A New Approach to Meusnier’s Theorem in Game Theory

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Abstract
In this paper strategy sphere for the continuous games inspired by Meusnier’s theorem is defined. Meusnier’s sphere of a regular surface, an equilibrium point of a saddle surface and differential game theory are used for defining strategy sphere.

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1 Introduction
The mathematical theory of games was invented by John von Neumann and Oskar Morgenstern (1944) “Theory of Games and Economic Behavior” [10].

In 1950, John F. Nash Jr. wrote three papers Equilibrium Points in n-person games [7], The Bergaining problems [8] and Non-cooperative Games [9].
Nash’s Ph. D. Thesis “Non-cooperative games,” is one of the foundational paper of game theory. The concept of an equilibrium for non-cooperative games “Nash Equilibrium” is introduced. John Nash indicated that every game in which the set of actions available to each player is finite has at least one mixed-strategy equilibrium.

In the literature, there are several books and papers about “static non-cooperative infinite games, dynamics game and nash and saddle-point equilibria of infinite dynamic games [1].

A continuous game has a surface for pay-off function. We can give a surface in $\mathbb{R}^3$ with parametric curves. For example, let $\varphi(u, v)$ be a topological surface in $\mathbb{R}^3$ then $\varphi(u_0, v)$ and $\varphi(u, v_0)$ are the curves in $\varphi(u, v)$ surface, where $u_0$ and $v_0$ are fixed values. Every parameter curve is a pure strategy for players. If payoff function of a game is $P(u, v)$ then the surface form of the game is $\varphi(u, v, P(u, v))$. For every $v = v_0$ : fixed, the curves $\varphi(u, v_0, P(u, v_0))$ are pure strategies of players, etc.

The pure strategies curves, velocity, tangent and normal vectors, curvature can be studied in game theory. At this point Meusnier theorem and Meusnier sphere are important in game theory.

We study in this article Meusnier theorem and Meusnier sphere for two person zero-sum differential games.

## 2 Meusnier’s Theorem

Meusnier’s theorem states that all curves on a surface passing through a given point $p$ and having the same tangent line at $p$ also have the same normal curvature at $p$ and their osculating circles form a sphere. The theorem was first announced by Jean Baptiste Meusnier in 1776 [5].

Let $M$ be a regular surface in $\mathbb{R}^3$, $\{\alpha(s)\}$ be a curve in $M$ and $k_1(s)$ and $k_2(s)$ be curvatures of $\alpha(s)$. Suppose that $\alpha$ is not asymptotic in $M$, i.e.,

$$II(\alpha'(s), \alpha'(s)) = <\alpha'(s), \alpha'(s)> \neq 0$$

where $II$ is a second fundamental form of $M$. We know that

$$\frac{d^2\alpha}{ds^2} = k_1 \overrightarrow{n}$$

$$k_1 \langle n_p, N_p \rangle = \langle \alpha''(s), N_p \rangle$$
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\[ k_1 = \frac{\langle \alpha''(s), N_p \rangle}{\langle n_p, N_p \rangle} \]

\[ k_1 = -\frac{II(\alpha', \alpha')}{\langle n, N \rangle} \]

In this study we assume that all curves we study are not asymptotic.

If the surface normal \( N_p \) and \( II \) second fundamental form is fixed at the point \( p \in \alpha \), then the value of the curvature \( k_1(s) \) is belong to \( n_p \). So we have “All curves lying on a surface \( M \) and having the same tangent line at given point \( p \in M \) have the same normal curvature \( k_1 \) at this point”.

Let these curves be \( \alpha_i, 1 \leq i \leq n \). Then

\[ \frac{d\alpha_i}{dt} = t_i = t. \]

If \( \phi = \angle(n_i, N_p) \) then we have

\[ (k_i)_1 = -\frac{II(\frac{d\alpha_i}{ds}, \frac{d\alpha_i}{ds})}{\cos \phi_i} \]

and

\[ (k_i)_1 \cos \phi_i = -II(\frac{d\alpha_i}{ds}, \frac{d\alpha_i}{ds}) \]

Let \( \beta \) be intersecting curve of the plane \( sp\{t, N_p\} \) and the surface \( M \). We use \( n_\beta, \phi_\beta \) and \( k_{1\beta} \) for the curve \( \beta \). So we have

\[ k_{1\beta} = -II(\frac{d\alpha}{ds}, \frac{d\alpha}{ds})_p. \]

It is clear that \( k_{1\beta} = \min\{(k_i)_1\} \). Then

\[ R_\beta = \max\left\{ \frac{1}{(k_i)_1} = \max\{R_i\} \right\} \]
and

\[ \frac{1}{R_\beta} = -II(\alpha', \alpha'), \]

**Theorem 2.1 Meusnier’s Theorem** Curvature circles of all curves lying a surface \( M \subset E^3 \) and having the same tangent line at given point \( p \) in \( S^2_R(p + RN_p) \) [2, 5].

