Fair Evaluation of Life Insurance Policies with Periodic Rebalancing between Asset Portfolios and Interest Rate Guarantee

Massimo Costabile

Department of Economics, Statistics and Finance
University of Calabria, Italy

Marcellino Gaudenzi

Department of Economics and Statistics
University of Udine, Italy

Copyright © 2017 Massimo Costabile and Marcellino Gaudenzi. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

We consider the problem of evaluating at fair rates an innovative life insurance policy with a rebalancing scheme between different asset portfolios and an embedded interest rate guarantee. The premiums are invested in two portfolios of assets characterized by different levels of risk and sums are transferred from one fund to the other at some prefixed dates, depending on the performance of the funds. The dynamics of each fund is approximated by means of binomial lattices but, since the remixing feature makes the evolution of the riskier fund path-dependent, we propose a model based on “representative” values to keep the computational cost of the evaluation problem at a reasonable level. The usual backward induction coupled with linear interpolation allows us to determine the policy value at inception.

Mathematics Subject Classification: 91G60

Keywords: fair valuation, rebalancing scheme, binomial lattices
1 Introduction

In recent years, competition with financial institutions such as banks and open-ended investment companies has induced life insurers to offer increasingly sophisticated policies. In fact, as evidenced also by Gatzert and Schmeiser [5], beside traditional insurance contracts, new life insurance financial policies are becoming increasingly popular. The aim of such contracts is to capture the potential of higher returns from equity markets without abandoning the premium protection furnished by traditional insurance contracts. Among innovative contracts it is worth mentioning those in which the premiums paid by policyholders are distributed among portfolios characterized by different level of risk and where the allocation between portfolios is changed periodically in order to meet the embedded guarantees. Examples of such policies are the so-called hybrid products particularly diffused in the German market. The benefits of hybrid products are fully or partly guaranteed by investing a portion of the premiums in a conventional premium reserve stock while the remainder of the premiums is invested in a portfolio of risky assets. Hybrid products are said to be static when the allocation of premiums between the conventional premium reserve stock and the risky fund is established at the contract inception and it is not modified during the policy lifetime, whereas hybrid products are called dynamic when the asset allocation is changed periodically in order to meet the guarantees. Kochanski and Karnarski [7] developed a partial internal model to assess the solvency capital requirement for hybrid products while Bonhert and Gatzert [2] analyzed the impact of dynamic hybrid products on the fair valuation and risk assessment of an insurer with a portfolio consisting of traditional participating life insurance contracts and dynamic hybrid products. In both cases all the evaluations are conducted by using Monte Carlo simulations. Life insurance policies with benefit depending upon the performance of different portfolios and periodical rebalancing mechanism are becoming popular also in the Italian market. In general, these contracts are structured in such a way that a part of the premiums is invested in an internal fund directly managed by the insurer and characterized by low level of risk and stable returns, the so-called Gestione Separata. The remaining part of the premiums is invested in an external fund, usually riskier than the Gestione Separata, chosen by the policy-holder according to her/his risk attitude. The insurer periodically modifies the allocation of the invested premiums between the funds to capture the opportunity of higher returns under the constraint that the premiums invested in the Gestione Separata accrues interests at the guaranteed rate so as to protect the initial investment of the policyholder.

We now give a detailed description of a stylized version of such a policy offered in the Italian market. Ignoring mortality for the time being, a target interest rate and a target time horizon (not greater than the contract maturity) are
fixed. The policy is such that a single premium is paid at the outset of the contract. A part of the single premium, equal to the single premium discounted at the target interest rate over the target time horizon, is invested in the Gestione Separata (the low-volatility fund from now onward). The remaining part of the premium is invested in the riskier external fund (the high-volatility fund). The contract specifies also a set of rebalancing dates in correspondence of which sums are transferred from one fund to the other one according to the following scheme.

From the contract inception to the target date, if the rate of return of the low-volatility fund at any remix date is greater than the target interest rate, the surplus is transferred to the high-volatility fund. Vice versa, if the low-volatility fund rate of return is lower than the target rate, a sum is transferred from the high-volatility to the low-volatility fund so that the low-volatility fund value accrues interests at the target rate. The remixing scheme is such that at the target date the value of the low-volatility fund is equal to the single premium paid at inception by the policy-holder.

From the target date to the policy maturity, the target interest rate is set equal to zero, so that at each remaining remix date, if the low-volatility fund is greater than the single premium, the surplus is invested in the high-volatility fund while if the low-volatility fund value is lower than the single premium a sum is withdrawn from the high-volatility fund and is invested in the low-volatility fund to restore the level of the single premium. Finally, at maturity the policyholder receives the current value of the two funds. We underline that the target rate represents a minimum guaranteed interest rate. This implies that, if at any remix date the low-volatility fund rate of return is lower than the target rate, and the high-volatility fund is not sufficient to compensate for the drawdown, the insurer has to put extra money in the low-volatility fund to fulfill the guarantee. The cost of this downside protection is represented by a proportional fee applied periodically by the insurer.

