A Solution of the Nonlinear Exponential Diffusion Equation Using Lattice-Boltzmann and Tanh Solitary Wave Method

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Abstract

In this paper, we solve the exponential nonlinear diffusion equation using lattice-Boltzmann and a \( d1q3 \) lattice velocity scheme. Also, we find several families of solutions using a Tanh solitary wave method.

Keywords: Exponential Diffusion Equation, lattice-Boltzmann, Tanh method

1 Introduction

Anomalous diffusion are processes characterized with particle mean squared displacement proportional to temporal power laws \([1]-[3]\). This behavior can be found in systems such anisotropic media \([4]\), fluid flows in human body \([5]\), and studies in financial risks\([6]\). One of those nonlinear behavior is characterized by the exponential diffusion equation \([4]\) and \([7]-[8]\). We use the lattice-Boltzmann, a very versatile technique in its application, and put it into practice from fluid dynamics \([9]\) till dynamic systems \([11]\), and also solitary wave solutions \([12]\) in order to solve the exponential diffusion equation.

This work is organized like follows. Section 2, presents the equilibrium distribution function, based on \( d1q3 \) scheme, that holds the EDEq. Also, section (3), presents the moments of the distribution. In addition, section (4), we get
two versions of the EDEq. Moreover, in section (5), the equilibrium distribution functions for both of the EDEq’s, are given. Furthermore, in section (6) and (7), we solve the EDEq using the Tanh method. Finally, section (8), gives results and conclusions.

2 The lattice Boltzmann model

The lattice Boltzmann equation is given by [9]:

$$f_j(x + v_j \epsilon, t + \epsilon) - f_j(x, t) = -\frac{1}{\tau} \left( f_j(x, t) - f_j^{eq}(x, t) \right)$$  \hspace{1cm} (1)

Here $f_j^{eq}(x, t)$ is the distribution function, $v_j$ is velocity $x$ position and $t$ time, and $\Delta t$ is the time step. Also, we have used the BG.K. approximation with $\tau$ a nondimensional relaxation time [10]. We expand in a Taylor series, the left-hand side of eq. (1), up to third order is:

$$f_j(x + v_j \epsilon, t + \epsilon) - f_j(x, t) = \epsilon \left( \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x} \right) f_j$$

$$+ \frac{\epsilon^2}{2} \left( \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x} \right)^2 f_j + \frac{\epsilon^3}{6} \left( \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x} \right)^3 f_j + O(\epsilon^4)$$  \hspace{1cm} (2)

Doing a perturbative expansion of the distribution function in powers of $\epsilon$ [9], we get:

$$f_j = f_j^{(0)} + \epsilon f_j^{(1)} + \epsilon^2 f_j^{(2)} + \epsilon^3 f_j^{(3)}$$  \hspace{1cm} (3)

And assuming:

$$f_j^{(0)} = f_j^{(eq)}$$  \hspace{1cm} (4)

The temporal scales are defined as:

$$t_0 = t ; t_1 = \epsilon t ; t_2 = \epsilon^2 t ; t_3 = \epsilon^3 t$$  \hspace{1cm} (5)

And the perturbative expansion in parameter $\epsilon$ of the temporal derivative operator

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \epsilon^3 \frac{\partial}{\partial t_3}$$  \hspace{1cm} (6)

Replacing eqs. (3)-(6) in eq. (2), we get at first order in $\epsilon$

$$\frac{\partial f_j^0}{\partial t_0} + v_j \frac{\partial}{\partial x} f_j^0 = -\frac{1}{\tau} f_j^1$$  \hspace{1cm} (7)
At second order in $\epsilon$

$$\frac{\partial f_j^0}{\partial t_1} - \tau (1 - \frac{1}{\tau}) \left( \frac{\partial}{\partial t_0} + v_j \frac{\partial}{\partial x} \right)^2 f_j = -\frac{1}{\tau} f_j^2$$

(8)

And the third order in $\epsilon$,

$$\frac{\partial f_j^0}{\partial t_2} + (1 - 2\tau) \left( \frac{\partial}{\partial t_0} + v_j \frac{\partial}{\partial x} \right) \frac{\partial f_j^0}{\partial t_1} + \left( \tau^2 - \tau + \frac{1}{6} \right) \left( \frac{\partial}{\partial t_0} + v_j \frac{\partial}{\partial x} \right)^3 f_j^0 = -\frac{1}{\tau} f_j^3$$

(9)

3 The moments of the distribution

The moments of the distribution are:

$$\sum_j f_j^{(0)} = \phi = \sum_j f_j^{(eq)}$$

(10)

$$\sum_j v_j f_j^{(0)} = 0$$

(11)

$$\sum_j v_{j,m} v_{j,n} f_j^{(0)} = \lambda \exp(\phi) \delta_{m,n}; \quad \sum_j v_{j,m} v_{j,n} f_j^{(0)} = \lambda \exp(-\phi) \delta_{m,n};$$

(12)

$$\sum_j f_j^{(k)} = 0, \text{ if } k \geq 1$$

(13)

Where $\delta_{mn}$ is the Kronecker’s delta.