**Definition 2.2** The sphere in Theorem 1 is called Meusnier’s sphere [2].

### 3 Game Theory-Continuous Games

The second tool for this article is continuous games. Nash equilibrium is a main concept in the theory of games and the most widely used method of predicting the outcome of a strategic interaction in the social sciences. A game contains the following three elements: a set of players, a set of actions (or pure-strategies) available to each player, and a payoff (or utility) function for each player. The payoff functions represent each player’s preferences over action profiles, where an action profile is simply a list of actions, each player. A pure-strategy is an action profile with the property that no single player can get a higher payoff by deviating unilaterally from this profile. The references of this section are [1, 3, 6, 7, 12].

**Definition 3.1** The system

\[ \Gamma = (X,Y,K) \]

where \( X \) and \( Y \) are nonempty sets and the function \( K : X \times Y \to R \), is called a two person zero-sum game in normal form [6].

The elements \( x \in X \) and \( y \in Y \) are called strategies of players 1 and 2, respectively, in the game \( \Gamma \), the elements of the cartesian product \( X \times Y \), the function \( K \) is payoff of player 1. Players 2’s payoff in situation \((x, y)\) is a set equal to \( K(x, y)\).

**Definition 3.2** Two-person zero-sum games in which both players have finite set of strategies are called matrix games [6].

**Definition 3.3** Two-person zero-sum games in which both players have infinite set of strategies and \( K \) is continuous are called continuous games [6].
In the theory of games it is supposed that the behavior of both players is rational. Suppose player 1 chooses strategy $x$, then, at worst case he will win $\min_y K(x, y)$. Therefore, player 1 can always guarantee himself the payoff $\max_x \min_y K(x, y)$.

$$\vartheta = \sup_{x \in X} \inf_{y \in Y} K(x, y)$$

is called the lower value of the game. For player 2 it is possible to provide similar reasonings. Suppose he chooses strategy $y$. Then, at worst, he will lose $\min_x K(x, y)$. Therefore, the second player can always guarantee himself the payoff $\max_y \min_x K(x, y)$. The number

$$\overline{\vartheta} = \inf_{y \in Y} \sup_{x \in X} K(x, y)$$

is called the upper value of the game $\Gamma$. In two-person zero-sum game $\Gamma$, $\vartheta \leq \overline{\vartheta}$.

In the game $\Gamma = (X, Y, K)$ it is natural to consider as optimal a situation $(x^*, y^*) \in X \times Y$ the deviation from which there is no advantage for both players. Such a point $(x^*, y^*)$ is called the equilibrium point and the optimality principle based on constructing an equilibrium point is called equilibrium principle.

**Definition 3.4** In the two-person zero-sum game $\Gamma = (X, Y, K)$ the point $(x^*, y^*)$ is called an equilibrium point, or saddle point if

$$K(x, y^*) \leq K(x^*, y^*)$$
$$K(x^*, y) \geq K(x^*, y^*)$$

for all $x \in X$ and $y \in Y$.

In the matrix game $\Gamma_A$ the equilibrium points are the saddle points of the payoff matrix $A$, i.e. the point $(i^*, j^*)$ for which for all $i \in M$ and $j \in N$ the following inequalities are satisfied

$$a_{ij^*} \leq a_{i^*j} \leq a_{i^*j^*}.$$

**Definition 3.5** Let $(x^*, y^*)$ be a saddle point the game $\Gamma$. Then the number

$$\vartheta = K(x^*, y^*)$$

is called the value of the game $\vartheta$. 
Let $D$ be a plane domain. Let $\varphi^*$ be at least $C^2$ class function defined on $\Gamma$. The function $\varphi^*$ then defines a curve $T$ in $(x, y, \varphi)$ space of which $\Gamma$ is a simply covered projection. Such a curve is said to satisfy a three point condition with constant $D$ provided that any plane which intersects it in three or more points has maximum inclination less than $D$. The concept of three point condition has been an essential feature in the theory of quasi-linear elliptic partial differential equations in two independent variables

$$a(x, y, p, q)\varphi_{xx} + 2b(x, y, p, q)\varphi_{xy} + c(x, y, p, q)\varphi_{yy} = 0$$  \hspace{1cm} (1)

where $p = \varphi_x, q = \varphi_y, ac - b^2 = 1$.

