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Abstract

In this paper, Cauchy problem for the nonlinear partial differential equations is considered. Approximate analytical solutions for mixed parabolic-hyperbolic equations are obtained using the hybrid scheme combined the well-known Laplace transform and variational iteration method. Moreover, the scheme is also used to approximate the generalized Burger’s-Fisher equation. It is shown that the nonlinear terms of differential equation can be easily handled by the present method. The method shows that it is in excellent agreement with the exact solution and it is better than the variational iteration method and some existing methods.

Keywords: Laplace Transform, Variational iteration method, Variational iteration transform method, Generalized Burger’s-Fisher equation

1 Introduction

Most of the problems in engineering and science are modeled by differential equations. Nonlinearity appears in a wide variety of scientific fields such as the gas dynamics, traffic flow, and in applied mathematics and physics applications. The nonlinear models of real-life problems are still very difficult to solve either numerically or theoretically.

An effective method is required to analyze the mathematical model which provides solutions conforming to physical reality. Recently, much attention has
been devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to non-linear models due to their potential applications [1–3].

Many methods have been developed for providing approximate solutions of nonlinear partial differential equations (NPDEs). Some of these methods include spectral collocation method [4], Adomian decomposition method (ADM) [5, 6], homotopy perturbation method (HPM) [7], reduced differential transform method (RDTM) [8, 9] and variational iteration method (VIM) [10]. The Laplace transformation method is also combined with other methods such as the HPM [11], the VIM [12-15] for solving linear and NPDEs.

The generalized non-linear Burger’s-Fisher equation is of high importance which models the interaction between reaction mechanisms, convection effects and diffusion transports. It arises often in applied mathematics and physics applications. Several different analytical as well as numerical methods were used for finding analytic or / and approximate solution of Burger’s-Fisher equation. For instance, VIM [16], ADM [17], homotopy analysis method (HAM) [18], HPM [19], tanh-coth method [20], Exp-function method [21], and so on are some of the analytical methods whereas Haar wavelet method [22], collocation method using radial basis [23], spectral domain decomposition approach [24], and so on fall in the category of numerical methods. But, to the best of our knowledge, the Burger’s–Fisher equation as well as some NPDEs of Cauchy problem has not been attempted by the Laplace-VIM so far. In this article, we will use the Laplace-VIM which we call the variational Iteration transformed method (VITM) to solve the Burger’s-Fisher equation and some of the nonlinear mixed parabolic-hyperbolic differential equations. The variety and broad applicability of parabolic-hyperbolic equations is the basic motivation of this work. Comparison of numerical results with the exact solutions, and the solutions obtained using some existing methods such as VIM and RDTM; show that the used method is fairly accurate and viable for solving such problems.

2 Variational Iteration Transform Method (VITM)

To illustrate the basic idea of this method, we consider a general nonlinear nonhomogeneous partial differential equation with initial conditions of the form

\[
\begin{align*}
\mathcal{D}u(x, t) + \mathcal{R}u(x, t) + \mathcal{N}u(x, t) &= g(x, t) \\
\hspace{1cm} u(x, 0) &= h(x), \\ u_t(x, 0) &= f(x) 
\end{align*}
\]

where \(\mathcal{D}\) is the second order linear differential operator \(\mathcal{D} = \frac{\partial^2}{\partial t^2}\), \(\mathcal{R}\) is linear differential operator of less order than \(\mathcal{D}\), \(\mathcal{N}\) represent the general nonlinear differential operator and \(g(x, t)\) is the source term. Taking the Laplace transform of both sides of Eq. (1), and then using its differentiation property, we get

\[
L[\mathcal{D}u(x, t)] + L[\mathcal{R}u(x, t)] + L[\mathcal{N}u(x, t)] = L[g(x, t)]
\]
An efficient variational iteration transform method

\[ s^2L[u(x, t)] - su(x, 0) - u_t(x, 0) + L[\mathcal{R}u(x, t)] + L[\mathcal{N}u(x, t)] = L[g(x, t)] \tag{3} \]

Taking the inverse Laplace transform of both sides of Eq. (3), yields

\[ u(x, t) = h(x) + tf(x) + \frac{1}{s^2}L^{-1}[L(g(x, t))] - \frac{1}{s^2}L^{-1}[L(\mathcal{R}u(x, t))] \]
\[ - \frac{1}{s^2}L^{-1}[L\mathcal{N}u(x, t)] \tag{4} \]

By taking the first partial derivative with respect to \( t \) of Eq. (4), yields

\[ u_t(x, t) = f(x) + \frac{\partial}{\partial t}L^{-1}\left(\frac{1}{s^2}L\{g(x, t)\}\right) - \frac{\partial}{\partial t}L^{-1}\left(\frac{1}{s^2}L\{\mathcal{R}u(x, t)\}\right) \]
\[ - \frac{\partial}{\partial t}L^{-1}\left(\frac{1}{s^2}L\{\mathcal{N}u(x, t)\}\right) \tag{5} \]

