Applied Mathematical Sciences, Vol. 11, 2017, no. 9, 407 - 424
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2017.612282

Two Dimensional Inertial Flow of a Viscous Fluid in a Corner

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Abstract

This paper investigates the inertial flow of a viscous fluid between two intersecting planes. The nonlinear equations describing the flow are formulated, and analytical solutions for the first and second-order inertial effects are obtained by using the recursive approach due to Langlois. A detailed analysis reflecting the effects of the variation of the angle of the scraper on the flow is presented. In addition, pressure, and the tangential and normal stresses are also computed. Finally, the difference between inertial and non-inertial flow behavior is shown by sketching the graphs of velocities, stresses and streamlines.

Mathematics Subject Classification: 76D05, 76D07

Keywords: Creeping flow, Corner, Viscous fluid, Scraper, Inertial flow

1 Introduction

Scraping of fluids from a solid surface such as cleaning of pipes that conduit fluids used in the processing of various foods is a common situation in everyday life and in industrial processes. Taylor scraper, the mathematical model used to represent the dynamics of the scraping of a viscous fluid was originally proposed by Taylor [10]. This model studies the flow of a viscous fluid in a corner made up of two intersecting planes where one moves steadily over the other without taking into account the inertia force.

Related flow problems have been investigated by several authors. Dean & Montagnon [4] studied the flow induced in a corner with fixed sides by a general motion at a large distance from the corner. Later Moffatt [9] considered the similar type of problem and showed that there exists an infinite sequence of eddies of decreasing size and strength, provided that the corner angle is less than some critical value. The non-inertial flow of a shear thinning fluid between intersecting planes was examined by Mansutti and Rajagopal [8]. It was shown that the sharp and pronounced boundary layers develop adjacent to the solid boundaries, even at zero Reynolds number. In addition, Anderson and Davis [1] studied the two-dimensional viscous flow of two fluids in a wedge made up by one rigid and one stress free boundary. It is worth noting that in almost all these investigations of the corner flows no attention is paid to the flow of viscous fluids with inertia.

Though the corner flow of viscous fluids in the presence of inertia force is one of the interesting problems in fluid mechanics, but there are only a few papers which deal with the flow of viscous fluids with inertia. The effects of inertia in forced corner flows were investigated by Hancock et al. [5]. They obtained the first-order inertial effect analytically in the form of a regular perturbation series for the stream function, and presented streamline plots indicating the first influence at distances from the corner where inertia forces become significant. Hills & Moffatt [6] considered the effects of inertia for a three-dimensional flow in a corner induced by the rotation in its plane of one of the boundaries. A local similarity solution valid in a neighborhood of the center of rotation was obtained, and it was shown that the inertial effects were significant in a small neighborhood of the plane of symmetry of the flow.

In this paper we extend the work of Hancock et al. [5] and obtain the second-order inertial effect analytically. In addition, we present the expressions for pressure, and normal and tangential stresses for both inertial and non-inertial flows which were absent in their work. We also study the effects of varying the angle of the scraper on the flow. The presence of inertia forces give rise to certain non-linear partial differential equations whose solution is obtained by employing the recursive approach introduced by Langlois [7]. This approach was basically proposed for slow and steady viscoelastic flows for which fluid inertia was neglected throughout. Here we tried to use this approach for the inertial flow and have shown that this approach can be applied to flows including fluid inertia. Our results subsume Taylor's and Hancock et al. results as special cases when the zeroth and first order solutions are considered respectively. Finally, the difference between inertial and non-inertial flow behavior is shown by sketching the graphs of velocities, stresses and streamlines.

2 Basic Flow Equations

The basic equations governing the motion of an incompressible steady viscous fluid, neglecting thermal effects and body forces are as follows:

$$\begin{aligned} \mathrm{div}\mathbf{V} &= 0. && (2.1)\\ \rho\left(\mathbf{V}.\nabla\right)\mathbf{V} &= \mathrm{div}\mathbf{T}, && (2.2)\end{aligned}$$

Here $\mathbf{V}$ is the velocity vector, ρ is the constant density of the fluid, and $\mathbf{T}$ is the Cauchy stress tensor given by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A_1} \qquad (2.3)$$

where μ is the viscosity, p is the pressure, $\mathbf{I}$ is the unit tensor, and

$$\mathbf{A_1} = \nabla\mathbf{V} + (\nabla\mathbf{V})^T \qquad (2.4)$$

the first Rivlin–Ericksen tensor.

Following the Langlois Recursive Approach [7], we expand the velocity $\mathbf{V}$, and pressure p in the following form

$$\begin{aligned} \mathbf{V} &= \epsilon\mathbf{V}^{(1)} + \epsilon^2\mathbf{V}^{(2)} + \epsilon^3\mathbf{V}^{(3)} && (2.5)\\ p &= \text{constant} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \epsilon^3 p^{(3)} && (2.6)\end{aligned}$$

where ϵ is a small dimensionless constant, and neglect all terms of the fourth or higher order in ϵ. From Eqs. (2.3) and (2.4), ignoring the constant pressure the stress tensor then takes the form

$$\mathbf{T} = \epsilon[-p^{(1)} + \mu\mathbf{A_1}^{(1)}] + \epsilon^2[-p^{(2)} + \mu\mathbf{A_1}^{(2)}] + \epsilon^3[-p^{(3)} + \mu\mathbf{A_1}^{(3)}] \qquad (2.7)$$

where

$$\begin{aligned} \mathbf{A_1}^{(1)} &= \nabla\mathbf{V}^{(1)} + (\nabla\mathbf{V}^{(1)})^T\\ \mathbf{A_1}^{(2)} &= \nabla\mathbf{V}^{(2)} + (\nabla\mathbf{V}^{(2)})^T\\ \mathbf{A_1}^{(3)} &= \nabla\mathbf{V}^{(3)} + (\nabla\mathbf{V}^{(3)})^T\end{aligned}$$

Using Eqs. (2.5), (2.6), and (2.7), the equation of continuity and momentum then becomes

$$\epsilon\nabla.\mathbf{V}^{(1)} + \epsilon^2\nabla.\mathbf{V}^{(2)} + \epsilon^3\nabla.\mathbf{V}^{(3)} = 0, \qquad (2.8)$$

$$\begin{aligned}\epsilon^2\rho\left(\mathbf{V}^{(1)}.\nabla\mathbf{V}^{(1)}\right) + \epsilon^3\rho\left(\mathbf{V}^{(1)}.\nabla\mathbf{V}^{(2)} + \mathbf{V}^{(2)}.\nabla\mathbf{V}^{(1)}\right) &= \epsilon\left(\mu\nabla.\mathbf{A_1}^{(1)} - \nabla p^{(1)}\right)\\ &+\epsilon^2\left(-\nabla p^{(2)} + \mu\nabla.\mathbf{A_1}^{(2)}\right) + \epsilon^3\left(-\nabla p^{(3)} + \mu\nabla.\mathbf{A_1}^{(3)}\right) \qquad (2.9)\end{aligned}$$

3 Formulation of the problem

Consider the flow of a viscous fluid near the corner made by two rigid boundaries intersecting at a constant angle θ_0. Assume that one of the boundaries is sliding steadily with velocity ϵU at the other. Both radial and tangential velocities are further assumed to be zero at the stationary boundary, we call the horizontal moving boundary a plate and the fixed boundary at $\theta = \theta_0$, the scraper. We choose the plane polar coordinates (r, θ) to study the flow behavior of the fluid between the boundaries.

