The Generalized Viscosity Implicit Rules of Asymptotically Nonexpansive Mappings in Hilbert Spaces

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Abstract

In this paper, we introduce the generalized viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces. The strong convergence theorems of the implicit rules proposed are proved under certain assumptions imposed on the control parameters. The results presented in this paper improve and extend some recent corresponding results announced.

Keywords: viscosity; generalized implicit rule; asymptotically nonexpansive mapping; variational inequality; Hilbert space

1 Introduction

Firstly, in 1996, the viscosity solutions of minimization problems was introduced by Attouch [1]. In 2000, Moudafi [2] introduced one of the successful approximation methods for finding a fixed point of nonexpansive mappings in Hilbert spaces. Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$, $T : C \to C$
be a nonexpansive mapping with a nonempty fixed point set $F(T)$. The following iteration method is known as the viscosity approximation method: for arbitrarily chosen $x_0 \in C$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0, \quad (1.1)$$

where $f : C \to C$ is a contraction and $\{\alpha_n\}$ is a sequence in $(0, 1)$. Under some certain conditions, the sequence $\{x_n\}$ converges strongly to a point $z \in F(T)$ which solves the variational inequality $(VI)$

$$\langle (I - f)z, x - z \rangle \geq 0, \quad x \in F(T), \quad (1.2)$$

where $I$ is the identity of $H$.

At present, some authors studied iterative sequence for the implicit midpoint rule since it is a powerful method for solving ordinary differential equations; see [6-11] and the references therein. Recently, Xu et al [3] proposed the following viscosity implicit midpoing rule ($VIMR$) for nonexpansive mappings:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \quad (1.3)$$

In 2015, Ke and Ma [4] proposed the generalized viscosity implicit rules of non-expansive mappings in Hilbert spaces as follows:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(s_n x_n + (1 - s_n)x_{n+1}), \quad n \geq 0, \quad (1.4)$$

and

$$x_{n+1} = \alpha_n x_n + \beta_nf(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}), \quad n \geq 0, \quad (1.5)$$

They proved that the generalized viscosity implicit rules (1.3) and (1.4) converge strongly to a fixed point of $T$ under certain assumptions, which also solved the $VI(1.1)$.

In 2016, motivated by the work of Xu [3], Zhao et al [5] proposed the following implicit midpoint rule for asymptotically nonexpansive mappings:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T^n \left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \quad (1.6)$$

where $T$ is an asymptotically nonexpansive mapping. They proved that the sequence $\{x_n\}$ converges strongly to a fixed point of $T$, which, in addition, also solves the $VI(1.1)$.

In this paper, we introduce and study the generalized viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces. More precisely, we consider the following implicit iterative algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T^n (\beta_n x_n + (1 - \beta_n)x_{n+1}), \quad (1.7)$$

Under suitable conditions, we proved that the sequence $\{x_n\}$ converge strongly to a fixed point of the asymptotically nonexpansive mapping $T$, which also solves the $VI(1.1)$. As applications, we apply our results to solve the variational inequality problem and convexly constrained minimization problem.
2 Preliminaries

Throughout this paper we always assume that $H$ is a real Hilbert space and $C$ is a nonempty closed convex subset of $H$. In the sequel we denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the induced norm, respectively.

**Definition 2.1** $P_C : H \to C$ is called a metric projection if for each point $x \in H$, there corresponds a unique point in $C$, denoted by $P_C x$, such that

$$\| x - P_C x \| \leq \| x - y \|, \forall y \in C.$$  

It is well known that $P_C$ has the following properties (1)-(3):

(1) $\langle x - P_C x, y - P_C x \rangle \leq 0$;

(2) $P_C$ is nonexpansive, i.e., $\| P_C x - P_C y \| \leq \| x - y \|, \forall x, y \in H$;

(3) $\| P_C x - P_C y \|^2 \leq \langle x - y, P_C x - P_C y \rangle, \forall x, y \in H$.

