Buffon-Laplace Type Problem for a Lattice with Cell Composed by Four Triangles and a Rhombus

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Abstract
In this paper we study a Buffon-Laplace type problem for a lattice with a cell composed by four triangles and a rhombus. In other words, we compute the probability that a segment of a random position and of constant length intersects a side of the lattice.

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1 Introduction
Poincaré [6] and Stoka [7] have obtained the fundamental results for the most important problems of geometric problems and in particular the Buffon-Laplace
type problems. In the recent years various authors have considered several Buffon-Lapalce type problems for particular fundamental cells [1], [2], [3], [4] and [5]. Starting from these results, in this paper we consider as fundamental cell a lattice composed by four isosceles triangles and a rhombus and the Laplace type problem was solved, considering the geometric point of view. We computed the probability that a random segment of constant length intersects the fundamental cell represented in fig. 1.

2 Main Results

Let $\mathcal{R}(a, b; \alpha)$ be the lattice with fundamental cell $C_0$ represented in fig. 1

\[ |EF| = |FG| = |GH| = |HE| = \frac{1}{2} \sqrt{a^2 + b^2}; \quad (1) \]

\[ \overline{FEH} = 2\alpha, \quad \sin \alpha = \frac{a}{\sqrt{a^2 + b^2}}, \quad \cos \alpha = \frac{b}{\sqrt{a^2 + b^2}}; \quad (2) \]

\[ areaC_{01} = areaC_{02} = areaC_{03} = areaC_{04} = \frac{ab}{8}, \quad areaC_{05} = \frac{ab}{2}. \quad (3) \]

We want to compute the probability that a segment $s$ with random position and of constant length $l < \frac{a}{2}$ intersects a side of lattice $\mathcal{R}$, i.e. the probability $P_{\text{int}}$ that $s$ intersects a side of the fundamental cell $C_0$. 
The position of the segment $s$ is determined by its centre and by the angle $\varphi$ formed with the line $BC$ (or $AD$).

To compute the probability $P_{int}$ we considered the limiting positions of segment $s$, for a specified value of $\varphi$, in the cells $C_{0i}$ ($i = 1, 2, ..., 5$).

Thus we have fig. 2

![Diagram](image_url)

and the relationships

$$area \hat{C}_{01}(\varphi) = area \hat{C}_{03}(\varphi) = area C_{01} - \sum_{i=1}^{5} area a_i(\varphi),$$

(4)

$$area \hat{C}_{02}(\varphi) = area \hat{C}_{04}(\varphi) = area C_{02} - \sum_{i=1}^{5} area b_i(\varphi),$$

(5)

$$area \hat{C}_{05}(\varphi) = area C_{05} - \sum_{i=1}^{6} area c_i(\varphi).$$

(6)

By fig.2 we have:

$$area a_1(\varphi) = \frac{l^2}{4} \sin 2\varphi,$$

$$area a_2(\varphi) = \frac{al}{4} \cos \varphi - \frac{l^2}{4} \sin 2\varphi,$$

$$area a_4(\varphi) = \frac{l^2 \sin \varphi \sin(\varphi - \alpha)}{2 \sin \alpha},$$
\[
\text{areaa}_5 (\varphi) = \frac{bl}{4} \sin \varphi - \frac{l^2}{4} \sin 2\varphi - \frac{l^2}{2a} (l \sin \varphi - a \cos \varphi),
\]
\[
\text{areaa}_3 (\varphi) = \frac{l\sqrt{a^2 + b^2}}{4} \sin (\varphi - \alpha) - \frac{l^2 \sin \varphi \sin (\varphi - \alpha)}{2 \sin \alpha}.
\]
\[
A_1 (\varphi) = \sum_{i=1}^{5} \text{areaa}_i (\varphi) = \frac{bl}{2} \sin \varphi - \frac{l^2}{4} \sin 2\varphi - \frac{l^2}{2a} (b \sin \varphi - a \cos \varphi).
\]

For \(\text{areaA}_{02} (\varphi)\) we have:
\[
\text{areab}_1 (\varphi) = \frac{l^2 \cos \varphi \sin (\varphi + \alpha)}{2 \cos \alpha},
\]
\[
\text{areab}_2 (\varphi) = \frac{al}{4} \cos \varphi - \frac{l^2 \cos \varphi \sin (\varphi + \alpha)}{2 \cos \alpha},
\]
\[
\text{areab}_4 (\varphi) = \frac{l^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha},
\]
\[
\text{areab}_3 (\varphi) = \frac{bl}{4} \sin \varphi - \frac{l^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha},
\]
\[
\text{areab}_5 (\varphi) = \frac{l}{4} (b \sin \varphi + a \cos \varphi) - \frac{l^2}{4} \left[ 2 \sin 2\varphi + \left( \frac{a}{b} - \frac{b}{a} \right) \cos 2\varphi + \frac{a}{b} + \frac{b}{a} \right].
\]