The velocity field for the problem under consideration then has the form

$$\mathbf{V} = u(r,\theta)\mathbf{e}_r + v(r,\theta)\mathbf{e}_\theta \tag{3.1}$$

and the boundary conditions for the problem are

$$u = \epsilon U, \quad v = 0 \quad \text{at} \quad \theta = 0, \tag{3.2}$$

and

$$u = 0, \quad v = 0 \quad \text{at} \quad \theta = \theta_0, \tag{3.3}$$

Again following the recursive approach, we assume the velocity components u and v as of the form

$$u = \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \epsilon^3 u^{(3)} \tag{3.4}$$

$$v = \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \epsilon^3 v^{(3)} \tag{3.5}$$

Eqs. (3.1)–(3.3) then takes the form

$$\begin{aligned}\mathbf{V} &= \epsilon\left[u^{(1)}(r,\theta)\mathbf{e}_r + v^{(1)}(r,\theta)\mathbf{e}_\theta\right] + \epsilon^2\left[u^{(2)}(r,\theta)\mathbf{e}_r + v^{(2)}(r,\theta)\mathbf{e}_\theta\right] \\ &\quad + \epsilon^3\left[u^{(3)}(r,\theta)\mathbf{e}_r + v^{(3)}(r,\theta)\mathbf{e}_\theta\right]\end{aligned} \tag{3.6}$$

$$\epsilon u^{(1)} + \epsilon^2 u^{(2)} + \epsilon^3 u^{(3)} = \epsilon U, \quad \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \epsilon^3 v^{(3)} = 0 \qquad \text{at} \quad \theta = 0 \tag{3.7}$$

$$\epsilon u^{(1)} + \epsilon^2 u^{(2)} + \epsilon^3 u^{(3)} = 0, \quad \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \epsilon^3 v^{(3)} = 0 \qquad \text{at} \quad \theta = \theta_0 \tag{3.8}$$

Equating coefficients of $\epsilon, \epsilon^2, \epsilon^3$ from Eqs. (2.8)–(2.9), and (3.7)–(3.8), we obtain the following first, second, and third order boundary value problems

$O(\epsilon)$:

$$\nabla.\mathbf{V}^{(1)} = 0. \tag{3.9}$$

$$\mu\nabla.\mathbf{A_1}^{(1)} - \nabla p^{(1)} = 0 \tag{3.10}$$

$$u^{(1)} = U, \quad v^{(1)} = 0 \quad \text{at} \quad \theta = 0, \tag{3.11}$$

$$u^{(1)} = 0, \quad v^{(1)} = 0 \quad \text{at} \quad \theta = \theta_0, \tag{3.12}$$

$O(\epsilon^2)$:

$$\begin{aligned} \nabla.\mathbf{V}^{(2)} &= 0. && (3.13)\\ \rho\left(\mathbf{V}^{(1)}.\nabla\mathbf{V}^{(1)}\right) &= -\nabla p^{(2)} + \mu\nabla.\mathbf{A_1}^{(2)} && (3.14)\\ u^{(2)} = 0, \quad v^{(2)} &= 0 \quad \text{at} \quad \theta = 0, && (3.15)\\ u^{(2)} = 0, \quad v^{(2)} &= 0 \quad \text{at} \quad \theta = \theta_0, && (3.16) \end{aligned}$$

and

$O(\epsilon^3)$:

$$\begin{aligned} \nabla.\mathbf{V}^{(3)} &= 0. && (3.17)\\ \rho\left(\mathbf{V}^{(1)}.\nabla\mathbf{V}^{(2)} + \mathbf{V}^{(2)}.\nabla\mathbf{V}^{(1)}\right) &= -\nabla p^{(3)} + \mu\nabla.\mathbf{A_1}^{(3)} && (3.18)\\ u^{(3)} = 0, \quad v^{(3)} &= 0 \quad \text{at} \quad \theta = 0, && (3.19)\\ u^{(3)} = 0, \quad v^{(3)} &= 0 \quad \text{at} \quad \theta = \theta_0, && (3.20) \end{aligned}$$

3.1 First Order System (Non-inertial flow)

Recalling that the first order velocity and pressure fields for the problem under consideration have the form

$$\mathbf{V}^{(1)} = u^{(1)}(r,\theta)e_r + v^{(1)}(r,\theta)e_\theta, \quad p^{(1)} = p^{(1)}(r,\theta) \qquad (3.21)$$

Then equations (3.9) and (3.10) in terms of polar coordinates, assumes the form

$$\begin{aligned} \frac{\partial u^{(1)}}{\partial r} + \frac{u^{(1)}}{r} + \frac{1}{r}\frac{\partial v^{(1)}}{\partial \theta} &= 0, && (3.22)\\ \frac{\partial p^{(1)}}{\partial r} &= -\frac{\mu}{r}\frac{\partial \Omega^{(1)}}{\partial \theta}, && (3.23)\\ \frac{1}{r}\frac{\partial p^{(1)}}{\partial \theta} &= \mu\frac{\partial \Omega^{(1)}}{\partial r}, && (3.24) \end{aligned}$$

where

$$\Omega^{(1)} = \frac{\partial v^{(1)}}{\partial r} + \frac{v^{(1)}}{r} - \frac{1}{r}\frac{\partial u^{(1)}}{\partial \theta} \qquad (3.25)$$

is the non-zero component of the vorticity tensor.