Let $T : H \to H$ be a mapping and $F(T)$ be the set of fixed points of the mapping $T$, i.e., $F(T) = \{ x \in H : Tx = x \}$. The expressions $x_n \to x$ and $x_n \rightharpoonup x$ denote the strong and weak convergence of the sequence $\{ x_n \}$, respectively.

Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\| Tx - Ty \| \leq \| x - y \|, \forall x, y \in C.$$  

A mapping $T : C \to C$ is said to be asymptotically nonexpansive if there exists a sequence $\{ k_n \} \subset [1, +\infty)$ with $\lim_{n \to +\infty} k_n = 1$ such that

$$\| T^n x - T^n y \| \leq k_n \| x - y \|, \forall x, y \in C, n \geq 1.$$  

It is well known that if $T$ is an asymptotically nonexpansive, then $F(T)$ is always closed and convex. Further if, in addition, $C$ is bounded, then $F(T)$ is nonempty.

The following Lemmas are very useful for proving our main results.

**Lemma 2.1** ([12]) Let $H$ be a real Hilbert space, $C$ be a nonempty closed and convex subset of $H$, and $T : C \to C$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. If $\{ x_n \}$ is a sequence in $C$ such that

(1) $\{ x_n \}$ weakly converges to $x$;

(2) $\{ (I - T)x_n \}$ converges strongly to 0.

Then $x = Tx$.

**Lemma 2.2** ([8]) Let $H$ be a real Hilbert space, $x, y \in H$ and $\lambda \in [0, 1]$. Then

$$\| \lambda x + (1 - \lambda) y \|^2 \leq \lambda \| x \|^2 + (1 - \lambda) \| y \|^2 - \lambda(1 - \lambda) \| x - y \|^2.$$  

**Lemma 2.3** ([13]) Assume that $\{ a_n \}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \beta_n) a_n + \delta_n, \forall n \geq 0,$$  

where $\delta_n \to 0$ as $n \to +\infty$.
where \( \{ \beta_n \} \subset (0,1) \) and \( \{ \delta_n \} \) satisfy the following conditions:

1. \( \sum_{n=0}^{\infty} \beta_n = \infty \);
2. \( \limsup_{n \to \infty} \frac{\delta_n}{\beta_n} \leq \sum_{n=0}^{\infty} \beta_n \leq \infty \).

Then \( \lim_{n \to \infty} a_n = 0 \).

### 3 Main Results

**Theorem 3.1.** Let \( C \) be a nonempty closed convex subset of the real Hilbert space \( H \). Let \( T : C \to C \) be an asymptotically nonexpansive mapping with a sequence \( \{k_n\} \subset [1, +\infty) \), \( \lim_{n \to \infty} k_n = 1 \) and \( F(T) \neq \emptyset \). Let \( f \) be a contraction on \( C \) with coefficient \( \theta \in [0, 1) \), pick any \( x_0 \in C \), let \( \{x_n\} \) be a sequence generated by

\[
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T^n (\beta_n x_n + (1 - \beta_n) x_{n+1}),
\]

where \( \{\alpha_n\} \) and \( \{\beta_n\} \in (0,1) \) satisfy the following conditions:

1. \( \lim_{n \to \infty} \alpha_n = 0 \), and \( \sum_{n=0}^{\infty} \alpha_n = \infty \);
2. \( \lim_{n \to \infty} \frac{k_n^2 - 1}{\alpha_n} = 0 \);
3. \( 0 < \tau \leq \beta_n \leq \beta_{n+1} < 1 \), for all \( n \geq 0 \);
4. \( \lim_{n \to \infty} \|T^n x_n - x_n\| = 0 \).