We have:
\[
A_2 (\varphi) = \sum_{i=1}^{5} \text{areab}_i (\varphi) = \frac{l}{2} (a \cos \varphi + b \sin \varphi) - \frac{l^2}{4} \left[ 2 \sin 2\varphi + \left( \frac{a}{b} - \frac{b}{a} \right) \cos 2\varphi + \frac{a}{b} + \frac{b}{a} \right].
\]  

(7)

For \(\text{areaAC}_{03} (\varphi)\) we have:
\[
\text{areac}_1 (\varphi) = \frac{l^2 \sin (\varphi + \alpha) \sin (\varphi + 3\alpha)}{2 \sin 2\alpha},
\]
\[
\text{areac}_6 (\varphi) = \frac{l\sqrt{a^2 + b^2}}{4} \sin (\varphi + 3\alpha) - \frac{l^2 \sin (\varphi + \alpha) \sin (\varphi + 3\alpha)}{2 \sin 2\alpha},
\]
areac\(_2\) (\(\varphi\)) = \frac{l\sqrt{a^2 + b^2}}{4} \sin (\varphi + \alpha) - \frac{l^2}{2 \sin 2\alpha} \sin (\varphi + \alpha) \sin (\varphi + 3\alpha),

areac\(_3\) (\(\varphi\)) = \frac{l\sqrt{a^2 + b^2}}{4} \sin (\varphi + 3\alpha) - \frac{l^2 \sin (\varphi + \alpha) \sin (\varphi + 3\alpha)}{2 \sin 2\alpha},

areac\(_4\) (\(\varphi\)) = \frac{l^2 \sin (\varphi + \alpha) \sin (\varphi + 3\alpha)}{2 \sin 2\alpha}.

We have:

\[
A_5 (\varphi) = \sum_{i=1}^{6} \text{areac}_i (\varphi) = \frac{l}{2} (b \sin \varphi + a \cos \varphi) + \frac{l}{2 (a^2 + b^2)} \left[ b \left( b^2 - 3a^2 \right) \sin \varphi + a \left( 3b^2 - a^2 \right) \cos \varphi \right] - \frac{l^2}{2ab (a^2 + b^2)} \left[ 2ab \left( b^2 - a^2 \right) \sin 2\varphi + \frac{1}{2} \left( 6a^2b^2 - a^4 - b^4 \right) \cos 2\varphi + \frac{1}{2} \left( b^4 - a^4 \right) \right].
\]

Denoting by \(M_i (i = 1, 2, ..., 5)\) the set of segments \(s\) which have their centre in \(C_{0i}\) and with \(N_i\) the set of segments \(s\) contained in the cell \(C_{0i}\), we have (cf. [7]):

\[
P_{\text{int}} = 1 - \sum_{i=1}^{5} \frac{\mu (N_i)}{\sum_{i=1}^{5} \mu (M_i)}, \tag{8}
\]

where \(\mu\) is the Lebesgue measure in the Euclidean plane.

To compute the above measure \(\mu (M_i)\) and \(\mu (N_i)\) we used the Poincaré kinematic measure (cf. [6]):

\[
dk = dx \wedge dy \wedge d\varphi,
\]

where \(x, y\) are the coordinates of centre of \(s\) and \(\varphi\) the fixed angle.

By fig. 2 we have that for the cells \(C_{01}\) and \(C_{03}\) that \(\varphi \in [0, \alpha]\), for the cells \(C_{02}\) and \(C_{04}\) we have \(\varphi \in \left[ 0, \frac{\pi}{2} \right]\) and for the cell \(C_{05}\) we have \(\varphi \in [\alpha, \pi - \alpha]\).