4 The construction of the exponential diffusion equation

Summing on $j$ in eq. (7)

$$\frac{\partial \sum_j f_j^0}{\partial t_0} + \frac{\partial}{\partial x} \sum_j v_j f_j^0 = -\frac{1}{\tau} \sum_j f_j^1$$

(14)

We consider an irrotational fluid, and using eqs. (10)-(13), in eq. (14) is:

$$\frac{\partial \sum_j f_j^0}{\partial t_0} = \frac{\partial \phi}{\partial t_0} = 0$$

(15)
And now summing on $j$ in eq. (8) and multiplying by $\epsilon$

$$\frac{\partial}{\partial t_1} \sum_j f_j^0 - \epsilon \tau (1 - \frac{1}{\tau}) \sum_j \left( \frac{\partial}{\partial t_0} + v_j \frac{\partial}{\partial x} \right)^2 f_j = -\frac{1}{\tau} \sum_j f_j^{(2)}$$

(16)

And using eqs. (10-13), we have:

$$\epsilon \frac{\partial \phi}{\partial t_1} - \epsilon \tau (1 - \frac{1}{\tau}) \lambda \frac{\partial^2 \exp (\phi)}{\partial x^2} = 0, \quad \epsilon \frac{\partial \phi}{\partial t_1} - \epsilon \tau (1 - \frac{1}{\tau}) \lambda \frac{\partial^2 \exp (-\phi)}{\partial x^2} = 0$$

(17)

Summing eqs. (15) and (17), and defining $D = \epsilon \lambda (\tau - \frac{1}{2})$, we obtain:

$$\left( \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} \right) \phi = D \lambda \frac{\partial^2 \exp (\phi)}{\partial x^2}, \quad \left( \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} \right) \phi = D \lambda \frac{\partial^2 \exp (-\phi)}{\partial x^2}$$

(18)

And using eq. (6), [5] and [7], we obtain:

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \exp (\phi)}{\partial x^2}; \quad \frac{\partial \phi}{\partial t} = D \frac{\partial^2 \exp (-\phi)}{\partial x^2}$$

(19)

5 The equilibrium function in the d1q3 velocity scheme.

The d1q3 velocity one-dimensional scheme defining $e_\alpha = c\{0, 1, -1\}$ [9], as:

$$f^{(eq)}_{i,\alpha,\beta} = \begin{cases} \frac{-\lambda}{2\tau} \exp (\phi) + \phi & \rightarrow i = 0 \\ \frac{\lambda}{2\tau} \exp (\phi) & \rightarrow i = 1 \\ \frac{\lambda}{2\tau} \exp (\phi) & \rightarrow i = 2 \end{cases}$$

(20)
\[ f_{i,\alpha,\beta}^{(eq)} = \begin{cases} \frac{\Lambda}{c^2} \exp(-\phi) + \phi & \to i = 0 \\ \frac{\Lambda}{c^2} \exp(-\phi) & \to i = 1 \\ \frac{\Lambda}{c^2} \exp(-\phi) & \to i = 2 \end{cases} \] (21)

6 The Nonlinear Exponential Diffusion Equation

The exponential diffusion equation is:

\[ \nabla e^\phi = \frac{\partial \phi}{\partial t} \] (22)

in one dimension

\[ \frac{\partial^2 \phi}{\partial x^2} + \left( \frac{\partial \phi}{\partial x} \right)^2 = -\frac{\partial e^{-\phi}}{\partial t} \] (23)

Using the coordinate transformation

\[ \zeta = x - ct \] (24)

\[ \frac{d}{dx} = \frac{d}{d\zeta}; \quad \frac{d^2}{dx^2} = \frac{d^2}{d\zeta^2}; \quad \frac{d}{dt} = -c \frac{d}{d\zeta} \] (25)

Then

\[ \frac{\partial^2 \phi}{\partial \zeta^2} + \left( \frac{\partial \phi}{\partial \zeta} \right)^2 = -c \frac{\partial e^{-\phi}}{\partial \zeta} \] (26)

Defining the transformation

\[ v = e^{-\phi} \] (27)

Then

\[ \frac{d\phi}{d\zeta} = \frac{d\phi}{dv} \frac{dv}{d\zeta} = -\frac{1}{v} \frac{dv}{d\zeta}, \quad \frac{d^2\phi}{d\zeta^2} = \frac{1}{v^2} \left( \frac{dv}{d\zeta} \right)^2 - \frac{1}{v} \frac{d^2v}{d\zeta^2} \] (28)

And replacing in eq. (26)

\[ v \frac{d^2v}{d\zeta^2} - 2 \left( \frac{dv}{d\zeta} \right)^2 - cv \frac{dv}{d\zeta} = 0 \] (29)

Now, we introduce a new independent variable [12]:
\[ Y = \tanh(\zeta) \]  

