^{\infty} \mu_{12}(t) dt \right], \quad (5.12) \]

\[ W^{(2,2)}(x, z) = W^{(2,2)}(0, z) \exp \left[ -\left( \lambda - \lambda C(z) \right) x - \int_{0}^{\infty} \mu_{22}(t) dt \right], \quad (5.13) \]

where \( W^{(1,1)}(0, z), W^{(1,2)}(0, z), W^{(2,1)}(0, z) \) and \( W^{(2,2)}(0, z) \) have been obtained above in (5.6), (5.7), (5.8) and (5.9) respectively.

We again integrate (5.10), (5.11), (5.12) and (5.13) with respect to \( x \) and obtain

\[ W^{(1,1)}(z) = W^{(1,1)}(0, z) \left( \frac{1 - \overline{B}_{11} \left( \lambda - \lambda C(z) \right)}{\lambda - \lambda C(z)} \right), \quad (5.14) \]

\[ W^{(1,2)}(z) = W^{(1,2)}(0, z) \left( \frac{1 - \overline{B}_{12} \left( \lambda - \lambda C(z) \right)}{\lambda - \lambda C(z)} \right), \quad (5.15) \]

\[ W^{(2,1)}(z) = W^{(2,1)}(0, z) \left( \frac{1 - \overline{A}_{21} \left( \lambda - \lambda C(z) \right)}{\lambda - \lambda C(z)} \right), \quad (5.16) \]

\[ W^{(2,2)}(z) = W^{(2,2)}(0, z) \left( \frac{1 - \overline{A}_{22} \left( \lambda - \lambda C(z) \right)}{\lambda - \lambda C(z)} \right), \quad (5.17) \]

where \( \overline{B}_{1j} \left( \lambda - \lambda C(z) \right) = \int_{0}^{\infty} \exp[-\left( \lambda - \lambda C(z) \right) x] d B_{1j}(x) \) is the Laplace-Stieltjes transform of \( S_{1j} \), the service time of type \( j \) first essential service and \( \overline{A}_{2j} \left( \lambda - \lambda C(z) \right) = \int_{0}^{\infty} \exp[-\left( \lambda - \lambda C(z) \right) x] d A_{2j}(x) \) is the Laplace-Stieltjes transform of \( S_{2j} \), the service time of type \( j \) additional optional service, \( j = 1, 2 \).

Now, we multiply equations (5.10), (5.11), (5.12) and (5.13) by \( \mu_{11}(x), \mu_{12}(x), \mu_{21}(x) \) and \( \mu_{22}(x) \) respectively and integrate them with respect to \( x \) and use (2.1) and (2.2). Thus we obtain
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\[ \int_0^\infty W^{(1,1)}(x, z) \mu_{11}(x) \, dx = W^{(1,1)}(0, z) \overline{B}_{11}(\lambda - \lambda C(z)). \]  
\[ (5.18) \]

\[ \int_0^\infty W^{(1,2)}(x, z) \mu_{12}(x) \, dx = W^{(1,2)}(0, z) \overline{B}_{12}(\lambda - \lambda C(z)). \]  
\[ (5.19) \]

\[ \int_0^\infty W^{(2,1)}(x, z) \mu_{21}(x) \, dx = W^{(2,1)}(0, z) \overline{A}_{21}(\lambda - \lambda C(z)). \]  
\[ (5.20) \]

\[ \int_0^\infty W^{(2,2)}(x, z) \mu_{22}(x) \, dx = W^{(2,2)}(0, z) \overline{A}_{22}(\lambda - \lambda C(z)). \]  
\[ (5.21) \]

Next, we use (5.18), (5.19), (5.20) and (5.21) into equations (5.5), (5.6), (5.7), (5.8) and (5.9), simplify and get

\[ V(z) = \delta(1 - \theta) [W^{(1,1)}(0, z) \overline{B}_{11}(\lambda - \lambda C(z)) + W^{(1,2)}(0, z) \overline{B}_{12}(\lambda - \lambda C(z))] + \delta [W^{(2,1)}(0, z) \overline{A}_{21}(\lambda - \lambda C(z)) + W^{(2,2)}(0, z) \overline{A}_{22}(\lambda - \lambda C(z))]. \]  
\[ (5.22) \]