Introducing the first order stream function $\psi^{(1)}(r,\theta)$ such that

$$u^{(1)} = \frac{1}{r}\frac{\partial \psi^{(1)}}{\partial \theta}, \quad v^{(1)} = -\frac{\partial \psi^{(1)}}{\partial r}, \qquad (3.26)$$

and then eliminating the pressure from Eqs. (3.23) and (3.24), we end up with the following first order system for creeping viscous flow

$$\nabla^4\psi^{(1)} = 0, \qquad (3.27)$$

with boundary conditions

$$\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta} = U, \qquad \frac{\partial\psi^{(1)}}{\partial r} = 0, \quad \text{at} \quad \theta = 0 \tag{3.28}$$

$$\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\theta} = 0, \qquad \frac{\partial\psi^{(1)}}{\partial r} = 0, \quad \text{at} \quad \theta = \theta_0 \tag{3.29}$$

3.2 Solution of the first order system (Non-inertial flow)

As discussed in [10], the solution of the system (3.27) – (3.29) is given by

$$\psi^{(1)} = Urf_1(\theta) \tag{3.30}$$

where

$$f_1(\theta) = \frac{1}{\theta_0^2 - \sin^2\theta_0}\{\theta_0^2 \sin\theta - \theta\cos\theta\sin^2\theta_0 - (\theta_0 - \sin\theta_0\cos\theta_0)\theta\sin\theta\} \tag{3.31}$$

By using Eqs. (3.25), (3.26), and (3.31), the velocity components $u^{(1)}$, $v^{(1)}$ and vorticity $\Omega^{(1)}$ in terms of $f_1(\theta)$, respectively, can be obtained in following forms:

$$u^{(1)} = \frac{1}{r}\frac{\partial(Urf_1)}{\partial\theta} = Uf_1', \;\; v^{(1)} = -\frac{\partial(Urf_1)}{\partial r} = -Uf_1 \tag{3.32}$$

and

$$\Omega^{(1)} = -U\frac{f_1 + f_1''}{r} = -U\frac{g}{r} \tag{3.33}$$

where

$$g(\theta) = f_1 + f_1''. \tag{3.34}$$

The pressure field obtained from Eqs. (3.23) and (3.24), has the form

$$p^{(1)}(r,\theta) = p_0 - \frac{\mu}{r}Ug' \tag{3.35}$$

where p_0 is a positive constant determined by conditions far from the corner. Here we note that pressure becomes singular as $r \to 0$.

The normal and tangential stresses $T_n^{(1)}$ and $T_t^{(1)}$ to the scraper at distance r from the point of contact are then obtained as

$$T_n^{(1)} = \frac{\mu}{r}Ug' \tag{3.36}$$

and

$$T_t^{(1)} = \frac{\mu}{r}Ug \tag{3.37}$$

We now turn to examine the first order inertial effects on the flow near the corner.

3.3 Second Order System (The first inertial correction)

We now consider the equations of motion (3.13) and (3.16), which include the inertia force and study its effects on the flow under consideration. Here we assume that the second order velocity and pressure fields are of the form

$$\mathbf{V}^{(2)} = u^{(2)}(r,\theta)e_r + v^{(2)}(r,\theta)e_\theta, \quad p^{(2)} = p^{(2)}(r,\theta) \tag{3.38}$$

Equations of motion (3.13) and (3.16) for the second order system then takes the form

$$0 = \frac{\partial u^{(2)}}{\partial r} + \frac{u^{(2)}}{r} + \frac{1}{r}\frac{\partial v^{(2)}}{\partial \theta}, \tag{3.39}$$

$$\frac{\partial p^{*(2)}}{\partial r} = \rho v^{(2)}\Omega^{(2)} - \frac{\mu}{r}\frac{\partial \Omega^{(2)}}{\partial \theta}, \tag{3.40}$$

$$\frac{\partial p^{*(2)}}{\partial \theta} = \mu r\frac{\partial \Omega^{(2)}}{\partial r} + \rho r u^{(2)}\Omega^{(2)}, \tag{3.41}$$

where

$$p^{*(2)} = p^{(2)} + \frac{\rho}{2}\left((u^{(2)})^2 + (v^{(2)})^2\right) \tag{3.42}$$

is the modified pressure.

Now, we eliminate the pressure from Eqs. (3.40) and (3.41) by cross differentiation, and introduce the second order stream function $\psi^{(2)}(r,\theta)$ such that

$$u^{(2)} = \frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \theta}, \quad v^{(2)} = -\frac{\partial \psi^{(2)}}{\partial r}, \tag{3.43}$$

by virtue of which we get the following second order system of equations

$$\nu\nabla^4\psi^{(2)} = -\frac{1}{r}\frac{\partial(\psi^{(1)}, \nabla^2\psi^{(1)})}{\partial(r,\theta)}, \tag{3.44}$$

$$\frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \theta} = 0, \quad \frac{\partial \psi^{(2)}}{\partial r} = 0, \quad \text{at} \quad \theta = 0 \tag{3.45}$$

$$\frac{1}{r}\frac{\partial \psi^{(2)}}{\partial \theta} = 0, \quad \frac{\partial \psi^{(2)}}{\partial r} = 0, \quad \text{at} \quad \theta = \theta_0 \tag{3.46}$$

3.4 Solution of the Second order system

In order to solve the second order system (3.44) – (3.46), we assume the solution of the form

$$\psi^{(2)} = \frac{r^2U^2}{\nu}f_2(\theta) \tag{3.47}$$

Substituting values of ψ_1 and ψ_2 from Eqs. (3.30) and (3.47) into Eqs. (3.44)–(3.46), we get the fourth order ordinary differential equation of the form

$$f_2^{iv} + 4f_2'' = \beta_1 \sin 2\theta + \beta_2 \cos 2\theta + \beta_3 \theta \sin 2\theta + \beta_4 \theta \cos 2\theta, \tag{3.48}$$

where β's are the known constants and their expressions will be given later. The corresponding boundary conditions on f_2 are

$$f_2 = 0, \quad f_2' = 0 \quad \text{at} \quad \theta = 0 \tag{3.49}$$

$$f_2 = 0, \quad f_2' = 0 \quad \text{at} \quad \theta = \theta_0. \tag{3.50}$$

The solution of the Eq. (3.48) is then found by elementary techniques in the form

$$\begin{aligned} f_2(\theta) &= R_1 + R_2\theta + R_3 \cos 2\theta + R_4 \sin 2\theta + \beta_5 \theta \cos 2\theta \\ &\quad + \beta_6 \theta \sin 2\theta + \beta_7 \theta^2 \cos 2\theta + \beta_8 \theta^2 \sin 2\theta, \end{aligned} \tag{3.51}$$

