Restrained Independent 2-Domination in the Join and Corona of Graphs

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Abstract

A restrained independent 2-dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex not in $S$ is dominated at least twice and adjacent to at least one vertex not in $S$, and every pair of vertices in $S$ are not adjacent. In this paper, we characterized the restrained independent 2-dominating sets of the join and corona of graphs and calculate their restrained independent 2-domination numbers.

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1 Introduction and Preliminary Results

Let $G = (V(G), E(G))$ be an undirected graph. A set $S \subseteq V(G)$ is an independent set of $G$ if for every $x, y \in S$, $xy \notin E(G)$. A set $S \subseteq V(G)$ is a restrained set of $G$ if for every $x \in V(G) \setminus S$, there exists $y \in V(G) \setminus S$ such that $xy \in E(G)$.

A set $S \subseteq V(G)$ is a dominating set of $G$ if for every $v \in V(G) \setminus S$, there exists $u \in S$ such that $uv \in E(G)$. The domination number of $G$, denoted by
$\gamma(G)$, is the smallest cardinality of a dominating set of $G$. A set $S \subseteq V(G)$ is a 2-dominating set of $G$ if for every $v \in V(G) \setminus S$, $|S \cap N_G(v)| \geq 2$. The 2-domination number of $G$, denoted by $\gamma_2(G)$, is the smallest cardinality of a 2-dominating set of $G$. A set $S \subseteq V(G)$ is a restrained 2-dominating set of $G$ if $S$ is a restrained set and a 2-dominating set of $G$. The restrained 2-domination number of $G$, denoted by $\gamma_{r2}(G)$, is the smallest cardinality of a restrained 2-dominating set of $G$. A set $S \subseteq V(G)$ is an independent 2-dominating set of $G$ if $S$ is an independent set and a 2-dominating set of $G$. The independent 2-domination number of $G$, denoted by $i_2(G)$, is the smallest cardinality of an independent 2-dominating set of $G$. A set $S \subseteq V(G)$ is a restrained independent 2-dominating set of $G$ if $S$ is an independent set, a restrained set, and a 2-dominating set of $G$. The restrained independent 2-domination number of $G$, denoted by $i_{r2}(G)$, is the smallest cardinality of a restrained independent 2-dominating set of $G$.

The concept of 2-domination and restrained 2-domination was studied in [1] and [3], respectively. In [2] and [4], the 2-domination and the restrained 2-domination of the join and corona of graphs were studied. For more details on domination and its variations, see [5].

Remark 1.1 Every leaf of a graph is contained in a restrained independent 2-dominating set.

Remark 1.2 The complete graph has no restrained independent 2-dominating set.

2 Join of Graphs

The join of two graphs $G$ and $H$, denoted by $G + H$, is the graph with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Lemma 2.1 If $m$ and $n$ are positive integers both greater than or equal to two, then $i_{r2}(K_m + K_n) = n$.

Proof: Let $S$ be a minimum restrained independent 2-dominating set of $K_m + K_n$. By Remark 1.1, $K_m$ has no restrained independent 2-dominating set. Thus, $S = V(K_n)$ is a minimum restrained independent 2-dominating set of $K_n$. Hence, $i_{r2}(K_m + K_n) = |S| = n$. \qed

Lemma 2.2 Let $G$ be a graph with no isolates and $n \geq 2$ a positive integer. If $i_2(G)$ does not exists, then $i_{r2}(G + K_n) = n$. 
Proof: Let $S$ be a minimum restrained independent 2-dominating set of $G + K_n$. Since $G$ has no restrained independent 2-dominating set, $S = V(K_n)$. Thus, $S$ is a minimum restrained independent 2-dominating set of $K_n$. Hence, $i_{r2}(G + K_n) = |S| = n$. □

**Lemma 2.3** Let $G$ be a graph with isolates but $G \not\cong K_m$ for $m \geq 2$ and $n \geq 2$ a positive integer. If $i_2(G)$ exists, then $i_{r2}(G + K_n) = i_2(G)$.