The following example should further clarify how the policy works. In Table 1, we illustrate a possible scenario of the two fund rate of returns and the consequent dynamics of the policy value. We consider a target interest rate of 3.5% per annum and a target date of five years while the policy maturity is ten years. The single premium paid at the contract inception is 100 while the rebalancing mechanism acts at each anniversary of the contract. The fee applied by the insurer is 100 basis points and it is charged yearly. A guarantee is embedded into the contract assuring that the increment of the low-volatility fund value, between two consecutive remix dates, can not be lower than the target rate, i.e., the target rate represents a minimum guaranteed interest rate. The sum invested in the low-volatility fund is $100 \times (1 + 0.035)^{-5} = 84.2$, while the sum invested in the high-volatility fund is $100 - 84.2 = 15.8$. At the end of the first year, the low-volatility fund rate of return is 5.2% so that its value is
$84.2 \times 1.052 = 88.58$ that is reduced, after the charge of the fee, to $88.58 \times 0.99 = 87.69$. Because the initial value of the low-volatility fund, increased at the target rate, is $84.2 \times 1.035 = 87.14$, the surplus $87.69 - 87.14 = 0.55$ is transferred to the high-volatility fund whose current value becomes $15.8 \times 1.028 \times 0.99 + 0.55 = 16.63$. The total policy value is $87.14 + 16.63 = 103.77$. It is interesting to see what happens at the third anniversary of the contract. The low-volatility fund rate of return is 3.2% so that the corresponding fund value becomes $90.19 \times 1.032 \times 0.99 = 92.15$ that is lower than $90.19 \times 1.035 = 93.35$. The difference, equal to 1.2, has to be financed by the high-volatility fund whose value becomes $17.74 \times 1.04 \times 0.99 - 1.2 = 17.06$. The total policy value is $93.35 + 17.06 = 110.41$. From the end of the target period, e.g. year 5, to the policy maturity, the low-volatility portfolio value remains equal to 100, the single premium paid at inception. For example, at the end of year 8, the low-volatility fund value is $100 \times (1 - 0.03) \times 0.99 = 96.03$ and a sum equal to 3.97 must be transferred from the high-volatility fund to restore the guarantee. The high-volatility fund value becomes $4.11 \times (1 - 0.08) \times 0.99 = 3.74$, but it is not sufficient to assure that the low-volatility fund is still equal to the single premium. Hence, the high-volatility fund is exhausted and the minimum guarantee obliges the insurer to inject the amount $3.97 - 3.74 = 0.23$ in the low-volatility fund. At the end of year 9, the rate of return of the low-volatility fund is 3.2% and this allows to transfer a sum to the high-volatility fund whose value becomes 2.17. Finally, at maturity the policy value is given by the sum of the two fund values, i.e., $104.2 + 2.28 = 106.48$.

<table>
<thead>
<tr>
<th>year</th>
<th>low-volatility return</th>
<th>low-volatility fund value</th>
<th>high-volatility return</th>
<th>high-volatility fund value</th>
<th>policy value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>84.2</td>
<td>-</td>
<td>15.8</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>5.2%</td>
<td>87.14</td>
<td>2.8%</td>
<td>16.63</td>
<td>103.77</td>
</tr>
<tr>
<td>2</td>
<td>4.8%</td>
<td>90.19</td>
<td>6.4%</td>
<td>17.74</td>
<td>107.93</td>
</tr>
<tr>
<td>3</td>
<td>3.2%</td>
<td>93.35</td>
<td>4%</td>
<td>17.06</td>
<td>110.41</td>
</tr>
<tr>
<td>4</td>
<td>1.8%</td>
<td>96.62</td>
<td>2.7%</td>
<td>14.81</td>
<td>111.43</td>
</tr>
<tr>
<td>5</td>
<td>-0.5%</td>
<td>100.00</td>
<td>3.5%</td>
<td>10.35</td>
<td>110.35</td>
</tr>
<tr>
<td>6</td>
<td>-1.2%</td>
<td>100.00</td>
<td>1.3%</td>
<td>8.19</td>
<td>108.19</td>
</tr>
<tr>
<td>7</td>
<td>-2.7%</td>
<td>100.00</td>
<td>-4%</td>
<td>4.11</td>
<td>104.11</td>
</tr>
<tr>
<td>8</td>
<td>-3%</td>
<td>100.00</td>
<td>-8%</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>9</td>
<td>3.2%</td>
<td>100.00</td>
<td>-</td>
<td>2.17</td>
<td>102.17</td>
</tr>
<tr>
<td>10</td>
<td>4.2%</td>
<td>104.2</td>
<td>5%</td>
<td>2.28</td>
<td>106.48</td>
</tr>
</tbody>
</table>

Table 1: This table reports an example regarding to a possible scenario of the evolution of the policy value
We consider the problem of evaluating the policy at fair rates in a complete, frictionless market, free of arbitrage opportunities, as in the classical Brennan and Schwartz model [3]. The evolution of the two asset portfolios is described by two correlated geometric Wiener processes. Since the rebalancing scheme induces outflows and inflows from the portfolios, it is very difficult to obtain an analytical formula for the policy value at inception. This implies the need to resort to a numerical evaluation model to obtain an estimate of the requested solution.