\[ zW^{(1,1)}(0, z) = \alpha_1 + (1 - \delta) \left[ W^{(1,1)}(0, z) \overline{B}_{11}(\lambda - \lambda C(z)) + W^{(1,2)}(0, z) \overline{B}_{12}(\lambda - \lambda C(z)) \right] + V(z)K(z) + \lambda(C(Z) - 1)Q. \]  
\[ (5.23) \]

\[ zW^{(1,2)}(0, z) = \alpha_2 + (1 - \delta) \left[ W^{(2,1)}(0, z) \overline{A}_{21}(\lambda - \lambda C(z)) + W^{(2,2)}(0, z) \overline{A}_{22}(\lambda - \lambda C(z)) \right] + V(z)K(z) + \lambda(C(Z) - 1)Q. \]  
\[ (5.24) \]

\[ W^{(2,1)}(0, z) = \theta \beta_1 [W^{(1,1)}(0, z) \overline{B}_{11}(\lambda - \lambda C(z)) + W^{(1,2)}(0, z) \overline{B}_{12}(\lambda - \lambda C(z))]. \]  
\[ (5.25) \]

\[ W^{(2,2)}(0, z) = \theta \beta_2 [W^{(1,1)}(0, z) \overline{B}_{11}(\lambda - \lambda C(z)) + W^{(1,2)}(0, z) \overline{B}_{12}(\lambda - \lambda C(z))]. \]  
\[ (5.26) \]
Next, we solve equations (5.22) to (5.26) simultaneously, utilize equations (5.14) to (5.17), simplify a lot of algebra and obtain

\[
W^{(1,1)}(z) = \frac{\alpha_1 \left( \overline{B}_{11} (\lambda - \lambda C(z)) - 1 \right) Q}{z - \left\{ \alpha_1 \overline{B}_{11} (\lambda - \lambda C(z)) + \alpha_2 \overline{B}_{12} (\lambda - \lambda C(z)) \right\} A(z)}, \quad (5.27)
\]

\[
W^{(1,2)}(z) = \frac{\alpha_2 \left( \overline{B}_{12} (\lambda - \lambda C(z)) - 1 \right) Q}{z - \left\{ \alpha_1 \overline{B}_{11} (\lambda - \lambda C(z)) + \alpha_2 \overline{B}_{12} (\lambda - \lambda C(z)) \right\} A(z)}, \quad (5.28)
\]

\[
W^{(2,1)}(z) = \frac{\theta \beta_1 \left[ \alpha_1 \overline{B}_{11} (\lambda - \lambda C(z)) + \alpha_2 \overline{B}_{12} (\lambda - \lambda C(z)) \right] \left( \overline{A}_{21} (\lambda - \lambda C(z)) - 1 \right) Q}{z - \left\{ \alpha_1 \overline{B}_{11} (\lambda - \lambda C(z)) + \alpha_2 \overline{B}_{12} (\lambda - \lambda C(z)) \right\} A(z)}, \quad (5.29)
\]

\[
W^{(2,2)}(z) = \frac{\theta \beta_2 \left[ \alpha_1 \overline{B}_{11} (\lambda - \lambda C(z)) + \alpha_2 \overline{B}_{12} (\lambda - \lambda C(z)) \right] \left( \overline{A}_{22} (\lambda - \lambda C(z)) - 1 \right) Q}{z - \left\{ \alpha_1 \overline{B}_{11} (\lambda - \lambda C(z)) + \alpha_2 \overline{B}_{12} (\lambda - \lambda C(z)) \right\} A(z)}, \quad (5.30)
\]

\[
V(z) = \frac{\left[ \alpha_1 \overline{B}_{11} (\lambda - \lambda C(z)) + \alpha_2 \overline{B}_{12} (\lambda - \lambda C(z)) \right] \left[ (1 - \theta) + \delta \theta \left[ \beta_1 \overline{A}_{21} (\lambda - \lambda C(z)) + \beta_2 \overline{A}_{22} (\lambda - \lambda C(z)) \right] \right] Q}{z - \left\{ \alpha_1 \overline{B}_{11} (\lambda - \lambda C(z)) + \alpha_2 \overline{B}_{12} (\lambda - \lambda C(z)) \right\} A(z)}, \quad (5.31)
\]

Where

\[ A(z) = \left[ (1 - \delta) + \delta K(z) \right] \left[ (1 - \theta) + \theta \beta_1 \overline{A}_{21} (\lambda - \lambda C(z)) + \beta_2 \overline{A}_{22} (\lambda - \lambda C(z)) \right] \]

