Decrease of Degrees for Polynomials

 Obtained from Sums of the Same Powers

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Abstract

We show that the polynomial received as a result of transformation of the initial sum of the same powers, in certain cases becomes simpler in comparison with the well-known Faulhaber’s formula. Any of these cases is defined as follows: the number of terms of the initial sum is smaller its power parameter. Computational illustration of this statement is performed.

Keywords:  sums of the same powers, Stirling numbers, falling factorials, polynomials with non-standard and standard representation

1 Introduction

We will accept known combinatorial representation of the sum of the same powers [5] as the initial formula. The outcome of its transformation will be expressed by means of falling factorials and Stirling numbers. This outcome will be alternative to known representation of the sum of the same powers as to Faulhaber’s formula [2].

At first we remind the known formula of combinatorial expression of the sum of unweighted same powers [5]:

\[
\Phi(p, \nu) = \sum_{l=1}^{p} \nu^l = \sum_{l=1}^{\max(l)} \left( \sum_{q=1}^{l} (-1)^{l-q} C_q^l q^\nu \right) \left( \sum_{t=0}^{p-l} \right),
\]

\[
\max l = \min(p, \nu), \quad \nu, p \in \mathbb{N}.
\]
Expression in the brackets, occupying extreme right position in (1), can be transformed to the following aspect:

\[ \sum_{t=0}^{p-l} \binom{p}{p-t} = \binom{p+1}{p+1}. \] (2)

As it is known, the Stirling number of the second kind is the quantity

\[ S(v, l) = \frac{1}{l!} \sum_{q=1}^{l} (-1)^{l+q} C_q^q q^v. \] (3)

Formulas (2), (3) can be introduced into expression (1):

\[ \Phi(p, v) = \sum_{l=1}^{\max} (l!) S(v, l) \binom{l+1}{p+1}. \] (4)

Now we transform magnitude (4):

\[ \Phi(p, v) = \sum_{l=1}^{\max} S(v, l) \frac{1}{l+1} (p+1)(p+1), \] (5)

\((p)_l\) — number of arrangements [2], or a falling factorial [4];

\[ (p)_l = p(p-1)...(p-l+1) = \frac{p!}{(p-l)!}. \]

Thus receipt of magnitude (5) required to use expression

\[ \binom{l+1}{p+1} C_l = \frac{1}{l+1} (p+1)(p+1). \] (5)

Faulhaber’s formula [2] represents a standard form of a polynomial in \( p \) [1, 3] and expresses respectively the sum of powers of the whole positive numbers with multipliers. These multipliers are products of binomial factors on Bernoulli numbers [2]. Degree of a considered Faulhaber’s polynomial which we designate through \( F_{\nu+1} \), is equal to number \( \nu +1 \):
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$$F_{\nu+1} = \frac{1}{\nu+1} \sum_{g=1}^{\nu+1} C_{\nu+1}^{g-1} B_{\nu+1-g} p^g;$$

$B_{\nu+1-g}$ - Bernoulli number.

But according to expression (5) at any $\max l = p < \nu$ degree for falling factorial $(p+1)(p)_l$ from the given expression matters $\max l + 1 = p + 1 < \nu + 1$. It, at first, explains the power-law decrease in the polynomial representation of the initial sum of the same powers for cases $p < \nu$ and, secondly, shows the extent of the range of the lowered values $\max l$.

The purpose of this article is to establish the fact connected with decrease of degree of polynomial obtained by converting the sum of the same powers from non-standard to standard representation.

**2 Main results of researches**

Let’s introduce polynomial notation $P_{\max l+1}(p, \nu) = P_{\max l+1}$, corresponding to expression (5):

$$\Phi(p, \nu) = P_{\max l+1} = \sum_{l=1}^{\max l} \theta_l(p, \nu);$$

$$\theta_l(p, \nu) = S(\nu, l) \frac{1}{l+1} (p+1)(p)_l.$$  

Allocation of powers of the base number $p$ from product $(p+1)(p)_l$ allows to receive the following:

$$(p+1)_{l+1} = \sum_{m=1}^{l+1} a(l, m) p^m,$$

$$(6) a(l, m) = (-1)^{m-l} s(l, m-1) + (-1)^{m+l} s(l, m),$$

$s(l, m)$ — signless Stirling numbers of the first kind; $s(l, 0) = s(l, l+1) = 0$.