A possible approach relies upon lattice-based models to approximate the dynamics of fund values. This choice is motivated by the fact that lattice-based models are simple tools, easy to implement and popular also among practitioners. Moreover, they allow to manage straightforwardly the presence of exotic features such as the option to surrender the policy before maturity. For these reasons, lattice-based models have been extensively applied in life insurance to solve the evaluation problem relative to policies with complex embedded riders. Among others, we may cite Bacinello [1] who considered guaranteed participating contracts with embedded surrender options, Yang and Dai [8] who proposed a model to evaluate variable annuities with a guaranteed minimum withdrawal benefit, and Wei et al. [9] who considered the problem of evaluating ratchet equity-indexed annuities under stochastic interest rates.

In the present paper, the problem of evaluating at fair rates the policy above described has been tackled in the Cox-Ross-Rubinstein (CRR hereafter [4]) framework. Hence, the dynamics of both funds is approximated by means of two recombining binomial trees. A problem arises because, at the first remix date, the sums transferred between the two funds determine several possible shifts of the realized high-volatility fund values. The consequence is that the lattice approximating the high-volatility fund dynamics lacks its reconnecting structure and the evaluation problem becomes computationally burdensome, even when a small number of time steps is considered. To overcome this obstacle, instead of considering all possible high-volatility fund values at each time step, we build up sets of “representative” values so that the computational complexity of the pricing problem is kept at a reasonable level. The policy value at inception is then computed through the usual backward induction scheme coupled with linear interpolation. Finally, the value of the fee which makes the policy actuarially fair is determined by solving a nonlinear equation through a common iterative method.

The rest of the paper is organized as follows. In Section 2, we describe the evaluation problem and the model proposed to computed the fair value of the insurance fee. In Section 3, we illustrate numerical results of the proposed model while in Section 4 we draw conclusions.
2 The evaluation model

We consider a policy written at time $t_0 = 0$ with target period $\tau$ and maturity $T$ ($\tau \leq T$). We do not consider mortality, hence the policy may be considered as a pure financial contract. A single premium, $U$, is paid by the insured person at the contract inception. A part of the single premium, $U(1+i_g)^{-\tau}$, is invested in a low-volatility fund while the remaining part is invested in a high-volatility fund ($i_g$ is the annual target interest rate). The allocation of the invested premium between the two portfolios is modified at dates, $t_k, k = 1, \ldots, m$, according to the following scheme. From the contract inception to the target date, if the low-volatility fund accrues interests at a rate greater than the target rate, the insurer transfers the surplus into the high-volatility fund. Vice versa, if the rate of return of the low-volatility fund is lower than the target interest rate, the insurer transfers a sum from the high-volatility fund to the low-volatility fund so that the low-volatility fund value increases at the target interest rate. From the target date to the policy maturity, if the low-volatility fund value is greater than the single premium, the surplus is invested in the high-volatility fund. Vice versa, if the low-volatility fund is smaller than the single premium, a sum is transferred from the high-volatility fund in such a way that the initial investment of the policyholder is preserved. The insurer applies an annual percentage fee, $\alpha$, to both portfolios whose value is reduced accordingly.

We denote the value at time $t$ of the two funds, $F_L(t)$ and $F_H(t)$, respectively. The subscript $L$ highlights that the first portfolio is characterized by low volatility, $\sigma_L$, and hence is less risky than the second fund that has higher volatility, $\sigma_H$. We do not make assumptions on the composition of the two portfolios with respect to different asset classes such as bonds, equities, and real estate. We simply consider them as aggregate and assume that the total market value evolves according to the following equations under the risk-neutral probability measure,

$$dF_L(t) = F_L(t)(r dt + \sigma_L dZ_L(t)), \quad F_L(0) = U(1+i_g)^{-\tau}, \quad F_L(t_k^+) = F_L(t_k) + D_k$$

$$dF_H(t) = F_H(t)(r dt + \sigma_H dZ_H(t)), \quad F_H(0) = U(1 - (1 + i_g)^{-\tau}), \quad F_H(t_k^+) = \max(F_H(t_k) - D_k, 0),$$

where

$$D_k = F_L(t_{k-1}^+)(1 + i_g) - (1 - \alpha)(1 + r_L(t_k)),$$

$r$ is the risk-free force of interest, and $Z_L(t)$ and $Z_H(t)$ are two standard Brownian motions with correlation $\rho$. Moreover, $\alpha$ is the insurance fee, and $r_L(t_k) = F_L(t_k)/F_L(t_{k-1}^+) - 1$ is the rate of return of the low-volatility fund in the time interval $[t_{k-1}^+, t_k]$, i.e., just after the rebalancing at time $t_{k-1}$ and
just before the rebalancing at time $t_k$. As it is evident by looking at equations (1) and (2), the rebalancing scheme induces outflows and inflows from the two funds, and this makes it very difficult to obtain a closed form expression for the policy value at inception. To overcome this obstacle, we build up an evaluation model based on CRR binomial lattices to approximate the evolution of the fund values. We divide, at first, the policy maturity, $T$, into $N$ equally-spaced time intervals of length $\Delta t = T/N$. Starting from the low (high)-volatility fund at the contract inception, $F_L(0)$ ($F_H(0)$), at the end of each time interval, the low (high)-volatility fund value may jump up by the factor $u_L = e^{\sigma_L \sqrt{\Delta t}}$ ($u_H = e^{\sigma_H \sqrt{\Delta t}}$), or it may jump down by the factor $d_L = 1/u_L$ ($d_H = 1/u_H$). The risk-neutral probability of an upward movement of the fund with low (high) volatility is $q_L = (e^{r \Delta t} - d_L)/(u_L - d_L)$ ($q_H = (e^{r \Delta t} - d_H)/(u_H - d_H)$) while the probability of a downward movement is $1 - q_L$ ($1 - q_H$).