Now, in order to determine the only unknown \( Q \), we proceed as follows:

\[
W^{(1,1)}(I) = \lim_{z \to 1} W^{(1,1)}(z)
\]

\[
= \lim_{z \to 1} \frac{\alpha_1 \left( \overline{B}_{11} (\lambda - \lambda C(z)) - 1 \right) Q}{z - \left\{ \alpha_1 \overline{B}_{11} (\lambda - \lambda C(z)) + \alpha_2 \overline{B}_{12} (\lambda - \lambda C(z)) \right\} A(z)}
\]

\[
= \frac{\lambda \alpha_1 E(I) E(S_{11}) Q}{1 - \left[ \lambda E(I) \{ \alpha_1 E(S_{11}) + \alpha_2 E(S_{12}) + \theta \beta_1 E(S_{21}) + \theta \beta_2 E(S_{22}) \} + \delta \lambda d \right]}, \quad (5.32)
\]
where \( E(I) \) is the average batch size, \( E(S_{1j}) \) is the average service time of type \( j \) first essential service and \( E(S_{2j}) \) is the average service time of type \( j \) additional optional service, \( j=1, 2 \).

\[
W^{(1,2)}(1) = \lim_{z \to 1} W^{(1,2)}(z) = \lambda \alpha_2 E(I) E(S_{12}) Q \\
1 - \left[ \lambda E(I) \left( \alpha_1 E(S_{11}) + \alpha_2 E(S_{12}) + \theta \beta_1 E(S_{211}) + \theta \beta_2 E(S_{22}) \right) + \delta \lambda d \right],
\]

(5.33)

\[
W^{(2,1)}(1) = \lim_{z \to 1} W^{(2,1)}(z) = \lambda \beta_1 E(I) E(S_{21}) Q \\
1 - \left[ \lambda E(I) \left( \alpha_1 E(S_{11}) + \alpha_2 E(S_{12}) + \theta \beta_1 E(S_{211}) + \theta \beta_2 E(S_{22}) \right) + \delta \lambda d \right],
\]

(5.34)

\[
W^{(2,2)}(1) = \lim_{z \to 1} W^{(2,2)}(z) = \lambda \beta_2 E(I) E(S_{22}) Q \\
1 - \left[ \lambda E(I) \left( \alpha_1 E(S_{11}) + \alpha_2 E(S_{12}) + \theta \beta_1 E(S_{211}) + \theta \beta_2 E(S_{22}) \right) + \delta \lambda d \right],
\]

(5.35)

\[
V(1) = \lim_{z \to 1} V(z) = \lambda E(I) Q \\
1 - \left[ \lambda E(I) \left( \alpha_1 E(S_{11}) + \alpha_2 E(S_{12}) + \theta \beta_1 E(S_{211}) + \theta \beta_2 E(S_{22}) \right) + \delta \lambda d \right],
\]

(5.36)
Next, we use the results found in (5.32) to (5.36) in the normalizing condition:

\[ Q + W^{(1,1)}(I) + W^{(1,2)}(I) + W^{(2,1)}(I) + V(I) = 1. \]  

(5.37)

On simplifying, (5.37) yields

\[ Q = \frac{1 - [\lambda E(I) \{ \alpha_1 E(S_{11}) + \alpha_1 E(S_{12}) + \theta \beta_1 E(S_{21}) + \theta \beta_2 E(S_{22}) \} + \delta \lambda d]}{1 + \delta \lambda E(I) - \delta \lambda d}. \]  

(5.38)

The result (4.42) gives the probability that the server is idle and the stability condition that emerges from this equation is given by

\[ [\lambda E(I) \{ \alpha_1 E(S_{11}) + \alpha_1 E(S_{12}) + \theta \beta_1 E(S_{21}) + \theta \beta_2 E(S_{22}) \} + \delta \lambda d] < 1. \]  

(5.39)

Now, we define \( \rho \), the utilization factor of the system as the proportion of time the server is providing any kind of service and using results (4.38), (4.39) and (4.40) and simplifying, we get

\[ \rho = P^e(I) + P_1(I) + P_2(I) = \frac{\lambda E(I) \{ \alpha_1 E(S_{11}) + \alpha_1 E(S_{12}) + \theta \beta_1 E(S_{21}) + \theta \beta_2 E(S_{22}) \}}{1 + \delta \lambda E(I) - \delta \lambda d}. \]  

(5.40)

References


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