Proof: Let $S$ be a minimum restrained independent 2-dominating set of $G + K_n$. Suppose $x$ is an isolate of $G$. Then $x$ is a leaf of $G + K_n$ and hence, $x \in S$ because every leaf is contained in a restrained dominating set. Since $S$ is an independent set, $S \subseteq V(G)$. Consequently, $S$ is a minimum restrained independent 2-dominating set of $G$. Therefore, $i_{r2}(G + K_n) = i_2(G)$. □

**Theorem 2.4** Let $G$ and $H$ be graphs such that $G \not\cong K_m$ and $H \not\cong K_n$. Then $S \subseteq V(G + H)$ is a restrained independent 2-dominating set of $G + H$ if and only if $S$ is an independent 2-dominating set of $G$ or $S$ is an independent 2-dominating set of $H$.

Proof: Suppose $S \subseteq V(G + H)$ is a restrained independent 2-dominating set of $G + H$. Then either $S$ is an independent set of $G$ or $S$ is an independent set of $H$. Let $S \subseteq V(G)$. Suppose $S$ is not a 2-dominating set of $G$. Then there exists $v \in V(G) \setminus S$ such that $|S \cap N_G(v)| < 2$, which implies that $|S \cap N_{G+H}(v)| < 2$. This contradicts the assumption that $S$ is a 2-dominating set of $G + H$. Hence, $S$ is a 2-dominating set of $G$. Consequently, $S$ is an independent 2-dominating set of $G$. Similarly, if $S \subseteq V(H)$, then $S$ is an independent 2-dominating set of $G$.

Conversely, suppose $S$ is an independent 2-dominating set of $G$. Clearly, $S$ is an independent 2-dominating set of $G + H$. Next, consider the following cases:

Case 1. $G \cong K_m$.

Then $H \not\cong K_n$ and has no isolates (if $H$ has an isolate, then $S \subseteq V(H)$ contradicting the hypothesis). This implies that $S = V(G) = V(K_n)$. Let $u \in V(G + H) \setminus S = V(H)$. Since $H$ has no isolates, $u$ belongs to some components of $H$. Thus, there exists $v \in V(H) = V(G + H) \setminus S$ such that $uv \in E(G + H)$.

Case 2. $G \not\cong K_m$.

Then $G$ has a nontrivial component. Since $S$ is independent, there exists $z \in V(G) \setminus S$. By definition of $G + H$, $zw \in E(G + H)$ for all $w \in V(H) \subseteq V(G + H) \setminus S$.

Therefore, $S$ is a restrained independent 2-dominating set of $G + H$. Similarly, if $S$ is an independent 2-dominating set of $H$, then $S$ is a restrained independent 2-dominating set of $G + H$. □

The next result is a direct consequence of Theorem 2.4.
Corollary 2.5 Let $G$ and $H$ be graphs. Then $i_{r2}(G+H) = \min\{i_2(G), i_2(H)\}$.

Proof: Suppose $i_2(G) \leq i_2(H)$. Let $S$ be a minimum independent 2-dominating set of $G$. Then $|S| = i_2(G)$. By Theorem 2.4, $S$ is a restrained independent 2-dominating set of $G + H$. Thus,

$$i_{r2}(G + H) \leq |S| = i_2(G).$$

Next, suppose $S'$ is a minimum restrained independent 2-dominating set of $G + H$. Then $i_{r2}(G + H) = |S'|$. By Theorem 2.4, $S'$ is an independent 2-dominating set of $G$. Hence,

$$i_{r2}(G + H) = |S'| \geq i_2(G).$$

Therefore, $i_{r2}(G + H) = i_2(G)$. Similarly, if we assume that $i_2(H) \leq i_2(G)$, then $i_{r2}(G + H) = i_2(H)$. Consequently, $i_{r2}(G + H) = \min\{i_2(G), i_2(H)\}$. □

3 Corona of Graphs

Let $G$ and $H$ be graphs of order $m$ and $n$, respectively. The corona of two graphs $G$ and $H$ is the graph $G \odot H$ obtained by taking one copy of $G$ and $m$ copies of $H$, and then joining the $i$th vertex of $G$ to every vertex of the $i$th copy of $H$.