Then the sequence \( \{x_n\} \) converges strongly to \( x^* = P_{F(T)} f(x^*) \), which is also the unique solution of the following variational inequality

\[
\langle (I - f)x, y - x \rangle \geq 0, \forall y \in F(T).
\]

**Proof.** We divided the proof into six steps.

**Step 1.** Firstly, we show that the generalized viscosity implicit rule (3.1) is well-defined.

Let \( S_n x = \alpha_n f(x_n) + (1 - \alpha_n) T^n (\beta_n x_n + (1 - \beta_n) x) \). For any \( x, y \in C \), we have

\[
\|S_n x - S_n y\| = \|\alpha_n f(x_n) + (1 - \alpha_n) T^n (\beta_n x_n + (1 - \beta_n) x) - [\alpha_n f(x_n) + (1 - \alpha_n) T^n (\beta_n x_n + (1 - \beta_n) y)]
\]

\[
= (1 - \alpha_n) \|T^n (\beta_n x_n + (1 - \beta_n) x) - T^n (\beta_n x_n + (1 - \beta_n) y)\|
\]

\[
\leq (1 - \alpha_n) k_n (1 - \beta_n) \|x - y\|.
\]

Since \( \lim_{n \to \infty} \alpha_n = 0 \), \( \lim_{n \to \infty} k_n = 1 \) and \( 0 < \tau \leq \beta_n \leq \beta_{n+1} < 1 \) for all \( n \geq 0 \), we may assume that \( (1 - \alpha_n) k_n (1 - \beta_n) \leq 1 - \tau \) for all \( n \geq 0 \). This implies that \( S_n \) is a contraction for each \( n \). Therefore there exists a unique fixed point for \( S_n \) by Banach contraction principle, which also implies that (3.1) is well-defined.

**Step 2.** We show that \( \{x_n\} \) is bounded.
In fact, let \( p \in F(T) \), we have

\[
\|x_{n+1} - p\| = \|\alpha_n f(x_n) + (1 - \alpha_n) T^n (\beta_n x_n + (1 - \beta_n) x_{n+1}) - p\|
\leq \alpha_n \|f(x_n) - p\| + (1 - \alpha_n) \|T^n (\beta_n x_n + (1 - \beta_n) x_{n+1}) - p\|
\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1 - \alpha_n) k_n \|\beta_n x_n + (1 - \beta_n) x_{n+1} - p\|
\]

It follows that

\[
[1 - (1 - \alpha_n) k_n (1 - \beta_n)] \|x_{n+1} - p\| \leq (\alpha_n \theta + (1 - \alpha_n) k_n \beta_n) \|x_n - p\| + \alpha_n \|f(p) - p\|.
\]

Since \( \alpha_n, \beta_n \in (0, 1) \), \( \lim \inf_{n \to \infty} (1 - \alpha_n) k_n (1 - \beta_n) < 1 \), and by condition (3), for any given positive number \( \epsilon, 0 < \epsilon < 1 - \theta \), there exists a sufficient large positive integer \( n_0 \), such that for any \( n \geq n_0 \), we have

\[
k_n^2 - 1 \leq \beta_n \epsilon \alpha_n, \quad \text{and} \quad k_n - 1 \leq \frac{k_n + 1}{\beta_n} (k_n - 1) \leq \frac{k_n^2 - 1}{\beta_n} \leq \epsilon \alpha_n.
\]

Moreover, by (3.3) we get

\[
\|x_{n+1} - p\| \leq \frac{\alpha_n \theta + (1 - \alpha_n) k_n \beta_n}{1 - (1 - \alpha_n) k_n (1 - \beta_n)} \|x_n - p\| + \frac{\alpha_n}{1 - (1 - \alpha_n) k_n (1 - \beta_n)} \|f(p) - p\|
\]

By induction, we obtain

\[
\|x_{n+1} - p\| \leq \max \{\|x_0 - p\|, \frac{1}{1 - \theta - \epsilon} \|f(p) - p\|\}, \forall n \geq 0.
\]
Hence \{x_n\} is bounded. Consequently, we deduce that \{f(x_n)\}, \{T^n(\beta_n x_n + (1 - \beta_n)x_{n+1})\} are bounded.

Step 3. Next we prove that \(\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0\).