where R's and β's, depending upon angle θ_0, are constants that are given by,

$$\left.\begin{aligned} R_1 &= \frac{2X(\theta_0) - 2X_1(\theta_0)\theta_0 - (\beta_5 - X_1(\theta_0)) \sin 2\theta_0 - 2(X(\theta_0) - \beta_5\theta_0) \cos 2\theta_0}{4(\cos 2\theta_0 + \theta_0 \sin 2\theta_0 - 1)}, \\ R_2 &= \frac{X_1(\theta_0) + \beta_5 - (X_1(\theta_0) + \beta_5) \cos 2\theta_0 - 2X(\theta_0)) \sin 2\theta_0)}{2(\cos 2\theta_0 + \theta_0 \sin 2\theta_0 - 1)}, \\ R_3 &= -\frac{2X(\theta_0) - 2X_1(\theta_0)\theta_0 - (\beta_5 - X_1(\theta_0)) \sin 2\theta_0 - 2(X(\theta_0) - \beta_5\theta_0) \cos 2\theta_0}{4(\cos 2\theta_0 + \theta_0 \sin 2\theta_0 - 1)}, \\ R_4 &= \frac{\beta_5 - X_1(\theta_0) + (X_1(\theta_0) - \beta_5) \cos 2\theta_0 + 2(X(\theta_0) - \beta_5\theta_0) \sin 2\theta_0}{4(\cos 2\theta_0 + \theta_0 \sin 2\theta_0 - 1)}, \\ k &= \theta_0 - \sin\theta_0 \cos\theta_0, \quad \beta_1 = \frac{\sin^4\theta_0 - k^2 - 2\theta_0^2 \sin^2\theta_0}{(\theta_0^2 - \sin^2\theta_0)^2}, \\ \beta_2 &= \frac{2k}{\theta_0^2 - \sin^2\theta_0}, \quad \beta_3 = \frac{4k \sin^2\theta_0}{(\theta_0^2 - \sin^2\theta_0)^2}, \quad \beta_4 = 2\frac{\sin^4\theta_0 - k^2}{(\theta_0^2 - \sin^2\theta_0)^2} \\ \beta_5 &= \frac{\beta_1}{16} - \frac{5}{64}\beta_4, \ \beta_6 = -\left(\frac{\beta_2}{16} + \frac{5}{64}\beta_3\right), \ \beta_7 = \frac{1}{32}\beta_3, \ \beta_8 = -\frac{1}{32}\beta_4, \\ X(\theta_0) &= \beta_5\theta_0 \cos 2\theta_0 + \beta_6\theta_0 \sin 2\theta_0 + \beta_7\theta_0^2 \cos 2\theta_0 + \beta_8\theta_0^2 \sin 2\theta_0. \\ X_1(\theta_0) &= \beta_5 \cos 2\theta_0 - 2\beta_5\theta_0 \sin 2\theta_0 + \beta_6 \sin 2\theta_0 + 2\beta_6\theta_0 \cos 2\theta_0 + 2\beta_7\theta_0 \cos 2\theta_0 \\ &\quad - 2\beta_7\theta_0^2 \sin 2\theta_0 + 2\beta_8\theta_0 \sin 2\theta_0 + 2\beta_8\theta_0^2 \cos 2\theta_0. \end{aligned}\right\} \tag{3.52}$$

Here, we notice that though we have used a different approach to find the solution for first inertial correction, but our results match with those obtained by Hancock et al. [5].

The velocity components $u^{(2)}$, $v^{(2)}$ and vorticity $\Omega^{(2)}$ in terms of $f_2(\theta)$, respectively, can be obtained by using Eqs. (3.43) and (3.47) in the following form:

$$u^{(2)} = \frac{1}{r}\frac{\partial(\frac{r^2U^2}{\nu}f_2)}{\partial\theta} = \frac{rU^2}{\nu}f_2', \quad v^{(2)} = -\frac{\partial(\frac{r^2U^2}{\nu}f_2)}{\partial r} = -\frac{2rU^2}{\nu}f_2 \tag{3.53}$$

and

$$\Omega^{(2)} = -\frac{U^2}{\nu}[4f_2 + f_2''] \tag{3.54}$$

The second order pressure field obtained from Eqs.(3.40) – (3.41), together with Eq. (3.42), then has the form

$$p^{(2)}(r,\theta) = \frac{\rho U^4 r^2}{\nu^2}\left[2f_2^2 + f_2 f_2'' - \frac{f_2'^2}{2}\right] + \frac{\mu U^2}{\nu}\ln r\,[4f_2' + f_2'''] \tag{3.55}$$

where f_2 is given by (3.51). Here we notice that the second order pressure field has a logarithmic singularity as $r \to 0$.

The normal and tangential stresses $T_n^{(2)}$ and $T_t^{(2)}$ are calculated as

$$T_n^{(2)} = -p^{(2)} - 2\frac{\mu U^2}{\nu}f_2', \tag{3.56}$$

$$T_t^{(2)} = \frac{\mu U^2}{\nu}f_2''. \tag{3.57}$$

3.5 Third Order System (The second inertial correction)

In order to obtain the second-order inertial effect analytically, recalling that the third order velocity and pressure fields for the problem under consideration have the form

$$\mathbf{V}^{(3)} = u^{(3)}(r,\theta)e_r + v^{(3)}(r,\theta)e_\theta, \quad p^{(3)} = p^{(3)}(r,\theta) \tag{3.58}$$

we write the equations of motion (3.17) and (3.20) in the plane polar coordinates form as

$$0 = \frac{\partial u^{(3)}}{\partial r} + \frac{u^{(3)}}{r} + \frac{1}{r}\frac{\partial v^{(3)}}{\partial\theta}, \tag{3.59}$$

$$\frac{\partial p^{*(3)}}{\partial r} = \rho\left[v^{(1)}\Omega^{(2)} + v^{(2)}\Omega^{(1)}\right] - \frac{\mu}{r}\frac{\partial\Omega^{(3)}}{\partial\theta}, \tag{3.60}$$

$$\frac{\partial p^{*(3)}}{\partial\theta} = \mu r\frac{\partial\Omega^{(3)}}{\partial\theta} - \rho r\left[u^{(1)}\Omega^{(2)} + u^{(2)}\Omega^{(1)}\right], \tag{3.61}$$

where

$$p^{*(3)} = p^{(3)} + \rho\left(u^{(1)}u^{(2)} + v^{(1)}v^{(2)}\right) \tag{3.62}$$

is the modified pressure.

Eliminating the pressure from Eqs. (3.60) and (3.61) by cross differentiation, and then introducing the third order stream function $\psi^{(3)}(r,\theta)$ such that

$$u^{(3)} = \frac{1}{r}\frac{\partial\psi^{(3)}}{\partial\theta}, \quad v^{(3)} = -\frac{\partial\psi^{(3)}}{\partial r}, \tag{3.63}$$

we get the following third order system of equations

$$\nu\nabla^4\psi^{(3)} = -\frac{1}{r}\left[\frac{\partial(\psi^{(1)},\nabla^2\psi^{(2)})}{\partial(r,\theta)} + \frac{\partial(\psi^{(2)},\nabla^2\psi^{(1)})}{\partial(r,\theta)}\right], \tag{3.64}$$

with boundary conditions

$$\psi^{(3)} = 0, \qquad \frac{1}{r}\frac{\partial\psi^{(3)}}{\partial\theta} = 0, \quad \text{at } \theta = 0 \tag{3.65}$$

$$\psi^{(3)} = 0, \qquad \frac{1}{r}\frac{\partial\psi^{(3)}}{\partial\theta} = 0, \text{ at } \quad \theta = \theta_0 \tag{3.66}$$

where ψ_1 and ψ_2 are already obtained, and their expressions are given in Eqs. (3.30) and (3.47).