Theorem 3.1 Let $G$ and $H$ be graphs each of order at least 2, where $G$ has no isolates if $H \cong K_n$. Then $C \subseteq V(G \odot H)$ is a restrained independent 2-dominating set of $G \odot H$ if and only if $C = \bigcup_{v \in V(G)} S^v$, where $S^v$ is an independent 2-dominating set of $H^v$ for all $v \in V(G)$.

Proof: Suppose $C \subseteq V(G \odot H)$ is a restrained independent 2-dominating set of $G \odot H$. Clearly, $C \cap V(H^v)$ is an independent 2-dominating set of $H^v$ for all $v \in V(G)$. For each $v \in V(G)$, let $S^v = C \cap V(H^v)$. Then, $C = \bigcup_{v \in V(G)} S^v$, where $S^v$ is an independent 2-dominating set of $H^v$ for all $v \in V(G)$.

Conversely, suppose $C = \bigcup_{v \in V(G)} S^v$, where $S^v$ is an independent 2-dominating set of $H^v$ for all $v \in V(G)$. Then, $C = \bigcup_{v \in V(G)} S^v$ is an independent 2-dominating set of $G \odot H$. Consider the following cases:

Case 1. $H \cong K_n$.

Then $S^v = V(H^v)$ for all $v \in V(G)$. Clearly, $v \in V(G \odot H) \setminus C$. Since $G$
has no isolates, there exists \( w \in V(G) \) such that \( vw \in E(G) \subseteq E(G \circ H) \).

Case 2. \( H \not\cong \overline{K_n} \).

Then \( H^v \) has a nontrivial component for all \( v \in V(G) \). Since \( S^v \) is independent, there exists \( z \in V(H) \setminus S^v \) for all \( v \in V(G) \). This implies that \( z \in V(G \circ H) \setminus C \). Thus \( zv \in E(G \circ H) \) for all \( v \in V(G) \subseteq V(G \circ H) \setminus C \).

Therefore, \( C \) is a restrained independent 2-dominating set of \( G \circ H \). \( \square \)

The next corollary is a direct consequence of Theorem 3.1.

**Corollary 3.2** Let \( G \) and \( H \) be graphs each of order at least 2, where \( G \) has no isolates if \( H \cong \overline{K_n} \). Then \( i_r^2(G \circ H) = |V(G)| \cdot i_2(H) \).

*Proof*: Let \( S \) be a minimum independent 2-dominating set of \( H \). For each \( v \in V(G) \), let \( S^v \subseteq V(H^v) \) be an independent 2-dominating set of \( H^v \) such that \( |S^v| = |S| \). Then \( i_2(H^v) = |S| = i_2(H) \). By Theorem 3.1, \( C = \bigcup_{v \in V(G)} S^v \) is a restrained independent 2-dominating set of \( G \circ H \). Thus,

\[
i_r^2(G \circ H) \leq |C| = \sum_{v \in V(G)} |S^v| = |V(G)| \cdot i_2(H).
\]

Next, suppose \( C' \) is a minimum restrained independent 2-dominating set of \( G \circ H \). Then \( i_r^2(G \circ H) = |C'| \). By Theorem 3.1, \( C' = \bigcup_{v \in V(G)} S^v \), where \( S^v \) is an independent 2-dominating set of \( H^v \) for all \( v \in V(G) \). Hence,

\[
i_r^2(G \circ H) = |C'| = \sum_{v \in V(G)} |S^v| \geq |V(G)| \cdot i_2(H).
\]

Therefore, \( i_r^2(G \circ H) = |V(G)| \cdot i_2(H) \). \( \square \)

The following results follows from Corollary 3.2.

**Corollary 3.3** Let \( G \) be a connected graph of order \( m \geq 2 \) and \( n \geq 2 \) a positive integer. Then \( i_r^2(G \circ \overline{K_n}) = mn \).

**Corollary 3.4** Let \( G \) be a graph with no isolates of order \( m \geq 2 \) and \( n \geq 2 \) a positive integer. Then \( i_r^2(G \circ \overline{K_n}) = mn \).

**Corollary 3.5** Let \( m \) be a positive integer and \( H \) a graph such that \( H \not\cong \overline{K_n} \). Then \( i_r^2(K_m \circ H) = m \cdot i_2(H) \).
References


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