We can discover Stirling numbers according to well-known modes [2]. Expressions (5), (6) already define transformation of initial sums of the type (1) to equivalent polynomials with maximum power parameters ($\max l + 1$).

Now, to facilitate addition of similar members in these polynomials, we change the order of the arrangement of their parts:
Formula (7) ensures completion of transition to standard representation of the initial sum (1).

Further we will need to use difference expression

$$\Delta P( p + 1, \nu + 1) = P_{\nu + 1} - P_{p + 1},$$

where

$$P_{p + 1} = \sum_{l=1}^{p} \theta_l( p, \nu) = \sum_{m=1}^{p+1} \sum_{l=m-1}^{p} \varphi_l( m, \nu),$$

$$P_{\nu + 1} = \sum_{l=1}^{\nu} \theta_l( p, \nu) = \sum_{m=1}^{\nu+1} \sum_{l=m-1}^{\nu} \varphi_l( m, \nu).$$

We use introduced and transformed notation $P_{\text{max} + 1}$ to formulate the following statement.

**Theorem 2.1.** For numerical range $p < \nu$ the identity is fair

$$\Delta P( p + 1, \nu + 1) \equiv 0.$$  \hfill (8)

**Proof.** Really, according to expression (7) in the given numerical range the equality is observed

$$\Phi( p < \nu, \nu) = P_{p + 1} ( p < \nu).$$

However, on the other hand [2],

$$\Phi( p < \nu, \nu) = F_{\nu + 1}.$$

And, obviously, we have in this numerical range:

$$F_{\nu + 1} = P_{\nu + 1} \Rightarrow P_{p + 1} ( p, \nu) = P_{\nu + 1}.$$

From the given expression follows that

$$P_{p + 1} ( p < \nu) + \Delta P( p + 1, \nu + 1) \equiv P_{\nu + 1} \Rightarrow \Delta P( p + 1, \nu + 1) \equiv 0.$$
By means of the proved identity (8) we can formulate such statement.

**Theorem 2.2.** For numerical range \( p < \nu \) identities are fair

\[
\Delta P(\nu - k + 1, \nu - k + 2) = P_{\nu - k + 2} - P_{\nu - k + 1} \equiv 0, \quad k = 1, \ldots, \nu - p.
\]

**Proof.** Let’s prove this Theorem 2.2 by means of an inductive method. The statement of the Theorem 2.1 for \( k = 1 \)

\[
\Delta P|_{k=1} = \Delta P(\nu, \nu + 1) = P_{\nu + 1} - P_{\nu} \equiv 0
\]

is fair, as it here accurately corresponds to our Theorem 2.1.

The following step of an induction is that. We suppose that for arbitrary \( 1 < k < \nu - p \) the system of identities takes place

\[
\Delta P|_{k=2} \equiv 0, \\
\ldots \ldots \ldots \ldots \\
\Delta P|_{k=\nu - p - 1} \equiv 0.
\]

Then

\[
\Delta P|_{k=1} + \Delta P|_{k=2} + \ldots + \Delta P|_{k=\nu - p - 1} = \\
\Delta P(\nu - (\nu - p - 1) + 1, \nu - 1 + 2) = \\
\Delta P(p + 2, \nu + 1) = P_{\nu + 1} - P_{p+2} \equiv 0.
\]

On the other hand, we receive from the Theorem 2.1 for \( k = \nu - p \)

\[
k = \nu - p : \quad \Delta P(\nu - (p + 1), \nu + 1) \equiv P_{\nu + 1} - P_{p+1} \equiv 0.
\]

The difference of identities (9), (10) gives

\[
\forall p < \nu : \quad \Delta P(p + 1, p + 2) = P_{p+2} - P_{p+1} \equiv 0.
\]

In given numerical domain the identities

\[
\left( \Delta P|_{k=1} \equiv 0, \ldots, \Delta P|_{k=\nu - p - 1} \equiv 0 \right), \Delta P|_{k=\nu - p} \equiv 0
\]

are always satisfied, and therefore our Theorem 2.2 can be taken for granted.
3 Numerical results

Example 3.1. To find the value of polynomial \( P_{\max,l+1} \) making \( 1^5 + 2^5 + 3^5 = 276 \) for \( p = 3, \nu = 5 \).