It follows from (3.1) that
\[
\|x_{n+1} - x_n\| \\
\leq \|x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\
= \|\alpha_n f(x_n) + (1 - \alpha_n)T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - T^n x_n\| + \|T^n x_n - x_n\| \\
\leq \alpha_n \|f(x_n) - T^n x_n\| + (1 - \alpha_n)\|T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - T^n x_n\| \\
+ \|T^n x_n - x_n\| \\
\leq \alpha_n \|f(x_n) - T^n x_n\| + (1 - \alpha_n)k_n\|\beta_n x_n + (1 - \beta_n)x_{n+1} - x_n\| \\
+ \|T^n x_n - x_n\| \\
\leq \alpha_n \|f(x_n) - T^n x_n\| + (1 - \alpha_n)k_n (1 - \beta_n)\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \\
\leq \alpha_n M_1 + (1 - \alpha_n)k_n (1 - \beta_n)\|x_{n+1} - x_n\| + \|T^n x_n - x_n\|,
\]
where \(M_1 = \sup \{\|f(x_n) - T^n x_n\|, n \geq 1\}\). It turns out that
\[
(1 - (1 - \alpha_n)k_n (1 - \beta_n))\|x_{n+1} - x_n\| \leq \alpha_n M_1 + \|T^n x_n - x_n\|,
\]
i.e.
\[
\|x_{n+1} - x_n\| \leq \frac{\alpha_n}{1 - (1 - \alpha_n)k_n (1 - \beta_n)} M_1 + \frac{1}{1 - (1 - \alpha_n)k_n (1 - \beta_n)} \|T^n x_n - x_n\|.
\]
Since \(1 - (1 - \alpha_n)k_n (1 - \beta_n) \geq \tau\), by virtue of the conditions (1) and (4), we have
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\tag{3.4}
\]

Step 4. Now, we prove that \(\lim_{n \to \infty} \|x_n - T x_n\| = 0\). In fact, since
\[
\|x_n - T^{n-1} x_n\| \\
= \|\alpha_{n-1} f(x_{n-1}) + (1 - \alpha_{n-1})T^{n-1}(\beta_{n-1} x_{n-1} + (1 - \beta_{n-1})x_{n-1}) - T^{n-1} x_n\| \\
\leq \alpha_{n-1} \|f(x_{n-1}) - T^{n-1} x_n\| + (1 - \alpha_{n-1})k_n \|\beta_{n-1} x_{n-1} + (1 - \beta_{n-1})x_{n-1} - x_n\| \\
\leq \alpha_{n-1} \|f(x_{n-1}) - T^{n-1} x_n\| + (1 - \alpha_{n-1})k_n \|\beta_{n-1} x_{n-1} - x_{n-1}\| \\
\leq \alpha_{n-1} M_1 + (1 - \alpha_{n-1})k_n \|x_{n-1} - x_{n-1}\|,
\]
by condition (1) and (3.4), we have
\[
\lim_{n \to \infty} \|x_n - T^{n-1} x_n\| = 0.
\]
Hence, we can get
\[
\|x_n - T x_n\| \leq \|x_n - T^n x_n\| + \|T^n x_n - T x_n\| \\
\leq \|x_n - T^n x_n\| + k_1 \|T^{n-1} x_n - x_n\| \to 0, \quad (n \to \infty). \tag{3.5}
\]
Then, it follows from (3.4) and (3.5) that
\[\|T(\beta_n x_n + (1 - \beta_n)x_{n+1}) - x_n\|\]
\[\leq \|T(\beta_n x_n + (1 - \beta_n)x_{n+1}) - Tx_n\| + \|Tx_n - x_n\|\]
\[\leq k_n \|\beta_n x_n + (1 - \beta_n)x_{n+1} - x_n\| + \|Tx_n - x_n\|\]
\[= k_n (1 - \beta_n)\|x_{n+1} - x_n\| + \|Tx_n - x_n\|\]
\[\leq k_n \|x_{n+1} - x_n\| + \|Tx_n - x_n\| \to 0, \quad n \to \infty.\quad (3.6)\]

Step 5. We claim that \(\limsup_{n \to \infty} \langle x^* - f(x^*), x^* - x_n \rangle \leq 0\), where \(x^* = P_{F(T)} f(x^*)\).