By the correction function of the variational iteration method, we get

\[ u_{n+1}(x, t) = u_n(x, t) \]
\[ - \int_0^t (u_n)_\xi(x, \xi) \]
\[ + \frac{\partial}{\partial \xi} \left[L^{-1}\left(\frac{1}{s^2}L\{\mathcal{R}u(x, \xi)\}\right) + L^{-1}\left(\frac{1}{s^2}L\{\mathcal{N}u(x, \xi)\}\right) \right. \]
\[ - \left. L^{-1}\left(\frac{1}{s^2}L\{g(x, \xi)\}\right)\right] - f(x) \right) d\xi \tag{6} \]

Finally, the solution \( u(x, t) \) is given by

\[ u(x, t) = \lim_{n \to \infty} u_n(x, t) \tag{7} \]

3 The generalized Burger’s-Fisher equation

Consider the generalized Burger’s–Fisher equation [19]:

\[ \frac{\partial u}{\partial t} + \alpha u^\sigma \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u^\sigma), 0 \leq x \leq 1, t \geq 0 \tag{8} \]

with the following initial condition

\[ u(x, 0) = \left(\frac{1}{2} + \frac{1}{2} \tanh \left(\frac{-\alpha \sigma}{2(\sigma + 1)}\right)\right)^{\frac{1}{\sigma}}, \tag{9} \]

Where \( \alpha, \beta \geq 0 \) and \( \sigma > 0 \) are given constants. If \( \sigma = 1 \), Eq. (8) is called the Burger’s–Fisher equation. When \( \alpha = 0, \sigma = 1 \), Eq. (8) is reduced to the Huxley equation [1]. Generalized Burger’s equation will be obtained when \( \beta = 0 \).
The exact solution of Eq. (8) is given by:

\[ u(x, t) = \left( \frac{1}{2} + \frac{1}{2} \tanh \left[ \frac{-\alpha \sigma}{2(\sigma + 1)} \left( x - \left( \frac{\alpha}{\sigma + 1} + \frac{\beta(\sigma + 1)}{\alpha} \right) t \right) \right] \right)^{\frac{1}{\sigma}} \quad (10) \]

4 Applications

In this section, three examples will be given to show the efficiency of the present VITM. The first two examples are considering the solutions of parabolic-hyperbolic NPDEs of Cauchy problem, whereas the third example is regarding the generalized Burger’s–Fisher equation.

Example 1: Consider the following differential equation:

\[
\frac{\partial^3 y}{\partial t^3} - \frac{\partial^3 y}{\partial t \partial x^2} - \frac{\partial^4 y}{\partial x^2 \partial t^2} + \frac{\partial^4 y}{\partial x^4} = - \left( \frac{1}{3} \frac{\partial^2 y}{\partial x^2} \right)^2 + \left( \frac{1}{6} \frac{\partial^2 y}{\partial t^2} \right)^3 - 16y \quad (11)
\]

with the following condition:

\[ y(x, 0) = -x^4, \frac{\partial y}{\partial t}(x, 0) = 0, \frac{\partial^2 y}{\partial t^2}(x, 0) = 0 \quad (12) \]

The exact solution of above problem is given by

\[ y(x, t) = -x^4 + 4t^3 \]

Taking Laplace transform of both sides of Eq. (11) then using the differentiation property of Laplace transform and the initial condition (12) yields,

\[
L[y(x, t)] = \frac{1}{s}(-x^4) + \frac{1}{s^3}L \left[ \frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^4 y}{\partial x^4} \right] - \frac{1}{s^3}L \left[ \frac{1}{3} \frac{\partial^2 y}{\partial x^2} \right]^2
- \left( \frac{1}{6} \frac{\partial^2 y}{\partial t^2} \right)^3 + 16y \]

Taking the inverse Laplace of both sides of Eq.(13), and the partial derivative with respect to \( t \), we obtain,

\[
y_t(x, t) = \frac{\partial}{\partial t} \left( L^{-1} \left( \frac{1}{s^3}L \left[ \frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^4 y}{\partial x^4} - 16y \right] \right) - L^{-1} \left( \frac{1}{s^3}L \left[ \frac{1}{9} \left( \frac{\partial^2 y}{\partial x^2} \right)^2 \right] \right) \right)
- \frac{1}{216} \left( \frac{\partial^2 y}{\partial t^2} \right)^3 \quad (14)
\]