3.6 Solution of the third order system

For the third order system (3.64), we assume the following form of the stream function

$$\psi^{(3)} = \frac{r^3U^3}{\nu^2}f_3(\theta) \tag{3.67}$$

Using (3.67) together with expressions for ψ_1 and ψ_2 from Eqs. (3.30) and (3.47), the linear partial differential equation given by Eq. (3.64), after a fair amount of calculations, is converted to the fourth order ordinary differential equation of the form

$$\begin{aligned} f_3^{iv} + 10f_3'' + 9f_3 = {} & M_1\cos3\theta + M_2\sin3\theta + M_3\cos\theta + M_4\sin\theta + M_5\theta\sin3\theta \\ & + M_6\theta\cos3\theta + M_7\theta\cos\theta + M_8\theta\sin\theta + M_9\theta^2\sin\theta \\ & + M_{10}\theta^2\cos\theta + M_{11}\theta^2\sin3\theta + M_{12}\theta^2\cos3\theta, \end{aligned} \tag{3.68}$$

where M's are known constants and their expressions are given in Eq. (3.72).

Also, the boundary conditions (3.65) – (3.66) are reduced to

$$f_3 = 0, \quad f_3' = 0 \quad \text{at} \quad \theta = 0 \tag{3.69}$$

$$f_3 = 0, \quad f_3' = 0 \quad \text{at} \quad \theta = \theta_0. \tag{3.70}$$

After a considerable amount of work, solution of the Eq. (3.68) using boundary conditions (3.69)– (3.70) is then obtained as

$$\begin{aligned} f_3(\theta) = {} & C_1\cos\theta + C_2\sin\theta + C_3\cos3\theta + C_4\sin3\theta + M_{13}\theta\cos3\theta + M_{14}\theta\sin3\theta \\ & + M_{15}\theta\cos\theta + M_{16}\theta\sin\theta + M_{17}\theta^2\sin\theta + M_{18}\theta^2\cos\theta + M_{19}\theta^2\sin3\theta + M_{20}\theta^2\cos3\theta, \end{aligned} \tag{3.71}$$

where constants C's and M's are given by

$$
\left.\begin{aligned}
C_1 &= \frac{Y_1(\theta_0)\sin\theta_0 - 3Y(\theta_0)\cos\theta_0 + (M_{13}+M_{15})\sin 2\theta_0}{4\sin^2\theta_0},\\
C_2 &= -\frac{3Y_1(\theta_0)\cos\theta_0\sin^2\theta_0 + 3Y(\theta_0)\sin\theta_0(1-3\cos^2\theta_0) + (M_{13}+M_{15})(\cos^2\theta_0 - 2\cos^4\theta_0 + 1)}{4\sin^4\theta_0},\\
C_3 &= -\frac{Y_1(\theta_0)\sin\theta_0 - 3Y(\theta_0)\cos\theta_0 + (M_{13}+M_{15})\sin 2\theta_0}{4\sin^2\theta_0},\\
C_4 &= \frac{Y_1(\theta_0)\cos\theta_0\sin^2\theta_0 + Y(\theta_0)\sin\theta_0(1-3\cos^2\theta_0) - (M_{13}+M_{15})(2\cos^4\theta_0 - 3\cos^2\theta_0 + 1)}{4\sin^4\theta_0}\\
B &= \frac{\theta_0^2}{\theta_0^2 - \sin^2\theta_0},\ \ C = -\frac{\sin^2\theta_0}{\theta_0^2-\sin^2\theta_0},\ \ D = -\frac{k}{\theta_0^2-\sin^2\theta_0}\\
M_1 &= 4CR_3 - 6B\beta_7 - C\beta_6 - D\beta_5 - 4B\beta_6 - 4DR_4,\\
M_2 &= 4B\beta_5 + C\beta_5 - 6B\beta_8 - D\beta_6 + 4CR_4 + 4DR_3,\\
M_3 &= 4B\beta_6 + C\beta_6 - D\beta_5 + 6B\beta_7 + 4CR_1 - 2DR_2\\
M_4 &= 6B\beta_8 - 4B\beta_5 - C\beta_5 - D\beta_6 - 4BR_2 + 2CR_2 + 4DR_1,\\
M_5 &= 8D\beta_5 + 8B\beta_7 + 8C\beta_6 + 8C\beta_7 - 8D\beta_8,\ M_6 = 8C\beta_5 - 8B\beta_8 - 8D\beta_6 - 8C\beta_8 - 8D\beta_7,\\
M_7 &= 4C\beta_5 + 8B\beta_8 + 4D\beta_6 - 4C\beta_8 + 4D\beta_7,\ M_8 = 4C\beta_6 - 4D\beta_5 - 8B\beta_7 + 4C\beta_7 + 4D\beta_8,\\
M_9 &= 8C\beta_8 - 8D\beta_7,\ M_{10} = 8C\beta_7 + 8D\beta_8,\ M_{11} = 12C\beta_8 + 12D\beta_7,\ M_{12} = 12C\beta_7 - 12D\beta_8,\\
M_{13} &= \frac{1}{48}\left(M_2 - \frac{11}{12}M_6\right),\ \ M_{14} = -\frac{1}{48}\left(M_1 + \frac{11}{12}M_5\right),\ \ M_{15} = \frac{1}{16}\left(\frac{1}{4}M_7 - M_4\right),\\
M_{16} &= \frac{1}{16}\left(M_3 + \frac{1}{4}M_8\right),\ \ M_{17} = \frac{1}{32}M_7,\ \ M_{18} = -\frac{1}{32}M_8,\ \ M_{19} = -\frac{1}{96}M_6,\ \ M_{20} = \frac{1}{96}M_5,\\
Y(\theta_0) &= M_{13}\theta_0\cos 3\theta_0 + M_{14}\theta_0\sin 3\theta_0 + M_{15}\theta_0\cos\theta_0 + M_{16}\theta_0\sin\theta_0 + M_{17}\theta_0^2\sin\theta_0 + M_{18}\theta_0^2\cos\theta_0\\
&\quad + M_{19}\theta_0^2\sin 3\theta_0 + M_{20}\theta_0^2\cos 3\theta_0,\\
Y_1(\theta_0) &= M_{13}\cos 3\theta_0 - 3M_{13}\theta_0\sin 3\theta_0 + M_{14}\sin 3\theta_0 + 3M_{14}\theta_0\cos 3\theta_0 + M_{15}\cos\theta_0 - M_{15}\theta_0\sin\theta_0\\
&\quad + M_{16}\sin\theta_0 + M_{16}\theta_0\cos\theta_0 + 2M_{17}\theta_0\sin\theta_0 + +M_{17}\theta_0^2\cos\theta_0 + +2M_{18}\theta_0\cos\theta_0 - M_{18}\theta_0^2\sin\theta_0\\
&\quad + +2M_{19}\theta_0\sin 3\theta_0 + 3M_{19}\theta_0^2\cos 3\theta_0 + 2M_{20}\theta_0\cos 3\theta_0 - 3M_{20}\theta_0^2\sin 3\theta_0.
\end{aligned}\right\}
\tag{3.72}
$$

