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Option Pricing Model with Stepped Payoff

Hernán Garzón G.

Department of Mathematics
Universidad Nacional de Colombia
Bogotá, Colombia

Abstract

This paper shows how to obtain a valuation for an exotic call option using the well-known Black-Scholes model. The financial derivative that we will consider here is constructed from a European call option or (vanilla call option), where we have changed the final payment $S - K$ by one defined from a stepped function. Once the theoretical price of the contract has been determined, some specific cases are analyzed.

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1 Introduction

In May 1973 the economists Fisher Black and Myron Scholes published an paper in the Journal of Political Economy entitled "The Pricing of Options and Corporate Liabilities" [2], which is considered by many as the greatest theoretical support of the great industry of the financial markets, see [3, 6, 10]. Is defined as an exotic option any contract, which differs in some term of the European option or vanilla option. In recent years, a great variety of these instruments have been created and are currently traded in the different stock markets around the world. To evaluate these instruments, many methods have been proposed, see [4, 7, 11]. Some of them use numerical methods [1, 5], in others we can find written solutions in terms of non-elementary functions.

In this paper, a contract with a stepped payoff that allows a fairly flexible level of risk exposure, which depends on the risk profiles of the two contract parties, the buyer and the writer is shown.
This paper is divided into 5 sections. In the second section an option is modeled with a constant payoff; in the third section the main model is proposed, which is the objective of the paper; in the fourth section some examples are presented and finally in the last section some comments of the study are made.

2 Call option with a constant payoff

Let $V(S,t)$ be a non-negative function, that represents the price of a call option on an asset where, $S > 0$ is the price of the underlying asset, $0 \leq t \leq T$ is the time that elapses from the moment the contract is issued, $K > 0$ is the strike price, $T > 0$ is the Time to Expiration and $f(s)$ the payoff received by the contract holder if this is exercised at time T. As we can see, in this way we have defined an exotic option based on a european call option, where the final payment is expressed generically with a function of S. Suppose also that here also all the conditions that are required in the Black-Scholes model [2] are fulfilled. So the following model allows us to find the fire price of the call option.

$$\left.\begin{array}{l} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \\ V(S,T) = f(S) \end{array}\right\} \tag{2.1}$$

Where r and σ positive constants, correspond respectively to risk-free rate and volatility of the underlying asset. Note that if in (2.1) we do $f(S) = S - K$, we have the classic model for a vanilla option developed by Black and Scholes in [2]. For this reason we have called this contract "call option", although in the strict sense it is not. Now let's see what is the fair pricing of a binary option, see [6].

Theorem 2.1. *Be $V(S,t)$ the price of the call option defined in (2.1) with payoff $f(S)$ given by,*

$$f(S) = V(S,T) = \begin{cases} 0 & if \quad S < K \\ L & if \quad S \geq K \end{cases}$$

where L is a positive real number, then $V(S,t)$ can be expressed as,

$$V(S,t) = L\ e^{-r(T-t)}\ \Phi\left(\frac{ln\left(\frac{S}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right),$$

Where Φ is the normal standard distribution.

Proof. To start in the equation (2.1) let's do:

$$w(x,\tau)=\frac{1}{K}V(S,t),\quad x=ln\left(\frac{S}{K}\right),\quad \tau=\frac{1}{2}\sigma^2(T-t)\ \ and\ \ \lambda=\frac{2r}{\sigma^2} \tag{2.2}$$

Now, since $\tau(T)=0$ the boundary condition $V(S,T)$ becomes the initial condition,

$$w(x,0)=\begin{cases}0 & if \quad x<0\\ \frac{L}{K} & if \quad x\geq 0\end{cases} \tag{2.3}$$

And consequently problem becomes,

$$\left.\begin{array}{l}\frac{\partial w}{\partial \tau}-\frac{\partial^2 w}{\partial x^2}-(\lambda-1)\frac{\partial w}{\partial x}+\lambda w=0,\quad x\in\mathbb{R},\ \ \tau\in\left[0,T\frac{\sigma^2}{2}\right]\\ \\ w(x,0)=\begin{cases}0 & if \quad x<0\\ \frac{L}{K} & if \quad x\geq 0\end{cases}\end{array}\right\} \tag{2.4}$$

Now, let's make the following substitution,

$$w(x,\tau)=e^{-\frac{\lambda-1}{2}x-\frac{(\lambda+1)^2}{4}\tau}u(x,\tau), \tag{2.5}$$

so, the initial condition of (2.4) becomes:

$$u(x,0)=u_0(x)=\frac{L}{K}e^{\left(\frac{\lambda-1}{2}\right)x},\ if\ x\geq 0,\ and\ 0,\ if\ x<0$$

and most importantly, problem (2.4) is transformed into the widely known model of heat equation.

$$\left.\begin{array}{l}\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2},\ with\ x\in\mathbb{R},\ and\ \tau\in\left[0,T\frac{\sigma^2}{2}\right]\\ \\ u(x,0)=u_0(x)=\begin{cases}0 & if \quad x<0\\ \frac{L}{K}e^{\left(\frac{\lambda-1}{2}\right)x} & if \quad x\geq 0\end{cases}\end{array}\right\} \tag{2.6}$$