In order to complete the construction of the discrete approximating model we are left to define the joint transition probabilities. We label $q_{uu}$ the probability that both processes jump upward in a time interval $\Delta t$, while $q_{ud}$ is the probability that the low-volatility fund jumps upward and the high-volatility fund jumps downward, and so on. The joint probabilities are computed following the approach of Hull and White [6]. To do this, we start by considering the case of zero correlation. In this case, the transition probabilities are determined simply as the product of marginal transition probabilities, i.e., they are $q_{uu} = q_L q_H$, $q_{ud} = q_L (1 - q_H)$, $q_{du} = (1 - q_L) q_H$, and $q_{dd} = (1 - q_L) (1 - q_H)$. In the case of correlated processes the joint transition probabilities are modified by adding a term $\varepsilon$ in such a way that the marginal probabilities remain unaffected. More explicitly, we have $q_{uu} = q_L q_H + \varepsilon$, $q_{ud} = q_L (1 - q_H) - \varepsilon$, $q_{du} = (1 - q_L) q_H - \varepsilon$, and $q_{dd} = (1 - q_L) (1 - q_H) + \varepsilon$. Ignoring terms of order higher than $\Delta t$, the matching condition between the correlation of the two discrete approximating processes and the correlation of the corresponding continuous-time processes, reduces to

$$q_{uu} - q_{ud} - q_{du} + q_{dd} = \rho.$$ 

Considering that when $\Delta t$ tends to zero the marginal transition probabilities tend to $\frac{1}{2}$, we obtain $\varepsilon = \frac{\rho}{4}$. Hence,

$$q_{uu} = q_L q_H + \frac{\rho}{4}, \quad q_{ud} = q_L (1 - q_H) - \frac{\rho}{4},$$

$$q_{du} = (1 - q_L) q_H - \frac{\rho}{4}, \quad q_{dd} = (1 - q_L) (1 - q_H) + \frac{\rho}{4}.$$ 

It may happen, particularly when a strong positive or negative correlation is considered, that the joint probabilities are not legitimate, i.e., they do not fall into $[0,1]$. In these cases, the adjustment parameter $\varepsilon$ is set equal to the maximum value for which all the transition probabilities are non-negative. This
implies that the condition that guarantees the correlation matching is no more satisfied, but this distortion has a negligible impact on the precision of the evaluation model, because when $\Delta t$ is very small both $q_L$ and $q_H$ approximate the value $\frac{1}{2}$, and the joint transition probabilities fall in $[0, 1]$. In order to clarify the construction of the evaluation model we start by considering a simple example with two time steps. The policy premium paid at the contract inception is $U = 100$, the maturity is $T = 2$ years, the target date is $\tau = 1$ year, and the target rate is $i_g = 3.5\%$ per annum. The sum invested in the low-volatility fund is $100(1.035^{-1}) = 96.62$ while $3.38$ is the sum invested in the high-volatility fund. To construct the two-steps binomial lattices describing the evolution the fund values, we observe at first that $\Delta t = 1$ year. Assuming $\sigma_L = 0.05$ and $\sigma_H = 0.15$, we obtain $u_L = 1.05$ and $u_H = 1.16$. At the end of the first year, the low-volatility fund may assume the value $101.45$ in the case of an upward movement or $92.02$ in the case of a downward movement. The value of the low-volatility fund increased at the target rate is $100$. The possible values of the high-volatility fund are $3.92$, and $2.91$. Ignoring for simplicity the fees to be applied by the insurer, after the first time step the remix mechanism acts in the following way. In the case of an up step of the low-volatility fund, the surplus $1.45$ is transferred to the high-volatility fund generating two possible values, $5.37$ and $4.36$. Vice versa, in the case of a down step of the low-volatility fund, a sum equal to $7.98$ is needed to restore the level $100$. Because the high-volatility fund consistency is not enough to cover the sum, it is exhausted and the insurer has to guarantee the difference equal to $7.98 - 3.92 = 4.06$ in the case of an up jump of the high-volatility fund, or to $7.98 - 2.91 = 5.07$ in the case of a down jump of the high-volatility fund. At the next time step the low volatility fund may assume two possible values, $105$ if an up step occurs or $95.24$ if a down jump takes place. In contrast, the lattice describing the evolution of the high-volatility fund lacks its reconnecting property and assumes $5$ possible values, $6.23$ and $4.63$ stemming from $5.37$, $5.06$ and $3.76$ stemming from $4.36$, and $0$ in the case the fund is exhausted at the previous time step. After the second time step, the policy matures, and its value is given by the maximum of the sum of the two fund values and the single premium guaranteed by the insurer. Hence, we have seven possible policy values, $111.23$, $110.06$, $109.63$, $105$, $108.76$, $101.47$, $100.3$, greater than the guaranteed single premium and three possible policy values, $99.87$, $99$, and $95.24$ smaller than the guaranteed amount. In this case, the insurer has to restore the level of the guarantee. This example is illustrated in Figure 1.