Indeed, take a sequence \(\{x_n\}\) of \(\{x\}\) such that
\[\limsup_{n \to \infty} \langle x^* - f(x^*), x^* - x_n \rangle = \lim_{n_i \to \infty} \langle x^* - f(x^*), x^* - x_{n_i} \rangle.\]

Since \(\{x_n\}\) is bounded, there exists a subsequence of \(\{x_n\}\) which converges weakly to \(p\), without loss of generality, we may assume that \(x_{n_i} \to p\). From (3.5) and Lemma 2.1, we have \(p = Tp\), that is \(p \in F(T)\). This together with the property of the metric projection implies that
\[\limsup_{n \to \infty} \langle x^* - f(x^*), x^* - x_n \rangle = \lim_{n_i \to \infty} \langle x^* - f(x^*), x^* - x_{n_i} \rangle\]
\[= \langle x^* - f(x^*), x^* - p \rangle \leq 0.\]

Step 6. Finally, we prove that \(x_n \to x^* \in F(T)\) as \(n \to \infty\).
\[\|x_{n+1} - x^*\|^2\]
\[= \|\alpha_n f(x_n) + (1 - \alpha_n)T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - x^*\|^2\]
\[= \|\alpha_n f(x_n) - x^*\|^2 + (1 - \alpha_n)\|T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - x^*\|^2\]
\[= \alpha_n^2 \|f(x_n) - x^*\|^2 + (1 - \alpha_n)^2 \|T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - x^*\|^2\]
\[+ 2\alpha_n (1 - \alpha_n)\langle f(x_n) - f(x^*), T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - x^* \rangle\]
\[\leq \alpha_n^2 \|f(x_n) - x^*\|^2 + (1 - \alpha_n)^2k_n^2 \|\beta_n x_n + (1 - \beta_n)x_{n+1} - x^*\|^2\]
\[+ 2\alpha_n (1 - \alpha_n)\langle f(x_n) - f(x^*), T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - x^* \rangle\]
\[\leq (1 - \alpha_n)^2k_n^2 \|\beta_n x_n + (1 - \beta_n)x_{n+1} - x^*\|^2\]
\[+ 2\alpha_n (1 - \alpha_n)\|f(x_n) - f(x^*)\|\|T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - x^*\| + L_n\]
\[\leq (1 - \alpha_n)^2k_n^2 \|\beta_n x_n + (1 - \beta_n)x_{n+1} - x^*\|^2\]
\[+ 2\alpha_n (1 - \alpha_n)k_n \|x_n - x^*\|\|\beta_n x_n + (1 - \beta_n)x_{n+1} - x^*\| + L_n\]

Where
\[L_n = \alpha_n^2 \|f(x_n) - x^*\|^2 + 2\alpha_n (1 - \alpha_n)\langle f(x^*) - x^*, T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - x^* \rangle\]

It turns out that
\[(1 - \alpha_n)^2 k_n^2 \| \beta_n x_n + (1 - \beta_n) x_{n+1} - x^* \|^2 + L_n \\
+ 2\theta \alpha_n (1 - \alpha_n) k_n \| x_n - x^* \| \| \beta_n x_n + (1 - \beta_n) x_{n+1} - x^* \| - \| x_{n+1} - x^* \|^2 \geq 0.\]

Solving this quadratic inequality for \( \| \beta_n x_n + (1 - \beta_n) x_{n+1} - x^* \| \) yields

\[
\| \beta_n x_n + (1 - \beta_n) x_{n+1} - x^* \| \\
\geq \frac{1}{2(1 - \alpha_n)^2 k_n^2} \left\{ -2\theta \alpha_n (1 - \alpha_n) k_n \| x_n - x^* \| + \sqrt{4\theta^2 \alpha_n^2 (1 - \alpha_n)^2 k_n^2 \| x_n - x^* \|^2 - 4(1 - \alpha_n)^2 k_n^2 (L_n - \| x_{n+1} - x^* \|^2)} \right\}
\]

\[
= -\theta \alpha_n \| x_n - x^* \| + \frac{\sqrt{\theta^2 \alpha_n^2 \| x_n - x^* \|^2 - L_n + \| x_{n+1} - x^* \|^2}}{(1 - \alpha_n) k_n}.
\]