Making the correction function given by:

\[
y_{n+1}(x, t) = y_n + \int_0^t (y_n)_{\xi} \left( L^{-1} \left( \frac{1}{s^3}L \left[ \frac{\partial^3 y_n}{\partial t \partial x^2} + \frac{\partial^4 y_n}{\partial x^2 \partial t^2} - \frac{\partial^4 y_n}{\partial x^4} - 16y_n \right] \right) - L^{-1} \left( \frac{1}{s^3}L \left[ \frac{1}{9} \left( \frac{\partial^2 y_n}{\partial x^2} \right)^2 \right] \right) \right) d\xi \quad (15)
\]
Substituting Eq. (12) into Eq. (15) we obtain

\[ y_1(x, t) = -x^4 + 4t^3 \]

(17)

\[ y(x, t) = \lim_{n \to \infty} y_n(x, t) \]

(18)

Hence, the solution of \( y(x, t) \) is given by

\[ y(x, t) = -x^4 + 4t^3. \]

**Example 2:** Consider the following nonlinear PDE:

\[
\left( \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) y = y \frac{\partial y}{\partial t} + \frac{\partial^3 y}{\partial t^2 \partial x} \]

(19)

with the following conditions:

\[ y(x, 0) = \cos(x), \frac{\partial y}{\partial t}(x, 0) = -\sin(x), \frac{\partial^2 y}{\partial t^2}(x, 0) = -\cos(x) \]

(20)

The exact solution of above problem is given by \( y(x, t) = \cos(x + t) \).

Taking Laplace transform of both sides of Eq. (19) then using the differentiation property of Laplace transform and the initial condition (20) yields,

\[
L[y(x, t)] = \frac{\cos(x)}{s} - \frac{\sin(x)}{s^2} - \frac{\cos(x)}{s^3} + \frac{1}{s^3} L \left[ \frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^4 y}{\partial x^4} \right] + \frac{1}{s^3} L \left[ \frac{\partial y}{\partial t} + \frac{\partial^3 y}{\partial t^2 \partial x} \right] \]

(21)

Taking the inverse Laplace of both sides of Eq. (21), and the partial derivative with respect to \( t \), we obtain,

\[
y_t(x, t) = -\sin(x) - t \cos(x) + \frac{\partial}{\partial t} \left[ L^{-1} \left( \frac{1}{s^3} L \left[ \frac{\partial^3 y}{\partial t \partial x^2} + \frac{\partial^4 y}{\partial x^2 \partial t^2} - \frac{\partial^4 y}{\partial x^4} \right] \right) \right] + L^{-1} \left( \frac{1}{s^3} L \left[ \frac{\partial y}{\partial t} + \frac{\partial^3 y}{\partial t^2 \partial x} \right] \right) \]

(22)
Making the correction function to eq. (22), we get

\[
y_{n+1}(x, t) = y_n(x, t) + \int_0^t (y_n)_\xi \\
+ \frac{\partial}{\partial \xi} \left[ L^{-1} \left( \frac{1}{s^3} L \left[ \frac{\partial^3 y_n}{\partial \xi \partial x^2} + \frac{\partial^4 y_n}{\partial x^2 \partial \xi^2} - \frac{\partial^4 y_n}{\partial \xi^4} \right] \right) \\
+ L^{-1} \left( \frac{1}{s^3} \left[ y_n \frac{\partial y_n}{\partial \xi} + \frac{\partial^3 y_n}{\partial \xi^2 \partial x} \right] \right) \right] - \sin(x) - t \cos(x) \, d\xi
\]  
(23)

Substituting Eq. (20) into Eq. (23) we obtain

\[
y_1(x, t) = \cos(x) - t \sin(x) - \frac{t^2}{2!} \cos(x) + \int_0^t (- \sin(x) - \xi \cos(x)) \\
+ \frac{\partial}{\partial \xi} \left[ L^{-1} \left( \frac{1}{s^4} \sin(x) + \frac{1}{s^5} (\cos(x) + \sin(x)) + \frac{1}{s^6} \cos(x) \right) \\
+ L^{-1} \left( \frac{1}{s^3} \left[ \frac{1}{s} \cos(x) - \frac{1}{s^2} \sin(x) - \frac{1}{s^3} \cos(x) \right] \\
\times \left( - \frac{1}{s} \sin(x) - \frac{1}{s^2} \cos(x) \right) \right] + \frac{1}{s} \sin(x) \right] - \sin(x) \\
- \xi \cos(x) \, d\xi
\]  
(24)

\[
y_1(x, t) = \cos(x) - \frac{t^2}{2!} \cos(x) + \frac{t^4}{4!} \cos(x) - \cdots - t \sin(x) + \frac{t^3}{3!} \sin(x) \\
- \frac{t^5}{5!} \sin(x) + \cdots
\]  
(25)

\[
y(x, t) = \lim_{n \to \infty} y_n(x, t)
\]  
(26)

Hence, the solution of \( y(x, t) \) is given by

\[
y(x, t) = \cos(x + t).
\]

It is noted that the exact solutions of the above two examples were obtained from first order approximations.