As it is evident by looking at this simple example, the effect of the remixing mechanism on the lattices describing the fund values evolution is twofold. In fact, at each remix date the low-volatility lattice collapses into a single node and then it takes the usual binomial structure. In contrast, all nodes of the high-volatility fund are shifted downward and upward due to inflows and
outflows needed to assure that the low-volatility lattice accrues interest at the target rate. This implies that the high-volatility lattice lacks its reconnecting structure at the first remix date, and the evaluation model becomes readily unmanageable from a computational point of view, even when a small number of time steps is considered. To overcome this obstacle, since it is not possible to keep track of all possible values of the high-volatility fund, we associate a set of representative values with each possible node of the low-volatility fund. In details, we label \((i, j)\) a generic node of the low-volatility fund lattice, where the first index identifies one of the \(N\) time periods \(i \Delta t\) \((i = 0, \ldots, N)\) while the second index is relative to the vertical position of the fund value in the lattice, so that \((i, 0)\) represents the greatest low-volatility fund value at time \(i \Delta t\), \((i, 1)\) stands for the second greatest low-volatility fund value at time \(i \Delta t\), and so on. Moreover, we build up the lattice so that each time interval, \(t_k - t_{k-1}\) \((k = 1, \ldots, m)\), is divided into \(h\) time steps. Hence, starting from \(G(0) = F_L(0)\), at each remix date, \(t_k, k = 1, \ldots, m\), the low-volatility fund is reset to the guaranteed level, that is,

\[
G(k) = \begin{cases} 
F_L(0)(1 + i_g)^{t_k} & \text{if } 1 \leq t_k < \tau \\
U & \text{if } t_k \geq \tau.
\end{cases}
\]

Hence, the low-volatility fund may assume one of the values

\[
F_L(i, j) = G(\lceil i/h \rceil - 1) u_L^{i-\lceil i/h \rceil - 1} d_L^{i} (1 - \alpha I_{\{i=kh\}}),
\]
\[ i = 0, \ldots, N, \quad j = 0, \ldots, j_L(i) \equiv i - (\lceil i/h \rceil - 1)h, \]

where \( \lceil x \rceil \) returns the nearest integer greater than or equal to \( x \).

An example of the binomial lattice describing the evolution of the low-volatility fund value with \( T = 4 \) years, \( \tau = 2 \) years, \( h = 2 \), \( N = 8 \) time steps, and annual rebalancing is illustrated in Figure 2.

The dynamics of the high-volatility fund value is described by a recombining binomial lattice for the first \( h \) steps, i.e., until the first remix date is reached. Then, due to outflows and inflows, from \( (h + 1)\Delta t \) until maturity, the lattice is no more reconnecting. To keep the computational cost of the evaluation problem at a reasonable level, we consider at each time step a set of representative values of the high volatility fund. These sets are built up as follows. At first, we consider the maximum and the minimum possible values of the high volatility fund at each time step. The maximum possible high-volatility fund value at time \( i\Delta t \) is,

\[
F_H^M(i) = \{F_H^M(i - 1) + \max(F_L(i - 1,0) - G(\lceil i/h \rceil - 1))I_{(i-1=kh)}\}u_H(1-\alpha I_{(i=kh)}),
\]

while the minimum possible high-volatility fund value at time \( i\Delta t \) is,

\[
F_H^m(i) = \max(F_H^m(i-1)-G(\lceil i/h \rceil - 1) - F_L(i-1,\tilde{j}_L(i-1)))I_{(i-1=kh)},0)d_H(1-\alpha I_{(i=kh)}).
\]

The representative values of the high-volatility fund at a generic time step \( i \), \( i = 1, \ldots, N \), are labeled \( F_H(i,l) \), \( l = 0, \ldots, \tilde{j}_H(i) \), where \( F_H(i,0) = F_H^M(i) \), \( F_H(i,\tilde{j}_H(i)) = F_H^m(i) \) and defined as follows. For the first \( h \) time steps the representative values coincide with those generated by the recombining binomial tree describing the evolution of the high-volatility fund. From time step \( h + 1 \)
onward, the representative values are generated, starting from the greatest possible value \( F_H(i, 0) \), in such a way that the difference between two consecutive fund values is proportional to \( F_H(0) \Delta t \) (we will discuss this point later). Once the lattice describing the evolution of the low-volatility fund value and the sets of representative values of the high-volatility fund have been constructed, we need to identify the successors of each possible value of the low-volatility fund and of the high-volatility fund at each time step. This is easily done in the case of the low-volatility fund. In fact, each value \( F_L(i, j) \) \((i \neq hk)\) has two possible successors, \( F_L(i + 1, j) = F_L(i, j)u_L \) in case of an up jump, and \( F_L(i + 1, j + 1) = F_L(i, j)d_L \) in the case of a down-jump, while at each epoch coinciding with a remix date \((i = hk)\), the low-volatility fund, reset at level \( G(i/h) \), has successors \( F_L(i + 1, 0) \) and \( F_L(i + 1, 1) \) in the case of an up jump and of a down jump, respectively.