Then, the derivatives of \( \xi \), are:

\[
\frac{d}{d\zeta} = (1 - Y^2) \frac{d}{dY}, \quad \frac{d^2}{d\zeta^2} = -2Y(1 - Y^2) \frac{d}{dY} + (1 - Y^2)^2 \frac{d^2}{dY^2}
\]  

The solutions are postulated as:

\[
v(\zeta) = \sum_{i=1}^{m} a_i Y^i
\]  

Then replacing

\[
v(-2Y(1 - Y^2) \frac{dv}{dY} + (1 - Y^2)^2 \frac{d^2v}{dY^2})
\]

\[-2((1 - Y^2) \frac{dv}{dY})^2 - cv^2(1 - Y^2) \frac{dv}{dY} = 0
\]

Now, taking the homogeneous balance between the \( vv'' \) and \( v^2v' \) terms in eq. (22), we have:

\[
vY^4 \frac{d^2v}{dY^2} \rightarrow v^2 \frac{dv}{dY} \rightarrow m + m + 2 = 2m + m + 1 \rightarrow m = 1
\]

So

\[
v = a_0 + a_1 Y 
\]

Replacing in eq. (22)

\[
(a_0 + a_1 Y)(-2Y(1 - Y^2)a_1) - 2((1 - Y^2)a_1)^2
\]

\[-c(a_0 + a_1 Y)^2(1 - Y^2)a_1 = 0
\]

Then, doing some algebra we find

\[
f_1 = \left(a_1 = -2/c, a_0 = \sqrt{8}/c\right), \quad f_2 = \left(a_1 = -2/c, a_0 = -\sqrt{8}/c\right)
\]

\[
f_3 = \left(a_1 = -2/c, a_0 = 2/c\right), \quad f_4 = \left(a_1 = -2/c, a_0 = -2/c\right)
\]
The exponential diffusion equation is:

\[ \nabla e^{-\phi} = \frac{\partial \phi}{\partial t} \quad (38) \]

Also, in one dimension and using eqs. (30-31). Then

\[ \frac{\partial^2 \phi}{\partial \zeta^2} - (\frac{\partial \phi}{\partial \zeta})^2 = -c \frac{\partial e^\phi}{\partial \zeta} \quad (39) \]

Defining the transformation

\[ v = e^\phi \quad (40) \]

Then

\[ \frac{d\phi}{d\zeta} = \frac{d\phi}{dv} \frac{dv}{d\zeta} = \frac{1}{v} \frac{dv}{d\zeta}, \quad \frac{d^2 \phi}{d\zeta^2} = -\frac{1}{v^2} (\frac{dv}{d\zeta})^2 + \frac{1}{v} \frac{d^2 v}{d\zeta^2} \quad (41) \]

And replacing in eq. (39)

\[ v \frac{d^2 v}{d\zeta^2} - 2(\frac{dv}{d\zeta})^2 + cv^2 \frac{dv}{d\zeta} = 0 \quad (42) \]

Also, we using eqs. (24-25) as [8]:

Figure 2: Using two initial profiles given by eq. (49), the spatiotemporal, LB, evolution of \( \phi(x,t) \) is shown.

7 The Nonlinear Exponential Diffusion Equation

The exponential diffusion equation is:
\[ v(\zeta) = \sum_{i=1}^{m} a_i Y^i \] (43)

Then replacing

\[ 2v(-2Y(1-Y^2) \frac{dv}{dY} + (1-Y^2)^2 \frac{d^2v}{dy^2}) \] (44)

\[ -4((1-Y^2) \frac{dv}{dy})^2 + cv^2(1-Y^2) \frac{dv}{dy} = 0 \]

Now, taking the homogeneous balance between the \(vv''\) and \(v^2v'\) terms in eq. (22), we have:

\[ vY^4 \frac{d^2v}{dy^2} \rightarrow v^2 \frac{dv}{dy} \rightarrow m + m + 2 = 2m + m + 1 \rightarrow m = 1 \] (45)

So

\[ v = a_0 + a_1 Y \] (46)

Replacing in eq. (22)

\[ (a_0 + a_1 Y)(-2Y(1-Y^2)a_1) - 2((1-Y^2)a_1)^2 + c(a_0 + a_1 Y)^2(1-Y^2)a_1 = 0 \] (47)

Then, doing some algebra

\[ f_5 = (a_1 = 2/c, a_0 = \sqrt{8}/c), \quad f_6 = (a_1 = 2/c, a_0 = -\sqrt{8}/c) \] (48)

\[ f_7 = (a_1 = 2/c, a_0 = 2/c), \quad f_8 = (a_1 = 2/c, a_0 = -2/c) \]

8 Conclusions

We solved the exponential nonlinear diffusion equation, in two versions, using lattice-Boltzmann technique and tanh solitary wave method. We obtain eight families of solutions. The solutions are:

\[ \phi = \pm \ln (a_0 + a_1 \tanh (x + y - ct)) \] (49)

As a future work, we can extend the method to two or three dimensions and in spherical or cylindrical coordinates.

Acknowledgements. This research was supported by Universidad Nacional de Colombia in Hermes project (32501).
References


Received: December 21, 2017; Published: January 5, 2018