This implies that

\[
\beta_n \| x_n - x^* \| + (1 - \beta_n) \| x_{n+1} - x^* \| \\
\geq \frac{-\theta \alpha_n \| x_n - x^* \| + \sqrt{\theta^2 \alpha_n^2 \| x_n - x^* \|^2 - L_n + \| x_{n+1} - x^* \|^2}}{(1 - \alpha_n) k_n},
\]

namely,

\[
[(1 - \alpha_n) k_n \beta_n + \theta \alpha_n] \| x_n - x^* \| + (1 - \alpha_n) k_n (1 - \beta_n) \| x_{n+1} - x^* \| \\
\geq \sqrt{\theta^2 \alpha_n^2 \| x_n - x^* \|^2 - L_n + \| x_{n+1} - x^* \|^2}.
\]

Then

\[
\theta^2 \alpha_n^2 \| x_n - x^* \|^2 - L_n + \| x_{n+1} - x^* \|^2 \\
\leq [(1 - \alpha_n) k_n \beta_n + \theta \alpha_n]^2 \| x_n - x^* \|^2 + (1 - \alpha_n)^2 k_n^2 (1 - \beta_n)^2 \| x_{n+1} - x^* \|^2 \\
+ 2[(1 - \alpha_n) k_n \beta_n + \theta \alpha_n] (1 - \alpha_n) k_n (1 - \beta_n) \| x_n - x^* \| \| x_{n+1} - x^* \| \\
\leq [(1 - \alpha_n) k_n \beta_n + \theta \alpha_n]^2 \| x_n - x^* \|^2 + (1 - \alpha_n)^2 k_n^2 (1 - \beta_n)^2 \| x_{n+1} - x^* \|^2 \\
(1 - \alpha_n) k_n \beta_n + \theta \alpha_n (1 - \alpha_n) k_n (1 - \beta_n) (\| x_n - x^* \|^2 + \| x_{n+1} - x^* \|^2),
\]

which is reduced to the inequality

\[
[1 - (1 - \alpha_n)^2 k_n^2 (1 - \beta_n)^2 - ((1 - \alpha_n) k_n \beta_n + \theta \alpha_n) (1 - \alpha_n) k_n (1 - \beta_n)] \| x_{n+1} - x^* \|^2 \\
\leq [((1 - \alpha_n) k_n \beta_n + \theta \alpha_n)^2 + ((1 - \alpha_n) k_n \beta_n + \theta \alpha_n) (1 - \alpha_n) k_n (1 - \beta_n) \\
- \theta^2 \alpha_n^2] \| x_n - x^* \|^2 + L_n,
\]

that is,

\[
[1 - (k_n + \alpha_n (\theta - k_n)) (1 - \beta_n) (1 - \alpha_n) k_n] \| x_{n+1} - x^* \|^2 \\
\leq [((1 - \alpha_n) k_n \beta_n + \theta \alpha_n) (k_n + \alpha_n (\theta - k_n)) - \theta^2 \alpha_n^2] \| x_n - x^* \|^2 + L_n.
\]

(3.7)

It from (3.7) that

\[
\| x_{n+1} - x^* \|^2 \\
\leq \frac{[((1 - \alpha_n) k_n \beta_n + \theta \alpha_n) (k_n + \alpha_n (\theta - k_n)) - \theta^2 \alpha_n^2]}{1 - (k_n + \alpha_n (\theta - k_n)) (1 - \beta_n) (1 - \alpha_n) k_n} \| x_n - x^* \|^2 \\
+ \frac{L_n}{1 - (k_n + \alpha_n (\theta - k_n)) (1 - \beta_n) (1 - \alpha_n) k_n}.
\]

(3.8)
Let
\[ w_n := \frac{1}{\alpha_n} \left\{ 1 - \frac{(1 - \alpha_n)k_n \beta_n + \theta \alpha_n(k_n + \alpha_n(\theta - k_n)) - \theta^2 \alpha_n^2}{1 - (k_n + \alpha_n(\theta - k_n))(1 - \beta_n)(1 - \alpha_n)k_n} \right\} \]
\[ = \frac{1}{\alpha_n} \left\{ 1 - k_n^2 - 2\alpha_n k_n(\theta - k_n) - \alpha_n^2(\theta - k_n)^2 + \theta^2 \alpha_n^2 \right\} \alpha_n \frac{1 - (k_n + \alpha_n(\theta - k_n))(1 - \beta_n)(1 - \alpha_n)k_n}{1 - (k_n + \alpha_n(\theta - k_n))(1 - \beta_n)(1 - \alpha_n)k_n}. \]