Considering formula (3) we have

\[
\mu (M_1) = \mu (M_3) = \int_{0}^{\alpha} d\varphi \int_{\{(x,y)\in C_{01}\}} dxdy =
\]
\[
\int_0^{\alpha} (\text{area} C_{01}) \, d\varphi = \alpha \text{area} C_{01} = \frac{\alpha ab}{8},
\]

\[
\mu (M_2) = \mu (M_4) = \int_0^{\frac{\pi}{2}} \int \int_{\{(x,y) \in C_{02}\}} dxdy = 
\]

\[
\int_0^{\frac{\pi}{2}} (\text{area} C_{02}) \, d\varphi = \frac{\pi}{2} \text{area} C_{02} = \frac{\pi ab}{16},
\]

\[
\mu (M_5) = \int_{\alpha}^{\pi-\alpha} d\varphi \int \int_{\{(x,y) \in C_{05}\}} dxdy = 
\]

\[
\int_{\alpha}^{\pi-\alpha} (\text{area} C_{05}) \, d\varphi = (\pi - 2\alpha) \text{area} C_{05} = \left(\frac{\pi}{2} - \alpha\right) ab.
\]

Then

\[
\sum_{i=1}^{5} \mu (M_i) = (5\pi - 6\alpha) \frac{ab}{8}. \tag{9}
\]

\[
\mu (N_1) = \mu (N_3) = \int_0^{\alpha} d\varphi \int \int_{\{(x,y) \in \hat{C}_{01}\}} dxdy = 
\]

\[
\int_0^{\alpha} \left[ \text{area} \hat{C}_{01} (\varphi) \right] \, d\varphi = 
\]

\[
\int_0^{\alpha} [\text{area} C_{01} - A_1 (\varphi)] \, d\varphi = \frac{\alpha ab}{8} - \int_0^{\alpha} [A_1 (\varphi)] \, d\varphi,
\]

\[
\mu (N_2) = \mu (N_4) = \int_0^{\frac{\pi}{2}} \int \int_{\{(x,y) \in \hat{C}_{02}\}} dxdy = 
\]

\[
\int_0^{\frac{\pi}{2}} [\text{area} \hat{C}_{02} (\varphi)] \, d\varphi = \int_0^{\frac{\pi}{2}} [\text{area} C_{02} - A_2 (\varphi)] \, d\varphi = 
\]

\[
\frac{\pi ab}{16} - \int_0^{\frac{\pi}{2}} [A_2 (\varphi)] \, d\varphi,
\]

\[
\mu (N_5) = \int_{\alpha}^{\pi-\alpha} d\varphi \int \int_{\{(x,y) \in \hat{C}_{05}\}} dxdy = 
\]

\[
\int_{\alpha}^{\pi-\alpha} [\text{area} \hat{C}_{05} (\varphi)] \, d\varphi = \int_{\alpha}^{\pi-\alpha} [\text{area} C_{05} - A_5 (\varphi)] \, d\varphi = 
\]

\[
\left(\frac{\pi}{2} - \alpha\right) ab - \int_{\alpha}^{\pi-\alpha} [A_5 (\varphi)] \, d\varphi.
\]
Replacing we have

\[ \sum_{i=1}^{5} \mu(N_i) = \frac{(5\pi - 6\alpha)}{2} \left[ a + 2b - \frac{b(3a^2 - b^2)}{a^2 + b^2} \cos \alpha \right] \]

\[ l^2 \left[ \sin \alpha + \frac{b}{a} (\cos \alpha - 1) - \frac{3}{4} - \frac{1}{4} \cos 2\alpha - \frac{\pi (a^2 + b^2)}{4ab} + \right. \]

\[ \left. \frac{(6a^2b^2 - a^4 - b^4) \sin 2\alpha + 2(a^4 - b^4)}{8ab(a^2 + b^2)} \right]. \quad (10) \]

The formulas (8), (9), and (10) give

\[ P_{int} = \frac{8}{(5\pi - 6\alpha) ab} \left\{ l \left[ a + 2b - \frac{b(3a^2 - b^2)}{a^2 + b^2} \cos \alpha \right] + \right. \]

\[ \left. l^2 \left[ \sin \alpha + \frac{b}{a} (\cos \alpha - 1) - \frac{3}{4} - \frac{1}{4} \cos 2\alpha - \frac{\pi (a^2 + b^2)}{4ab} + \right. \right. \]

\[ \left. \left. \frac{(6a^2b^2 - a^4 - b^4) \sin 2\alpha + 2(a^4 - b^4)}{8ab(a^2 + b^2)} \right\} \right\}, \]

where \( \alpha = \arctg \frac{a}{b} \).

In particular if \( a = b \), \( \alpha = \frac{\pi}{4} \) hence the probability \( P_{int} \) become

\[ P_{int} = \frac{8}{7\pi} \left[ (6 + \sqrt{2}) \frac{l}{a} - (3 - 2\sqrt{2} + \pi) \left( \frac{l}{a} \right)^2 \right]. \]

References


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