The velocity components $u^{(3)}$, $v^{(3)}$ and vorticity $\Omega^{(3)}$ in terms of $f_3(\theta)$, respectively, can be obtained in following forms:

$$
u^{(3)} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{r^3U^3}{\nu^2}f_3\right) = \frac{r^2U^3}{\nu^2}f_3',\qquad v^{(3)} = -\frac{\partial}{\partial r}\left(\frac{r^3U^3}{\nu^2}f_3\right) = -\frac{3r^2U^3}{\nu^2}f_3,
\tag{3.73}
$$

and

$$
\Omega^{(3)} = -\frac{rU^3}{\nu^2}[9f_3 + f_3''],
\tag{3.74}
$$

where f_3 is given by Eq. (3.71).

The pressure field obtained from Eq.(3.62) together with Eqs.(3.60)– (3.61), has the form

$$p^{(3)}(r,\theta) = \frac{\mu r U^3}{\nu^2}\left[9f_3' + f_3'''\right] + \frac{\rho r U^3}{\nu}\left[4f_1 f_2 + f_1 f_2'' - f_1' f_2' + 2f_1'' f_2\right], \quad (3.75)$$

where f_1, f_2, and f_3 are given by Eqs. (3.31), (3.51), and (3.71) respectively.

Here we notice that unlike the first and second-order pressure fields, which have singularities at $r = 0$, the pressure field found here is defined for all r.

Finally, the normal and tangential stresses, $T_n^{(3)}$ and $T_t^{(3)}$ to the scraper are obtained as

$$T_n^{(3)} = -p^{(3)} - 4\frac{\mu r U^3}{\nu^2} f_3', \quad (3.76)$$

and

$$T_t^{(3)} = \frac{\mu r U^3}{\nu^2}\left(f_3'' - 3f_3\right). \quad (3.77)$$

4 Combined solution for the inertial flow

In this section, we combine the expressions of stream functions, velocity components, pressure fields and, normal and shear stresses for all the orders found in previous sections, to give the combined solution for the inertial flow.

Combining the expressions for $\psi^{(i)}$'s, $u^{(i)}$'s, $v^{(i)}$'s, $p^{(i)}$'s, $T_t^{(i)}$'s, and $T_n^{(i)}$'s, $(1 \leq i \leq 3)$, found in previous sections we get

$$\psi = \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)} + \epsilon^3\psi^{(3)}, \quad (4.1)$$

$$u = \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \epsilon^3 u^{(3)}, \quad (4.2)$$

$$v = \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \epsilon^3 v^{(3)}, \quad (4.3)$$

$$p = \text{constant} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} + \epsilon^3 p^{(3)}, \quad (4.4)$$

$$T_t = \epsilon T_t^{(1)} + \epsilon^2 T_t^{(2)} + \epsilon^3 T_t^{(3)}, \quad (4.5)$$

$$T_n = \epsilon T_n^{(1)} + \epsilon^2 T_n^{(2)} + \epsilon^3 T_n^{(3)}. \quad (4.6)$$

Taking $\epsilon \to 1$, and making use of Eqs. (3.30), (3.32), (3.35), (3.36), (3.37), (3.47), (3.53), (3.55), (3.56), (3.57), (3.67), (3.73), (3.75), (3.76), and (3.77) we get

$$\psi = Urf_1 + \frac{U^2 r^2}{\nu} f_2 + \frac{U^3 r^3}{\nu^2} f_3, \quad (4.7)$$

$$u = Uf_1' + \frac{rU^2}{\nu} f_2' + \frac{r^2 U^3}{\nu^2} f_3', \quad (4.8)$$

$$v = -(Uf_1 + 2\frac{rU^2}{\nu} f_2 + 3\frac{r^2 U^3}{\nu^2} f_3), \quad (4.9)$$

$$p = p_0 - \frac{\mu}{r} U g' + \frac{\rho U^4 r^2}{\nu^2} \left[2f_2^2 + f_2 f_2'' - \frac{f_2'^2}{2} \right] + \frac{\mu U^2}{\nu} \ln r \left[4f_2' + f_2''' \right]$$
$$+ \frac{\mu r U^3}{\nu^2} \left[9f_3' + f_3''' \right] + \frac{\rho r U^3}{\nu} \left[4f_1 f_2 + f_1 f_2'' - f_1' f_2' + 2f_1'' f_2 \right], \quad (4.10)$$

$$T_t = -p - 2\mu \left(\frac{U^2}{\nu} f_2' + 2\frac{U^3}{\nu^2} r f_3' \right), \quad (4.11)$$

$$T_n = \mu \left(\frac{U}{r} g + \frac{U^2}{\nu} f_2'' + \frac{U^3}{\nu^2} r \left[f_3'' - 3f_3 \right] \right). \quad (4.12)$$

where f_1, g, f_2, and f_3 are respectively given by Eqs. (3.31), (3.34), (3.51) and (3.71).

Here we notice that by taking $\nu^{-1} = \frac{\rho}{\mu} = 0$, we recover the results for non-inertial flow obtained by Taylor.

5 Discussion on Normal and Shear stresses

Let $T_{\parallel}$ and $T_{\perp}$ denote respectively, the components of the total stress parallel and perpendicular to the plate. Then $T_{\perp}$ and $T_{\parallel}$ are given by

$$T_{\perp} = T_n \cos\theta_0 + T_t \sin\theta_0, \quad (5.1)$$

$$T_{\parallel} = T_n \sin\theta_0 - T_t \cos\theta_0. \quad (5.2)$$

where tangential and normal stresses, T_t and T_n respectively, are given by Eqs. (4.11) and (4.12). We now construct tables of values to see the effects of $T_n, T_t, T_{\perp}$ and $T_{\parallel}$ for a set of values of angle θ_0 for both non-inertial and inertial flows.