Since \( 0 < \epsilon < 1 - \theta \), and the sequence \( \{ \beta_n \} \) satisfies \( 0 < \tau \leq \beta_n \leq \beta_{n+1} < 1 \) for all \( n \geq 0 \), \( \lim_{n \to \infty} \beta_n \) exists, assume that
\[ \lim_{n \to \infty} \beta_n = \beta^* > 0. \]

Then
\[ \lim_{n \to \infty} w_n \leq \frac{(2 - \beta^*)(1 - \theta)}{\beta^*} > 0. \]

Let \( 0 < \gamma_1 < \frac{(2 - \beta^*)(1 - \theta)}{\beta^*} \), then there exists an integer \( N_1 \) big enough such that \( w_n \geq \gamma_1 \) for all \( n \geq N_1 \). Hence, we have
\[ \frac{(1 - \alpha_n)k_n \beta_n + \theta \alpha_n(k_n + \alpha_n(\theta - k_n)) - \theta^2 \alpha_n^2}{1 - (k_n + \alpha_n(\theta - k_n))(1 - \beta_n)(1 - \alpha_n)k_n} \leq 1 - \gamma_1 \alpha_n, \forall n \geq N_1. \]

It turns out from (3.8) that
\[ \|x_{n+1} - x^*\|^2 \leq (1 - \gamma_1 \alpha_n)\|x_n - x^*\|^2 + \frac{L_n}{1 - (k_n + \alpha_n(\theta - k_n))(1 - \beta_n)(1 - \alpha_n)k_n}. \]

From (3.6), \( \lim_{n \to \infty} \alpha_n = 0 \) and step 4 we have
\[ \lim_{n \to \infty} \frac{L_n}{\gamma_1 \alpha_n[1 - (k_n + \alpha_n(\theta - k_n))(1 - \beta_n)(1 - \alpha_n)k_n]} \]
\[ = \lim_{n \to \infty} \frac{\alpha_n^2\|f(x_n) - x^*\|^2 + 2\alpha_n(1 - \alpha_n)\langle f(x^*) - x^*, T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - x^* \rangle}{\gamma_1 \alpha_n[1 - (k_n + \alpha_n(\theta - k_n))(1 - \beta_n)(1 - \alpha_n)k_n]} \]
\[ = \lim_{n \to \infty} \frac{\alpha_n\|f(x_n) - x^*\|^2 + 2(1 - \alpha_n)\langle f(x^*) - x^*, T^n(\beta_n x_n + (1 - \beta_n)x_{n+1}) - x^* \rangle}{\gamma_1[1 - (k_n + \alpha_n(\theta - k_n))(1 - \beta_n)(1 - \alpha_n)k_n]} \]
\[ \leq 0. \]  \hspace{1cm} (3.10)

From (3.9), (3.10) and Lemma 2.3, we can obtain
\[ \lim_{n \to \infty} \|x_{n+1} - x^*\| = 0, \]
namely, \( x_n \to x^* \) as \( n \to \infty \). This completes the proof.
The following result can be obtained from Theorem 3.1 immediately.

**Theorem 3.2.** Let $C$ be a nonempty closed convex subset of the real Hilbert space $H$. Let $T : C \to C$ be nonexpansive mapping with $F(T) \neq \emptyset$. Let $f$ be a contraction on $C$ with coefficient $\theta \in [0, 1)$, pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\beta_n x_n + (1 - \beta_n)x_{n+1}),
$$

where $\{\alpha_n\}$ and $\{\beta_n\} \in (0, 1)$ are same as in Theorem 3.1. Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{F(T)}f(x^*)$, which is also the unique solution of the variational inequality

$$
\langle (I - f)x, y - x \rangle \geq 0, \forall y \in F(T).
$$

**Remark 3.3.** It is well known that every nonexpansive mapping is an asymptotically nonexpansive mapping, so our result is an improvement and generalization of the main result in [4].