**Example 3:** Consider the generalized Burger’s–Fisher equation when \( \sigma = 1 \),

\[
\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u), \quad 0 \leq x \leq 1, \, t \geq 0
\]  
(27)

with the following initial condition:

\[
u(x, 0) = \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{\alpha x}{4} \right)
\]  
(28)
Taking Laplace transform of both sides of Eq. (27) then using the differentiation property of Laplace transform and the initial condition (28) yields,

\[
L[u(x, t)] = \frac{1}{2s} - \frac{1}{2s} \tanh \left( \frac{\alpha x}{4} \right) + \frac{1}{s} L[u_{xx}] - \frac{\alpha}{s} L[u u_x] - \frac{\beta}{s} L[u(u - 1)]
\] (29)

Taking the inverse Laplace of both sides of Eq. (29), and the derivative with respect to \( t \), we obtain,

\[
u_t(x, t) = \frac{\partial}{\partial t} \left[ L^{-1} \left( \frac{L}{s} u_{xx} \right) - \alpha L^{-1} \left( \frac{L}{s} u u_x \right) - \beta L^{-1} \left( \frac{L}{s} u(u - 1) \right) \right]
\] (30)

Making the correction function, we get

\[
u_{n+1}(x, t) = \nu_n(x, t) - \int_0^t \left( (\nu_n)_\xi (x, \xi) - \frac{\partial}{\partial \xi} \left[ L^{-1} \left( \frac{L}{s} (\nu_n)_{xx} \right) - \alpha L^{-1} \left( \frac{L}{s} (\nu_n) u_x \right) - \beta L^{-1} \left( \frac{L}{s} (\nu_n u - 1) \right) \right] \right) d\xi
\] (31)

Substituting Eq. (28) into Eq. (31) we obtain the approximate solution:

\[
u_1(x, t) = \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{\alpha x}{4} \right) + \int_0^t \left( \frac{\partial}{\partial \xi} \left[ L^{-1} \left( \frac{L}{s} - \frac{\alpha^2}{16} \text{sech}^2 \left( \frac{\alpha x}{4} \right) \tanh \left( \frac{\alpha x}{4} \right) \right) \right] - \alpha L^{-1} \left( \frac{L}{s} - \frac{\alpha}{16} \text{sech}^2 \left( \frac{\alpha x}{4} \right) + \frac{\alpha}{16} \tanh^2 \left( \frac{\alpha x}{4} \right) \tanh \left( \frac{\alpha x}{4} \right) \right) \right) d\xi
\] (32)

\[
u_1(x, t) = \frac{1}{2} - \frac{1}{2} \tanh \left( \frac{\alpha x}{4} \right) + \left( \frac{\alpha^2}{8} \left( \frac{\alpha x}{4} \right)^3 + \frac{\beta}{4} \right) t + \left( \frac{\alpha^2}{32} \left( \frac{\alpha x}{4} \right)^4 + \frac{\beta}{8} \left( \frac{\alpha x}{4} \right)^2 \right) t^2
\] (33)

Table (1) contains a comparison of the obtained results by VITM with exact solutions for various values of \( x \) and \( t \), RDTM [9] and VIM [17] when \( \alpha = \beta = 0.001 \), and \( \sigma = 1 \), whereas Table (2) shows the approximate solution of the generalized Burger’s–Fisher equation using the proposed VITM method when \( \alpha = \beta = 0.001 \), and \( \sigma = 2 \) against the exact solution and RDTM [9].
Table 1: A comparison of the approximate solution by VITM with exact solution, RDTM [9] and VIM [17] when $\alpha = \beta = 0.001$, and $\sigma = 1$.

<table>
<thead>
<tr>
<th>x</th>
<th>t</th>
<th>Exact sol.</th>
<th>VITM</th>
<th>RDTM[9]</th>
<th>VIM[17]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.02</td>
<td>0.5000038125</td>
<td>0.5000037500</td>
<td>0.5000050000</td>
<td>0.5025031108</td>
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<td>0.5100111940</td>
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Table 2: A comparison of the approximate solution by VITM with exact solution and RDTM [9] when $\alpha = \beta = 0.001$, and $\sigma = 2$.

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>Exact sol.</th>
<th>VITM</th>
<th>RDTM[9]</th>
</tr>
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<tr>
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5 Conclusion

In this study, VITM has been successfully implemented to solve nonlinear mixed parabolic-hyperbolic differential equations. The presented method reveals its capability of reducing the volume of the computational work and gives high accuracy in the numerical results. It is seen that the exact solutions were obtained from first-order approximations. On the other hand, comparison shows that the solution of Burger’s–Fisher equation by the VITM is in rather good agreement with the exact solutions and better than the existing methods such as RDTM and VIM.
References


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