With regards to the high-volatility fund values, the successors are easily identified for the first \( h \) time steps because the evolution of the fund value is described by a reconnecting lattice. Hence, \( F_H(i, l) \) jumps up to \( F_H(i + 1, l) = F_H(i, l)u_H \) or down to \( F_H(i + 1, l + 1) = F_H(i, l)d_H \), for \( i < h \). At time \( h \Delta t \), the remix mechanism acts for the first time, and from that date until maturity the model considers representative values of the high-volatility fund. Hence, when \( i \geq h \), it is not possible to identify the successors of a representative value, \( F_H(i, l) \), because, in general, \( F_H(i, l)u_H \) and \( F_H(i, l)d_H \) are not representative values of the high-volatility fund at the next time step, \( i + 1 \). To solve this problem we proceed as follows.

At each remix date, following the procedure previously described, a set of representative high-volatility fund values, \( F_H(i, l), l = 0, \ldots, l_H(i) \), is constructed just before the rebalancing. After the rebalancing, to take into account inflows and outflows from the fund, a new set of representative high-volatility fund values with elements \( F_H^+(i, l) = \max(F_H(i, l) + F_L(i, j) - G(i/h), 0) \), \( l = 0, \ldots, l_H(i), j = 0, \ldots, l_H(i) \), is generated. We label \( l_x(i, l, j) \) \((x = u, d)\) the value of the index \( l \in [0, l_H(i + 1) - 1] \) at time \( i + 1 \) such that \( F_H(i + 1, l_x(i, l, j) + 1) \leq F_H^+(i, l)x_H \leq F_H(i + 1, l_x(i, l, j)) \). In other words, we lock the “true” value \( F_H^+(i, l)x_H \), between two consecutive representative high-volatility fund values at time step \( i + 1 \). In the same way, at time step \( i \), not coinciding with a remix date, we define \( l_x(i, l) \) \((x = u, d)\), to be the index taking values in \([0, l_H(i + 1) - 1]\) such that \( F_H(i + 1, l_x(i, l) + 1) \leq F_H(i, l)x_H \leq F_H(i + 1, l_x(i, l)) \).

Associated with \( l_x(i, l, j) \) and \( l_x(i, l) \) we introduce the coefficients

\[
\omega_x(i, l, j) = \frac{F_H^+(i, l)x_H - F_H(i + 1, l_x(i, l, j) + 1)}{F_H(i + 1, l_x(i, l, j)) - F_H(i + 1, l_x(i, l, j) + 1)},
\]

and

\[
\omega_x(i, l) = \frac{F_H(i, l)x_H - F_H(i + 1, l_x(i, l) + 1)}{F_H(i + 1, l_x(i, l)) - F_H(i + 1, l_x(i, l) + 1)}.
\]
To compute the policy value at inception through the usual backward induction procedure, we start from maturity, where the policy value is set equal to
\[ W(N,j,l) = F_L(N,j) + F_H(N,l), \]
\[ j = 0, \ldots, N, l = 0, \ldots, L. \]
The policy value at previous time steps is computed as follows. At a generic time step \( i < N \), since it is not possible to find the successors of a high-volatility fund value, we use linear interpolation to obtain the missing values. Hence, at a time step not coinciding with a remix date, i.e. \( i \neq h k \), we compute the policy value as
\[
W(i,j,l) = e^{-r\Delta t}\{q_{uu}[\omega_u(i,l)W(i+1,j,l_u(i,l)) + (1-\omega_u(i,l))W(i+1,j,l_u(i,l)+1)] + q_{ud}[\omega_d(i,l)W(i+1,j,l_d(i,l)) + (1-\omega_d(i,l))W(i+1,j,l_d(i,l)+1)] + q_{du}[\omega_u(i,l)W(i+1,j+1,l_u(i,l)) + (1-\omega_u(i,l))W(i+1,j+1,l_u(i,l)+1)] + q_{dd}[\omega_d(i,l)W(i+1,j+1,l_d(i,l)) + (1-\omega_d(i,l))W(i+1,j+1,l_d(i,l)+1)]\}. \tag{3}
\]
At each time step coinciding with a remix date, i.e., when \( i = h k \), the backward induction scheme (3) has to be modified by substituting \( \omega_x(i,l) \) with \( \omega_x(i,l,j) \), and \( l_x(i,l) \) with \( l_x(i,l,j) \). Once the policy value at inception, \( W(0,0,0) \equiv W(\alpha) \), has been obtained, the insurance fee value that makes the policy actuarially fair, is computed by solving the non-linear equation \( W(\alpha) = U \). This can be done through common numerical root-finding schemes, such as the bisection method, or others.
The evaluation scheme may be easily modified to handle the case in which a surrender option is embedded into the contract. Following Bacinello [1], we assume that the policyholder decides to abandon the contract if it is financially convenient, and the evaluation model has to be modified in a way similar to that applied to American options. This is done by setting \( W(i,j,l) \), at each epoch where the policyholder may surrender the policy, equal to the maximum between the surrender value and the policy value in absence of surrender, computed through equation (3).