As we can see in [9], a solution to problem (2.6) is given by:

$$u(x,\tau)=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}u_0(y)e^{-\frac{(x-y)^2}{4\tau}}dy \tag{2.7}$$

Now, to express the solution given in (2.7) in a simple way, let's fix x and make the following substitution, $y=v\sqrt{2\tau}+x$, therefore $dy=\sqrt{2\tau}\ dv$. In addition, considering that u_0 is canceled when his argument is negative, we have to $x+v\sqrt{2\tau}>0$ and consequently of (2.7) it follows that,

$$u(x,\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty}u_0(v\sqrt{2\tau}+x)e^{-\frac{1}{2}v^2}dv \tag{2.8}$$

In this way we have the following,

$$u(x,\tau) = \frac{L}{K\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}[-(\lambda-1)(v\sqrt{2\tau}+x)+v^2]} dv \tag{2.9}$$

developing the expression (2.9) and completing square,

$$u(x,\tau) = \frac{L}{K\sqrt{2\pi}}\ e^{\frac{1}{2}(\lambda-1)x+\frac{1}{4}(\lambda-1)^2\tau} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}[v-\frac{\sqrt{2}}{2}(\lambda-1)\sqrt{\tau}]^2} dv \tag{2.10}$$

Making the substitution $\rho = v - \frac{1}{2}(\lambda-1)\sqrt{2\tau}$, it has to be $d\rho = dv$. Now, if we do $d = \frac{x}{\sqrt{2\tau}} - \left(-\frac{1}{2}(\lambda-1)\sqrt{2\tau}\right)$ we have,

$$u(x,\tau) = \frac{L}{K\sqrt{2\pi}}\ e^{\frac{1}{2}(\lambda-1)x+\frac{1}{4}(\lambda-1)^2\tau} \int_{-d}^{\infty} e^{-\frac{1}{2}\rho^2} d\rho \tag{2.11}$$

and therefore,

$$u(x,\tau) = \frac{L}{K}\ e^{\frac{1}{2}(\lambda-1)x+\frac{1}{4}(\lambda-1)^2\tau}\ \Phi(d) \tag{2.12}$$

Where Φ see [8], is the normal standard distribution. Returning the substitution (2.5) we have,

$$w(x,\tau) = e^{-\frac{\lambda-1}{2}x-\frac{(\lambda+1)^2}{4}\tau}\left[\frac{L}{K}e^{\frac{1}{2}(\lambda-1)x+\frac{1}{4}(\lambda-1)^2\tau}\Phi(d)\right] = \frac{L}{K}e^{-\lambda\tau}\Phi(d), \tag{2.13}$$

and from substitutions (2.2) you have to,

$$V(S,t) = L\ e^{-\lambda\tau}\ \Phi(d) = L\ e^{-\frac{2r}{\sigma^2}\left(\frac{1}{2}\sigma^2(T-t)\right)}\ \Phi(d) \tag{2.14}$$

and finally we have,

$$V(S,t) = L\ e^{-r(T-t)}\ \Phi(d) \tag{2.15}$$

Where,

$$d = \frac{ln\left(\frac{S}{K}\right) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{2.16}$$

3 Payoff with a finite number of steps

If in Problem posed by the theorem (2.1), we consider the parameters K and L as variables and rename the variable $V(S,t)$ as $C(S,t,K,L)$, we can rewrite the problem as,

$$\left.\begin{array}{l} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \\ \\ C(S,T,K,L) = \begin{cases} 0 & if \quad S < K \\ L & if \quad S \geq K \end{cases} \end{array}\right\} \tag{3.1}$$

So, for (2.15) and (2.16) we have that the solution of (3.1) is given by

$$C(S,t,K,L) = L\ e^{-r(T-t)}\ \Phi\left(\frac{ln\left(\frac{S}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \tag{3.2}$$

Theorem 3.1. *If $V(S,t)$ is the price of the call option defined in (2.1) where $f(S)$ is given by,*

$$f(S) = V(S,T) = \begin{cases} 0 & if \quad S < K_1 \\ L_1 & if \quad K_1 \leq S < K_2 \\ L_2 & if \quad S \geq K_2 \end{cases}$$

where $0 < K_1 < K_2$, and L_1, L_2 are two arbitrary real numbers, then $V(S,t) = C(S,t,K_1,L_1) + C(S,t,K_2,L_2 - L_1)$.