Table 1

θ_0^o	$\frac{T_n r}{2\mu U}$	$\frac{T_t r}{2\mu U}$	$\frac{T_\perp r}{2\mu U}$	$\frac{T_\parallel r}{2\mu U}$
0	∞	∞	∞	∞
15	43.67	3.83	43.17	7.6
30	10.84	1.93	10.35	3.75
45	4.75	1.3	4.28	2.44
60	2.62	0.99	2.17	1.77
75	1.62	0.80	1.19	1.36
90	1.07	0.68	0.68	1.07
105	0.73	0.59	0.38	0.86
120	0.50	0.53	0.21	0.70
135	0.33	0.47	0.10	0.57
150	0.20	0.42	0.04	0.46
165	0.09	0.37	0.01	0.38
180	0	0.32	0	0.32

Table 2

θ_0^o	$\frac{T_n r}{2\mu U}$	$\frac{T_t r}{2\mu U}$	$\frac{T_\perp r}{2\mu U}$	$\frac{T_\parallel r}{2\mu U}$
0	∞	∞	∞	∞
15	167.67	2.39	162.58	41.09
30	42.13	1.16	37.07	20.06
45	18.85	0.73	13.85	12.81
60	10.67	0.5	5.77	8.99
75	6.82	0.35	2.1	6.5
90	4.67	0.23	0.23	4.67
105	3.3	0.13	-0.73	3.22
120	2.32	0.04	-1.13	2.03
135	1.67	0.01	-1.17	1.19
150	2.76	0.38	-2.2	1.71
165	58.03	7.36	-54.15	22.13
180	∞	∞	∞	∞

Table 1 and 2 respectively give the values of T_n, T_t, $T_\perp$, and $T_\parallel$, divided by $2\mu U/r$ for a set of different values of θ_0 for non-inertial and inertial flows. We can see that for non-inertial flow, $T_\parallel$ decreases with an increase in θ_0, and attains its least value when $\theta_0 = \pi$, but the least value of $T_\parallel$ for inertial flow occurs at $\theta_0 = 5\pi/8$. It is interesting to observe that for non-inertial flow, $T_\perp$ does not change sign in the range $0 < \theta_0 < \pi$. But for the inertial flow, we can see that $T_\perp$ changes sign in the range $7\pi/12 < \theta_0 < 11\pi/12$. We further see that for non-inertial flow, the only singularity in the stress filed occurs at $\theta_0 = 0$, whereas we have two such points for inertial flow, i.e at $\theta_0 = 0$, and at $\theta = \pi$. This means that we cannot scrap the fluid for inertial flow if the scraper is held at these two positions, as an infinite force will be required there. A human intuitively holds the scraper in this position to scrape the fluid easily. But this type of analysis is necessary for the engineering point of view, because for doing the work on bulk basis, we need machines like robots, bulldozers etc. for the scraping purposes.

6 Graphs and Discussion

In this section some graphs are displayed for the behavior of velocity, stresses, and streamlines for both inertial and non-inertial flows. Figs.(6.1) and (6.2) presents the behavior of velocity components u and v against the radial distance r for the inertial flow. It is observed that the effect of the inertial forces near the corner is small and the rise in the velocity components is less there. While as we recede the corner, the contributions of the inertia forces comes into play and consequently, and a much rise is seen in the velocity graphs. Physically it is expected because the boundaries are close to each other near the corner, and hence viscous forces dominate there. Therefore, for low Reynolds number, the flow starts to creep and so velocity decreases. In mathematical analysis, we noted that the velocity components for the non-inertial flow become independent of the radial distance, so no change is noticed in the velocity against the radial distance for the non-inertial flow. However, the velocity components u and v are drawn against the velocity of the moving boundary in Figs. (6.3) and (6.4). And it is observed that, as expected, both the components increase in the negative direction with the boundary velocity U.

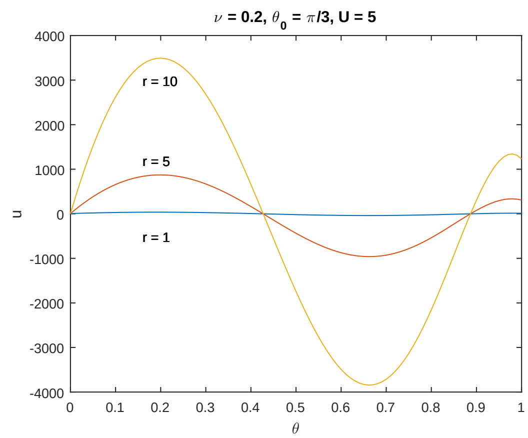

Figure 6.1: variation of velocity component u with radial distance r for inertial flow.

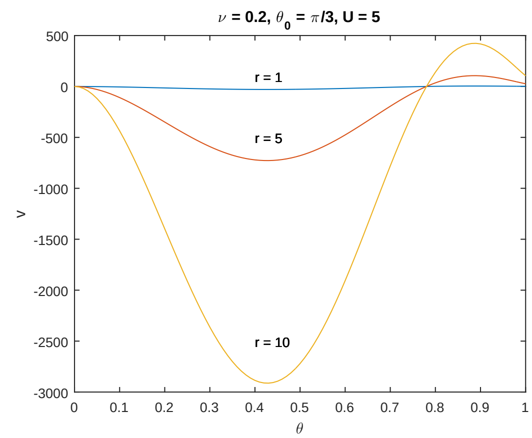

Figure 6.2: variation of velocity component v vs radial distance r for inertial flow.

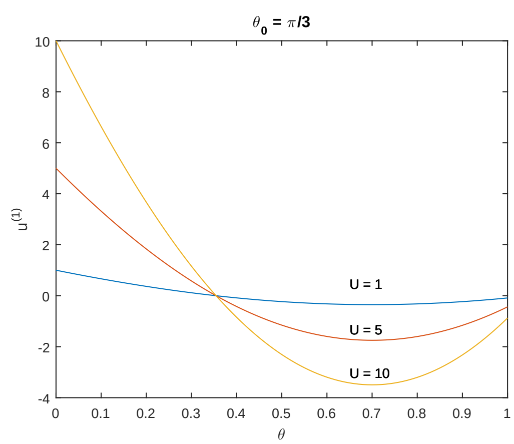

Figure 6.3: variation of velocity component $u^{(1)}$ with U for non-inertial flow.

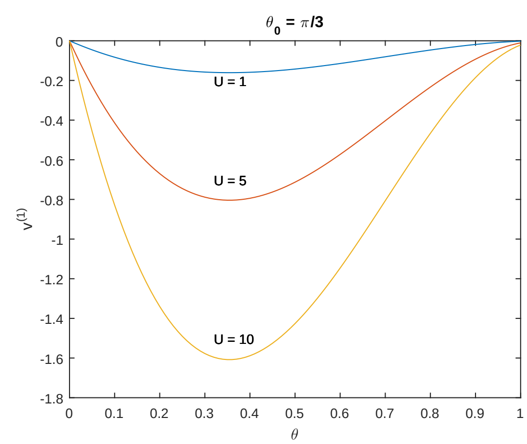

Figure 6.4: variation of velocity component $v^{(1)}$ with U for non- inertial flow.