3 Numerical Results

To assess the accuracy of the proposed model, we evaluate policies with different maturities, \( T = 2, 4, 6, 8, 10 \) years. In all cases, the target date, \( \tau \), is set equal to \( T/2 \), and the target rate, \( i_g \), is 3.5% per year. The remaining parameters are set as follows: the continuously compounded risk-free interest rate, \( r \), is 4% per year, the high volatility is \( \sigma_H = 0.3 \), the low volatility is \( \sigma_L = 0.05 \), and the correlation, \( \rho \), is 0. The number of steps, \( N \), used to compute the policy values has been determined as the integer number closest to 100 such that \( h = N/T \) is an integer. In other words, the number of steps
Life insurance policies with periodic rebalancing

<table>
<thead>
<tr>
<th>$T$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>100.7844</td>
<td>100.6940</td>
<td>100.5748</td>
<td>100.4954</td>
<td>100.4483</td>
</tr>
<tr>
<td>MC</td>
<td>100.7858</td>
<td>100.6939</td>
<td>100.5754</td>
<td>100.4836</td>
<td>100.4309</td>
</tr>
<tr>
<td>s.e.</td>
<td>(0.0042)</td>
<td>(0.0077)</td>
<td>(0.0123)</td>
<td>(0.0183)</td>
<td>(0.0265)</td>
</tr>
<tr>
<td>fee</td>
<td>0.0108</td>
<td>0.0021</td>
<td>0.0012</td>
<td>0.0008</td>
<td>0.0006</td>
</tr>
</tbody>
</table>

Table 2: in this table we consider policies with maturities, $T = 2, 4, 6, 8, 10$ years. In all cases, the target date, $\tau$, is set equal to $T/2$, and the target rate, $i_g$, is 3.5% per year. The remaining parameters are set as follows: the continuously compounded risk-free interest rate, is $r = 4\%$ per year, the high volatility is $\sigma_H = 0.3$, the low volatility is $\sigma_L = 0.05$, and the correlation is $\rho = 0$. The row labeled L contains the policy values at inception computed by the proposed model. The row labeled MC reports the corresponding values obtained by $10^7$ Monte Carlo simulations, while in the row labeled s.e., the standard error is reported in brackets. The last row, labeled fee, contains the fair fees computed with the proposed method.

used for computing the policy values at inception is roughly equal to 100 in all the cases considered. As usual when representative values are applied in lattice-based models, the high-volatility fund values are generated in such a way that the difference between two consecutive values is proportional to $\Delta t$. In particular, this difference is set equal to $\beta F_H(0)\Delta t$, where $\beta$ is a parameter used to control the fineness of the high-volatility fund values.\(^1\)

The policy values at inception, obtained by applying the proposed method, are reported in the row labeled L of Table 2. In the row labeled MC, are reported the corresponding policy values computed by the Monte Carlo method with 10,000,000 simulation runs. In the row labeled s.e., the corresponding standard error at the 95% confidence level is reported in brackets. In the last row, labeled fee, are reported the fair fees computed by the lattice-based model. We consider now the case when a surrender option is embedded in the contract. We assume that the surrender option can be exercised at each anniversary of the contract and in the case of an early termination of the contract the policyholder obtains a sum equal to the single premium paid at inception. All the parameters used are the same of those considered above for the case without surrender option. The resulting values are illustrated in Table 3 where the row labeled L contains the policy value at inception computed by the proposed

\(^1\)The parameter $\beta$ is set equal to 1 except when the high-volatility fund is greater than $F_H = 1000$. In this case, to save the computational effort of the evaluation model, $\beta$ is set equal to 5. Extensive numerical experiments have shown that this choice does not influence the accuracy of the proposed method.
Table 3: in this table we consider policies with an embedded surrender option. Different maturities, $T = 2, 4, 6, 8, 10$ years are considered. In all the cases, the target date, $\tau$, is set equal to $T/2$, and the target rate, $i_g$ is 3.5% per annum. The remaining parameters are set as follows: the continuously compounded risk-free interest rate, is $r = 4\%$ per annum, the high volatility is $\sigma_H = 0.3$, the low volatility is $\sigma_L = 0.05$, and the correlation is $\rho = 0$. The row labeled $L$ contains the policy values at inception computed by the proposed model, while the row labeled fee reports the corresponding fair fees.

$$
\begin{array}{cccccc}
T & 2 & 4 & 6 & 8 & 10 \\
L & 100.7844 & 100.8291 & 100.9297 & 101.0642 & 101.3039 \\
fee & 0.0108 & 0.0029 & 0.0024 & 0.0023 & 0.0024 \\
\end{array}
$$

lattice-based model while the row labeled fee contains the corresponding fees that make the policy actuarially fair. As it is obvious, the presence of the surrender option makes the policy more valuable and induces an increment of the corresponding fee.\(^2\)

In Figure 3, we illustrate how the low volatility influences the fair policy fee when the high volatility is kept fixed at level $\sigma_H = 0.3$. We have considered a maturity $T = 4$ years, a target date $\tau = T/2$, and a target interest rate $i_g = 3.5\%$ per annum. The continuously compounded risk-free interest rate, $r$, is 4\% per annum, and the correlation is $\rho = 0$. It emerges that the higher the low volatility, the higher the fair policy fee. This is due to the fact that an increment of the low volatility increases the probability that the corresponding fund performs very well or very bad. Since the unfavorable outcomes are covered by the guarantee of the insurer, this makes the policy more valuable. A similar effect on the fair fee is observed when, other parameters being fixed, the high volatility is increased.