Proof. Consider a portfolio P consisting of long positions in call options with constant payoff as those described in the theorem (2.1) with the same underlying asset and the same expiration time T. Suppose that the portfolio is formed by a unit of each of two options $P = \{O_1, O_2\}$, where O_1 is an option with strike K_1 and payoff L_1 and O_2 is an option with strike K_2 and payoff $L_2 - L_1$. Consider another portfolio $Q = \{O_3\}$, formed by a unit of the option whose payoff is given in the theorem (3.1). The payoff of portfolio P is the sum of f_1 and f_2.

$$f_1(S) = \begin{cases} 0 & if \quad S < K_1 \\ L_1 & if \quad S \geq K_1 \end{cases} \qquad f_2(S) = \begin{cases} 0 & if \quad S < K_2 \\ L_2 - L_1 & if \quad S \geq K_2 \end{cases}$$

And the payoff of portfolio Q is,

$$f_3(S) = \begin{cases} 0 & if \quad S < K_1 \\ L_1 & if \quad K_1 \leq S < K_2 \\ L_2 & if \quad S \geq K_2 \end{cases}$$

So we can see that $f_3(S) = f_1(S) + f_2(S)$ this is, the two portfolios have the same payoff, and consequently by the principle of non-arbitrage, see [6] the portfolios P and Q have the same value. So finally we have to

$$V(S,t) = C(S,t,K_1,L_1) + C(S,t,K_2,L_2 - L_1).$$

That which was to be demonstrated.

Let's consider now a finite partition of $\mathbb{K} = \{k_1, k_2, ..., k_m\}$ of interval (S, ∞) and a set $\mathbb{L} = \{l_1, l_2, ..., l_m\}$ of real numbers, and suppose that the function $f(S) = f_m(S)$ is given by,

$$f_m(S) = \begin{cases} 0 & if \quad S < k_1 \\ l_1 & if \quad k_1 \leq S < k_2 \\ l_2 & if \quad k_2 \leq S < k_3 \\ . & \\ . & \\ . & \\ l_m & if \quad S \geq k_m \end{cases} \tag{3.3}$$

And again consider the call option given in equation (2.1), where the boundary condition is given by (3.3). If we denominate as $V(S,t)$ the price of the option thus defined then, doing induction with the result obtained in Theorem (3.1), we have

$$V(S,t) = C(S,t,k_1,l_1) + \sum_{n=2}^{m} C(S,t,k_n,l_n - l_{n-1}).$$

4 Example of application

Here are two examples. In both cases, be $T = 1$, $r = 0.03$ and $s = 0.15$. Let's name $C_1(S,t)$ the price of a call option with strikes at $k_1 = 10$, $k_2 = 12$ and $k_3 = 13.5$, with staggered payments $l_1 = 1$, $l_2 = 1$ and $l_3 = 3$. In the second example, let's name $C_2(S,t)$ the price of a call option with strikes at $k_1 = 10$, $k_2 = 11$ and $k_3 = 12$, with staggered payments $l_1 = 1$, $l_2 = -2$ and $l_3 = 3$.

The following figure shows the graphs of $V_1(S,t)$ and $V_2(S,t)$ call options with three-step payoff.

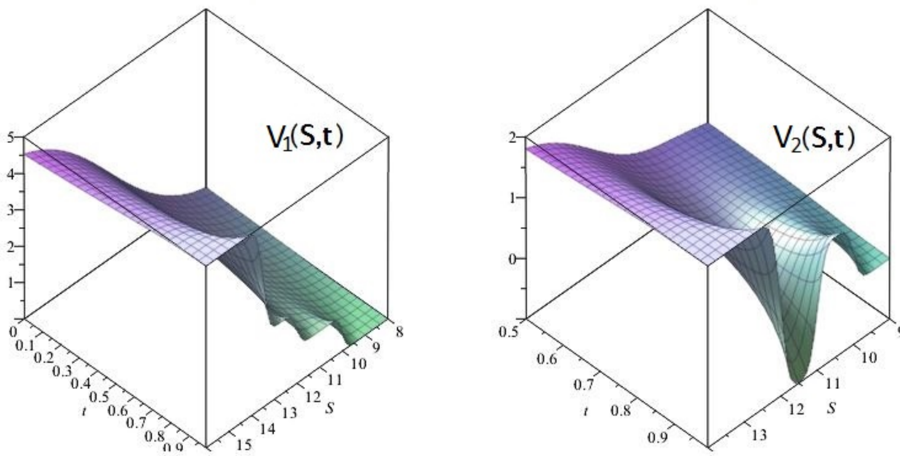

Figure 1: V_1 *and* V_2 *call options with three-step payoff*

For V_1 we can observe as the times are approaching in which the payments are presented, the price of the option accelerates quickly and assumes said payment. V_2 the second case is still more strange. Since the second payment is negative ($l_2 = -2$), the price of the option temporarily becomes negative. In this case we can see that if the payments are not close to the $S - K$ value, the contract should not be called a call option, it would be more convenient to call it a financial derivative, more appropriate instrument for speculation, which would allow to assume short positions.

5 Conclusions

The model considered here shows us that financial instruments allow great flexibility. Since the values that make up the set $\mathbb{L}$ can be negative or positive, allows the parties to take long or short positions according to the profiles managed by the investors. If $\|\{k_1, k_2, ..., k_m\}\| \to 0$, what it implies as a necessary condition that $m \to \infty$, then we can observe that practically any condition of boundary $f(s)$ can be approximated by a stepped function like that presented in equation (3.3).

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