To make sure in equality of found number to the value \( F_{\nu+1} = F_6 \). To make sure in equality to zero of differences \( \Delta P(4,5)_{p=3}, \Delta P(5,6)_{p=3} \) at set \( p, \nu \).

Here \( \max l = \min(p, \nu) = 3 \). Use of the formula (7) allows us to execute following calculations of values \( \varphi_i(m, \nu) \), see Table 1.

<table>
<thead>
<tr>
<th>( l )</th>
<th>( m )</th>
<th>( p^1 )</th>
<th>( p^2 )</th>
<th>( p^3 )</th>
<th>( p^4 )</th>
<th>( p^5 )</th>
<th>( p^6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1/2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>15/3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>25/4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-6</td>
<td>5</td>
<td>-5</td>
<td>1</td>
<td>10/5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>24</td>
<td>-26</td>
<td>-15</td>
<td>25</td>
<td>-9</td>
<td>1</td>
<td>1/6</td>
</tr>
</tbody>
</table>

To receive required value \( \varphi_i(m, \nu) \) any number from the given matrix it is multiplied, at first, by the corresponding number from external right column and, secondly, — by the corresponding number from external bottom line. Value \( \theta_i(p, \nu) \) is formed at the expense of summation of all numbers \( \varphi_i(m, \nu) \) in the line of given matrix with an index \( l \).

Summarizing values \( \theta_i(p, \nu), l = 1, 2, 3 \) we receive number

\[
P_4(3,5) = \sum_{l=1}^{3} S(5,l) \frac{1}{l+1} (p+1)(p) \bigg|_{p=3} = \frac{p}{4} \left( 3^2 - 2^3 p - 3 \ p^2 + 2 \ 5 \ p^3 \right) \bigg|_{p=3} = 276.
\]

We discover value of polynomial \( F_{\nu+1} = F_6 \) according to well-known expression:
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\[ F_6 \big|_{p=3} = \frac{1}{6} \sum_{g=1}^{6} C_{g-1} B_{6-g} p^g \bigg|_{p=3} = \frac{p^2}{2} \left( -\frac{1}{2} + \frac{5}{2} p^2 + \frac{6}{2} p^3 + p^4 \right) \bigg|_{p=3} = 276. \]

Further, if you use the formula (7) also after reaching number \( l \) specified value \( \text{max} \ l \) (see Table 1), it is possible to calculate the differences

\[ \Delta P(4,5) \big|_{p=3} = (-6 + 5 \cdot 3 + 5 \cdot 3^2 - 5 \cdot 3^3 - 3^4) \cdot 3 = 0, \]
\[ \Delta P(5,6) \big|_{p=3} = (24 - 26 \cdot 3 - 15 \cdot 3^2 + 25 \cdot 3^3 - 9 \cdot 3^4 + 3^5) \cdot 3 = 0. \]

It is easy to see that values \( P_4(3,5) \) and \( F_6 \) coincide among themselves, and any of differences \( \Delta P(4,5) \big|_{p=3}, \Delta P(5,6) \big|_{p=3} \) is equal to zero.

As we see, degree of our polynomial in standard form for \( p < \nu \) turn out below, than in Faulhaber’s polynomial.

4 Conclusion

So, within new approach we presented the introduced combinatorial polynomial with decrease in its degree and according to simplification of its record in comparison with Faulhaber's formula [2] for \( p < \nu \). That is we offer combinatorial alternative to known representation of the powers sum.

Two theorems on vanishing of differences of combinatorial polynomials for values from \( p < \nu \) to \( \nu \) are proved. The method of calculating our polynomial on the basis of use of its parts is specified (7). Comparison of the values found according to the proposed formula of the combinatorial polynomial and Faulhaber's formula made it possible to verify their coincidence.

References


https://doi.org/10.1017/cbo9780511987045
https://doi.org/10.14498/vsgtu1099

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