To see the significance of this condition, we need only observe that if $\varphi(x, y)$ is a solution of (1), then

$$\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2 = -\frac{1}{a}(a\varphi_{xx} + 2b\varphi_{xy}\varphi_{yy} + c\varphi_{yy}^2) \leq 0.$$  \hspace{1cm} (2)

Thus every solution of (1) represents a surface of non-positive Gaussian curvature [4].

## 4 Saddle Surfaces

Let $\varphi(x, y)$ be any two-variable function. Then $(x_0, y_0)$ is a saddle point of $\phi$ if at the same time $\phi(x, y_0)$ assumes its maximum at $x = x_0$ and $\phi(x_0, y)$ assumes its minimum at $y = y_0$ [2, 11].

**Definition 4.1** Let $\varphi : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a differentiable surface, if equation (2) is true at $\varphi(x_0, y_0)$ then $\varphi(x_0, y_0)$ is called a saddle point of surface $\varphi$ [6,8].

**Definition 4.2** A surface has at least one saddle point the surface is called saddle surface [2, 11].

For example, $\varphi(u, v) = (u, v, u^2 - v^2), \varphi(u, v) = (u, v, uv)$ are saddle surfaces with saddle point $\varphi(0, 0)$.

Also saddle surfaces are called hyperbolic paraboloid, because of parameter curve of a saddle surface being parabolas and hyperbolas. A saddle point of hyperbolic paraboloid (saddle surface) an important point, such that this point is minimum of maximum points of parabola and maximum of minimum points of hyperbolas.
5 Strategy Sphere

Using Meusnier’s theorem in game theory, we give a theorem as follows.

**Theorem 5.1** Let $S$ be a saddle surface and $\{\alpha_i\} \subset S$ be all curves set on $S$, and $P \in S$ be a saddle point. Let a relation $\sim$ be defined as $\alpha_i \sim \alpha_j \iff \alpha_i$ and $\alpha_j$ are passing from $P$. The relation $\sim$ is an equivalent relation on $\{\alpha_i\}$.

**Proof 5.2** It is obviously that, for $\forall i, j$

- Reflexive property: $\alpha_i \sim \alpha_i$
- Symmetric property: $\alpha_i \sim \alpha_j \iff \alpha_j \sim \alpha_i$
- Transitive property: $\alpha_i \sim \alpha_j$ and $\alpha_j \sim \alpha_k \implies \alpha_i \sim \alpha_k$

are satisfied.

Let $G$ be a continuous game whose surface forms a saddle surface. Then $\{\alpha_i\}$ equivalent class is the set of all strategies that have equilibrium point. Let $A$ be a subset of $\{\alpha_i\}$ which tangent vectors are same at equilibrium point. Then we can write Meusnier’s theorem for these curves.

**Theorem 5.3** Let $G$ be a continuous game whose surface forms a saddle surface. Then curvature circle of strategies has an equilibrium point and same tangent vector at equilibrium point in $S^2_R(p + RN_p)$.

**Definition 5.4** The sphere $S^2_R(p + RN_p)$ is called strategy sphere of the game $G$.

For example, let $\varphi(u, v) = (u, v, v^2 - u^2)$ be surface of a continuous game $G$.

Then for $u = 0, v = 0$, $\varphi(0, 0)$ is a saddle point, i.e., equilibrium point. The normal surface is $N = (0, 0, 1)$ at $\varphi(0, 0)$ and $R = \frac{1}{2}$. So, $S^2_\frac{1}{2}(0, 0, \frac{1}{2})$ is strategy sphere of the game $G$.

6 Conclusion

Meusnier’s theorem is valid for $c^2$-class surface. A continuous game whose surface forms a saddle surface has an equilibrium point. In addition Meusnier’s sphere as strategy sphere is defined. This is a new approach to Meusnier’s theorem in game theory.
References


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