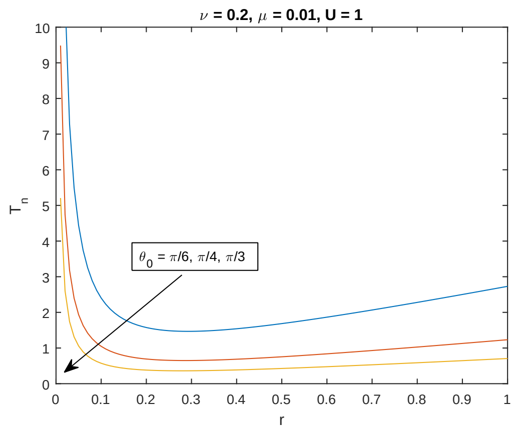

Figure 6.5: normal stress vs angle θ_0 for inertial flow at fixed boundary.

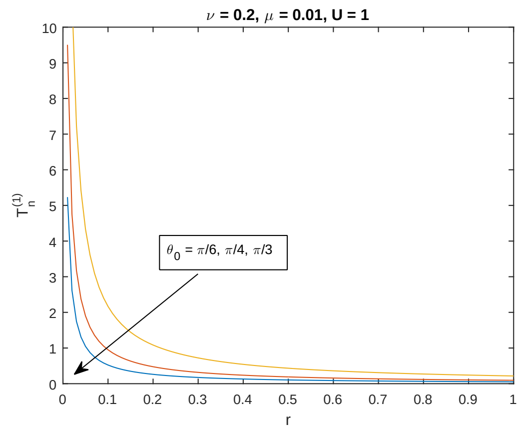

Figure 6.6: normal stress vs angle θ_0 for non-inertial flow at fixed boundary.

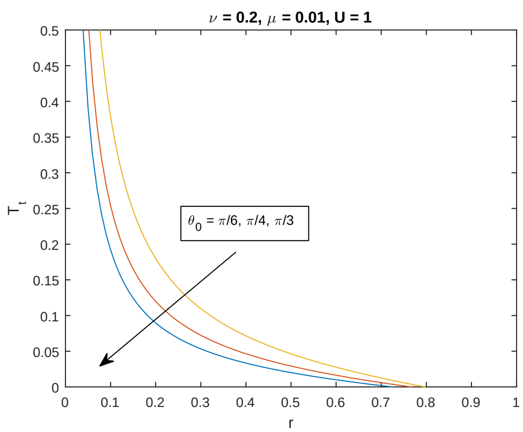

Figure 6.7: tangential stress vs angle θ_0 for inertial flow at fixed boundary.

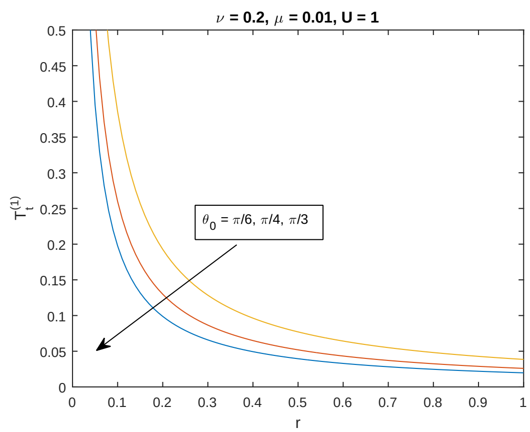

Figure 6.8: tangential stress vs angle θ_0 for non-inertial flow at fixed boundary.

In Figs. (6.5) and (6.7) normal and tangential stresses at the fixed boundary for the inertial flow are sketched against angle θ_0. Here we see that stresses decrease as we increase θ_0. It is further noticed that very close to the corner i.e for $r \to 0$, stresses shoots up very rapidly. It is a good agreement with mathematical analysis, because there exist a singularity at this point in the stress field. In Figs. (6.6) and (6.8) similar behavior has been seen for tangential and normal stresses for non-inertial flow, but the difference is that, for non-inertial flow, values of tangential stresses are higher whereas the values of normal stresses are smaller than those for non-inertial flow, for the similar kind of data.

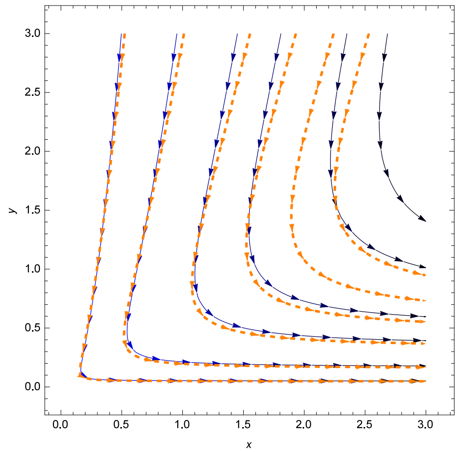

Figure 6.9: streamlines with $\theta_0 = \pi/2$; solid curves are streamlines of non-inertial flow ψ_1 = constant; dashed lines are the streamlines (including the first inertial correction) $\psi_1 + \psi_2$=constant.

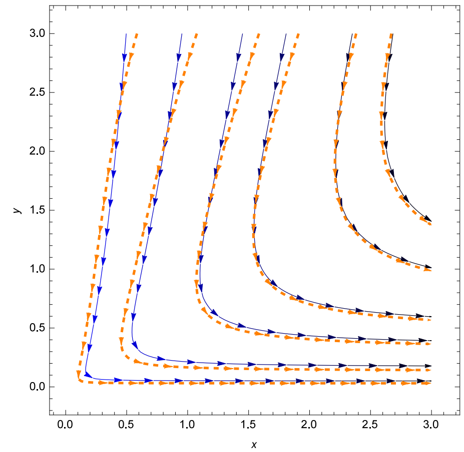

Figure 6.10: streamlines with $\theta_0 = \pi/2$; solid curves are streamlines ψ_1=constant of non-inertial flow; dashed lines are the streamlines $\psi_1 + \psi_2 + \psi_3$ = constant, including the first and second inertial corrections.

Finally, Figs. (6.9) and (6.10) respectively compare the streamlines $\psi_1 =$ *constant* (solid curves) for non-inertial flow, and streamlines $\psi_1 + \psi_2$=constant (dashed lines) including the first inertial correction, and $\psi_1+\psi_2+\psi_3$ = constant (dashed lines) including the first and second inertial corrections. We observe that the streamlines tend to get compressed towards the fixed boundary $\theta = \theta_0$, under the influence of inertia.

7 Conclusion

In the present work, corner flow of a viscous fluid in the presence of inertia has been analyzed, and the resulting nonlinear partial differential equations are solved to obtain the expressions for the velocity field, pressure field and stress field. By the comparison of inertial and non-inertial flows, we came to the conclusion that close to the corner the effects of inertia can be neglected. But as we move away, inertia forces dominate and cannot be ignored, and this analysis is an attempt to throw some light on the importance of inertia forces for such flows.

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Received: January 2, 2017; Published: February 14, 2017