## 4 Applications

### 4.1 Variational Inequality Problems

A monotone variational inequality problem (VIP) is formulated as the problem of finding a point $x^* \in C$ with the property:

$$
\langle Ax^*, z - x^* \rangle \geq 0, \quad \forall z \in C,
$$

where $A : C \to H$ is a nonlinear monotone operator.

Notice that the VIP(4.1) is equivalent to the fixed point problem, for any $\gamma > 0$,

$$
Tx^* = x^*, \quad Tz := PC(I - \gamma A)z.
$$

If in addition $A$ is $\kappa$-inverse strongly monotone mapping, then the map $T := PC(I - \gamma A)$ is nonexpansive for all $\gamma \in (0, 2\kappa)$. Applying Theorem 3.2 we can get the following result:

**Corollary 4.1.** Assume $A$ be a $\kappa$-inverse strongly monotone mapping, the map $T := PC(I - \gamma A)$ is nonexpansive for all $\gamma \in (0, 2\kappa)$. Let $f$ be a contraction on $C$ with coefficient $\theta \in [0, 1)$. Define a sequence $\{x_n\}$ by the viscosity implicit rule:

$$
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)PC(I - \gamma A)(\beta_n x_n + (1 - \beta_n)x_{n+1}),
$$

where $\{\alpha_n\}$ and $\{\beta_n\} \in (0, 1)$ are same as in Theorem 3.1. Then $\{x_n\}$ converges to a solution $x^*$ of the VIP(4.1) which is also a solution to the VIP

$$
\langle (I - f)x^*, z - x^* \rangle, \quad z \in A^{-1}(0).
$$
4.2 Convexly Constrained Minimization Problem

Consider the optimization problem
\[
\min_{x \in C} h(x),
\]  
(4.4)
where \( h : H \to \mathbb{R} \) is a convex and differentiable function. Assume (4.4) is consistent, and let \( \Gamma \) denote its set of solutions.

The gradient projection algorithm generates a sequence \( \{x_n\} \) via the iterative procedure:
\[
x_{n+1} = P_C(x_n - \gamma \nabla h(x_n)),
\]  
(4.5)
where \( \nabla h \) stands for the gradient of \( h \). The following lemma can be found in [14].

**Lemma 4.2.** [14] A necessary condition of optimality for a point \( x^* \in C \) to be a solution of the minimization problem (4.4) is that \( x^* \) solves the variational inequality
\[
\langle \nabla h(x^*), z - x^* \rangle \geq 0, \quad z \in C.
\]  
(4.6)

The \( x^* \in C \) is also the solution of the (4.5)
\[
x^* = P_C(x^* - \gamma \nabla h(x^*)),
\]
if \( \nabla h \) is \( \kappa \)-inverse strongly monotone mapping and \( \gamma \in (0, 2\kappa) \). Thus, we can apply the previous results to (4.4) by taking \( A = \nabla h \).

**Corollary 4.3.** Let \( C \) be a nonempty closed convex subset of the real Hilbert space \( H \). Assume \( h \) is a convex and differentiable function, \( \nabla h \) is a \( \kappa \)-inverse strongly monotone mapping and \( \gamma \in (0, 2\kappa) \). Let \( f \) be a contraction on \( C \) with coefficient \( \theta \in (0, 1) \), pick any \( x_0 \in C \), let \( \{x_n\} \) be a sequence generated by
\[
x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) P_C(I - \gamma \nabla h)(\beta_n x_n + (1 - \beta_n)x_{n+1}),
\]
where \( \{\alpha_n\} \) and \( \{\beta_n\} \in (0, 1) \) are same as in Theorem 3.1. Then \( \{x_n\} \) converges to a solution \( x^* \) of the (4.4) which is also a solution to the VIP
\[
\langle (I - f)x^*, z - x^* \rangle, \quad z \in \Gamma.
\]  
(4.7)

**References**


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