In Figure 4, we illustrate the effect of the correlation between the two funds on the policy fee. As before, we have considered a maturity $T = 4$ years, a target date $\tau = T/2$, and a target interest rate $i_g = 3.5\%$ per annum. The remaining parameters are set as follows: the continuously compounded risk-free interest rate is $r = 4\%$ per annum, the high volatility is $\sigma_H = 0.3$, the low volatility is $\sigma_L = 0.05$. It emerges that the higher the correlation, the higher the policy fee. This may be explained by observing that an increasing correlation determines an increasing probability that both funds performs very well.

\(^2\)In the case with maturity $T = 2$ years the value of the policy with surrender option is the same of the value of the policy without surrender option. This is obvious because the surrender is possible only at the target date, $\tau = 1$ year, where both the policy value without surrender and the surrender value are equal to the single premium.
Life insurance policies with periodic rebalancing

Figure 3: this figure illustrates how the low volatility, ranging in the interval [0.05, 0.20], influences the fair fee. We have considered a maturity $T = 4$ years, a target date $\tau = T/2$, and a target interest rate $i_g = 3.5\%$ per annum. The remaining parameters are set as follows: the continuously compounded risk-free interest rate, $r = 4\%$ per annum, the high volatility is $\sigma_H = 0.3$, and the correlation is $\rho = 0$.

well or very bad. As before, since the unfavorable outcomes are covered by the guarantee of the insurer, this makes the policy more valuable.

In Figure 5, we illustrate the impact of the target rate on the policy fee. We have considered a maturity $T = 4$ years and a target date $\tau = T/2$. The remaining parameters are set as follows: the continuously compounded risk-free interest rate is $r = 4\%$ per annum, the high volatility is $\sigma_H = 0.3$, the low volatility is $\sigma_L = 0.05$, and the correlation is $\rho = 0$. It emerges that increasing the target rate induces a reduction in the fair policy fee and that the magnitude of the reduction is negligible because it consists of few basis points.

In Figure 6, we illustrate the impact of the risk-free interest rate on the fair fee. We have considered a maturity $T = 4$ years, a target date $\tau = T/2$, and a target interest rate $i_g = 3.5\%$ per annum. The remaining parameters are set as follows: the high volatility is $\sigma_H = 0.3$, the low volatility is $\sigma_L = 0.05$, and the correlation is $\rho = 0$. It emerges that the fair fee declines as the risk-free interest rate increases. This is in line with the well-known relation already observed in the case of equity-linked policies with interest rate guarantee between the fair value of the contract at inception and the risk-free rate.
Figure 4: this figure illustrates how the correlation, ranging in the interval $[-1, 1]$, influences the fair fee. We have considered a maturity $T = 4$ years, a target date $\tau = T/2$, and a target interest rate $i_g = 3.5\%$ per annum. The remaining parameters are set as follows: the continuously compounded risk-free interest rate is $r = 4\%$ per annum, the high volatility is $\sigma_H = 0.3$, and the low volatility is $\sigma_L = 0.05$.

4 Conclusions

We have considered the problem of evaluating at fair rates a life insurance policy with a remixing mechanism which acts, at some fixed dates, in a way that sums are transferred between two funds with different level of risk. The dynamics of the two funds is approximated by means of binomial lattices. Since the rebalancing mechanism makes the evaluation problem strongly path-dependent, it is not possible to keep track of all the fund values. We overcome this obstacle by considering fictitious representative values of the fund with the higher volatility. The policy value at inception is computed through the usual backward induction scheme coupled with linear interpolation. Finally, the corresponding insurance fee is obtained by solving the non-linear equation that makes the policy actuarially fair. Numerical results illustrate the consistency of the method.
Life insurance policies with periodic rebalancing

Figure 5: this figure illustrates how the target rate, ranging in the interval $[0.01, 0.04]$, influences the fair fee. We have considered a maturity $T = 4$ years and a target date $\tau = T/2$. The remaining parameters are set as follows: the continuously compounded risk-free interest rate is $r = 4\%$ per annum, the high volatility is $\sigma_H = 0.3$, the low volatility is $\sigma_L = 0.05$, and the correlation is $\rho = 0$.

Figure 6: this figure illustrates how the risk-free rate, ranging in the interval $[0.02, 0.05]$, influences the fair fee. We have considered a maturity $T = 4$ years, a target date $\tau = T/2$, and a target interest rate $i_g = 3.5\%$ per annum. The remaining parameters are set as follows: the high volatility is $\sigma_H = 0.3$, the low volatility is $\sigma_L = 0.05$, and the correlation is $\rho = 0$. 
References


Received: November 4, 2017; Published: December 17, 2017